Conformal Blocks, Entanglement Entropy & Heavy States

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The Institute of Mathematical Scinces, Chennai

“Higher-point conformal blocks and entanglement entropy of heavy states”

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The Broad Picture

- Holography has interesting universal features.
- Some particular conformal blocks in 2D CFT
  $\equiv$ geodesic lengths in asymptotic AdS_3
- **Goal** : To show that conformal blocks with two heavy & arbitrary number of light operators factorize & to find its dual bulk picture.
- **Application** : Relevant in the context of EE for excited states with multiple intervals.

The Plan..

- Introduction
- Boundary Computation
- The Bulk Picture
- EE : An Application
- Conclusions

Introduction

What are Conformal Blocks?

- Consider a p-point correlator

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\cdots\mathcal{O}(z_p)\rangle$$

- Insert $p-3$ resolutions of the identity

$$\sum_{\alpha,\beta,\xi,\ldots} \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle \alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle \zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

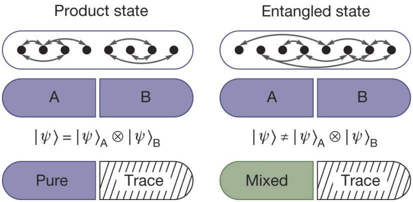
- A typical term of this sum is called conformal block

$$\mathcal{F}_p(z_i, h_i, \tilde{h}_i) := \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle \alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle \zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

- These are building blocks of CFT correlators.

What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- EE is Von Neumann entropy of reduced density matrix $\rho_A = \text{Tr}_B(\rho_{tot})$



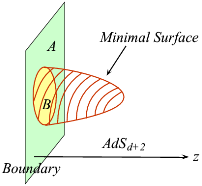
[Courtesy : R. Islam et al.]

$$S_{\partial A} = -\text{Tr}_A(\rho_A \log \rho_A)$$

- A measure of entanglement between subsystems. Vanishes for pure states.

What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- EE is a geometric quantity



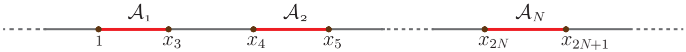
[Courtesy : T. Nishioka et al.]

$$S_{\mathcal{A}} = \frac{\min[\gamma_{\mathcal{A}}]}{4G_N}$$

- A measure of entanglement between subsystems. Vanishes for pure states.

What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- Disjoint intervals in 1+1 dimensional systems



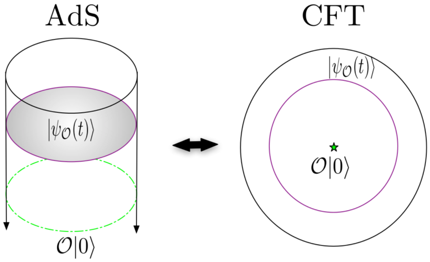
$$S_{\partial A} = -\mathrm{Tr}_A(\rho_A \log \rho_A).$$

- A measure of entanglement between subsystems. Vanishes for pure states.

Which excited states?

- Using the state-operator correspondence

$$|\psi\rangle = \mathcal{O}_H(0)|0\rangle \quad \text{and} \quad \langle\psi| = \lim_{z,\bar{z}\to\infty} \bar{z}^{2h_H} z^{2h_H} \langle 0|\mathcal{O}_H(z,\bar{z}).$$



[Courtesy : J. Kaplan]

- $\mathcal{O}_H(0)$ has very large scaling dimension. Corresponding states are heavy states.

Boundary Computation

Heavy-light correlators

- We are interested in

$$\langle \mathcal{O}_H(z_1,\bar{z}_1) \prod_{i=2}^{m+1} \mathcal{O}_L(z_i,\bar{z}_i)\, \mathcal{O}_H(z_{m+2},\bar{z}_{m+2})\rangle$$

- In terms of cross-ratios $x_i = \frac{(z_{m+2}-z_i)(z_2-z_1)}{(z_{m+2}-z_2)(z_i-z_1)}$

$$\Big\langle \mathcal{O}_H(\infty) \left[\mathcal{O}_L(1) \prod_{i=3}^{m+1} \mathcal{O}_L(x_i)\right] \mathcal{O}_H(0) \Big\rangle .$$

- We work in $c \to \infty$ limit for which

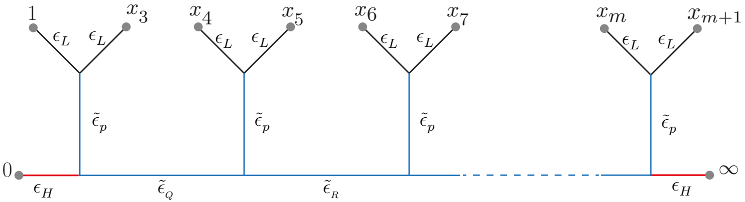
$$\mathcal{F}_{(p)}(z_i,h_i,\tilde{h}_i) = \exp\left[-\frac{c}{6} f_{(p)}(z_i,\epsilon_i,\tilde{\epsilon}_i)\right].$$

Heavy-light correlators

- We shall also work in the heavy-light limit

$$\epsilon_H = \frac{6h_H}{c} \sim \mathcal{O}(1)\ , \quad \epsilon_L = \frac{6h_L}{c} \ll 1$$

- And this particular OPE channel



Monodromy method

[Julian's Talk]

- Recall a typical conformal block looks like

$$\mathcal{F}_p(z_i, h_i, \tilde{h}_i) := \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle\alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

- Let's insert an additional operator, $\hat{\psi}(z)$

$$\begin{aligned}\Psi(z, z_i) &:= \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle\alpha|\hat{\psi}(z)\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle \\ &\approx \psi(z, z_i)\exp\left[-\frac{c}{6}f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)\right]\end{aligned}$$

- Choose that $\hat{\psi}(z)$ obeys the null-state condition at level 2

$$\left[L_{-2} - \frac{3}{2(2h_\psi+1)}L_{-1}^2\right]|\Psi\rangle = 0, \qquad \text{with, } h_\psi \overset{c\to\infty}{=} -\frac{1}{2} - \frac{9}{2c}$$

Monodromy method

- The differential operator representation gives an ODE

$$\frac{d^2\psi(z)}{dz^2} + T(z)\psi(z) = 0, \qquad \text{with, } T(z) = \sum_{i=1}^{p}\left[\frac{\epsilon_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}\right]$$

- Here, $\epsilon_i = 6h_i/c$ and c_i are the accessory parameters

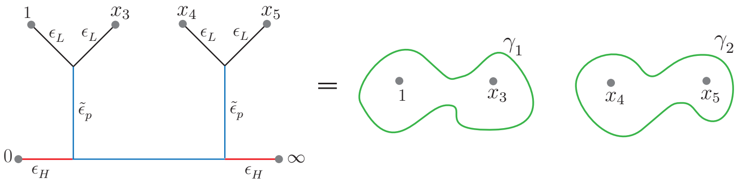
$$c_i = -\frac{\partial f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)}{\partial z_i} \quad \text{satisfying} \quad \frac{\partial c_i}{\partial z_j} = \frac{\partial c_j}{\partial z_i}$$

- Solve for the c_i, by using the monodromy properties of the solution $\psi(z)$ around the singularities of $T(z)$.

- Monodromy around a contour γ_k = info about the resultant operator which arises upon fusing the operators within γ_k

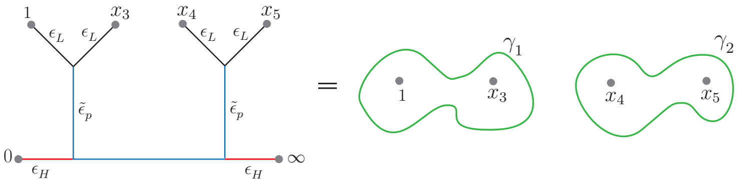
Monodromy method

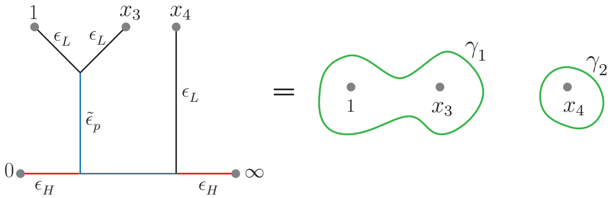
Choice of monodromy contour = Choice of OPE channel



Monodromy method

Choice of monodromy contour = Choice of OPE channel





Monodromy method

- The monodromy conditions for all the contours form a coupled system of equations for the accessory parameters.
- Performing the exercise for 5- and 6-point blocks provides sufficient intuition to guess the solutions.

Monodromy method

The accessory parameters can now be used to obtain the conformal block

$$c_i = -\frac{\partial f_{(p)}(z_i,\epsilon_i,\tilde{\epsilon}_i)}{\partial z_i} \qquad \mathcal{F}_{(p)}(z_i,h_i,\tilde{h}_i) = \exp\left[-\frac{c}{6}f_{(p)}(z_i,\epsilon_i,\tilde{\epsilon}_i)\right]$$

Even-point conformal blocks

- The $(m+2)$-point block factorizes into a product of $m/2$ 4-point conformal blocks

$$\begin{aligned}\mathcal{F}_{(m+2)}(\{x_i\};\epsilon_L,\epsilon_H;\tilde{\epsilon}_p) &= \prod_{\boldsymbol{\Omega}_i\mapsto\{(p,q)\}} \exp\left[-\frac{c}{6}f_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_p)\right] \\ &= \prod_{\boldsymbol{\Omega}_i\mapsto\{(p,q)\}} \mathcal{F}_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_p).\end{aligned}$$

$\boldsymbol{\Omega}_i$: Indicates the OPE channels / monodromy contours.

Monodromy method

Odd-point conformal blocks

- The $(m+2)$-point block factorizes into a product of $(m-1)/2$ 4-point conformal blocks and a 3-point function

$$
\begin{aligned}
\mathcal{F}_{(m+2)}(\{x_i\};\epsilon_L,\epsilon_H;\tilde{\epsilon}_p) &= x_s^{-\epsilon_L} \prod_{\mathbf{\Omega}_i^A \mapsto \{(p,q)\}} \exp\left[-\frac{c}{6} f_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_p)\right] \\
&= x_s^{-\epsilon_L} \prod_{\mathbf{\Omega}_i^A \mapsto \{(p,q)\}} \mathcal{F}_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_p).
\end{aligned}
$$

where, $f_{(4)}(x_i,x_j;\epsilon_L,\epsilon_H;\epsilon_p) = \epsilon_L\left((1-\alpha)\log x_i x_j + 2\log\frac{x_i^\alpha - x_j^\alpha}{\alpha}\right) + 2\tilde{\epsilon}_p \log\left[4\alpha\frac{x_j^{\alpha/2}+x_i^{\alpha/2}}{x_j^{\alpha/2}-x_i^{\alpha/2}}\right]$

with $\alpha = \sqrt{1-24h_H/c}$..

Caveats

This factorization is true only . . .

1. at large central charge
2. in the heavy-light limit
3. for this specific choice of OPE channels

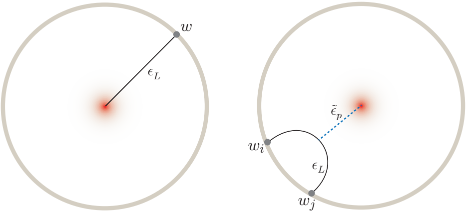
Bulk Picture

The dual geometry

- The heavy excited state is dual to the conical defect geometry

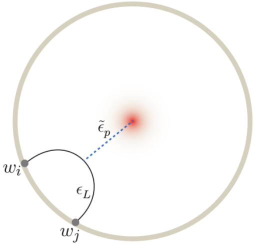
$$ds^2 = \frac{\alpha^2}{\cos^2\rho}\left(-dt^2 + \frac{1}{\alpha^2}d\rho^2 + \sin^2\rho\, d\phi^2\right), \quad \text{with } \alpha = \sqrt{1 - 24h_H/c}.$$

- The conformal blocks can be reproduced by considering lengths of suitable worldline configurations in the bulk.



4-point block from bulk

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(x_i)\mathcal{O}_L(x_j)\mathcal{O}_H(0)\rangle$$

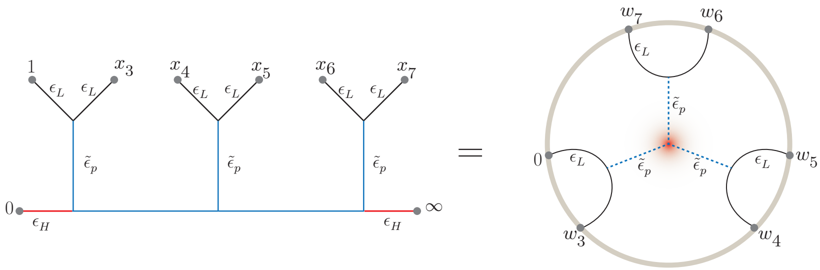


- The worldline action : $S = \epsilon_L l_L + \tilde{\epsilon}_p l_p$
- From cylinder to plane : $x_i = e^{iw_i}$ and $x_j = e^{iw_j}$

$$\mathcal{F}_{(4)}(x_i, x_j) = x_i^{-h_L} x_j^{-h_L} \times \; e^{-\frac{c}{6}S(w_i,w_j)}\Big|_{w_{i,j}=-i\,\log\, x_{i,j}}$$ *(Matches!)*

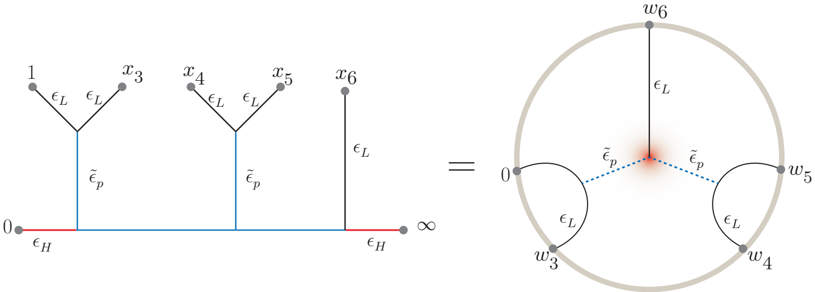
Higher point block from bulk

Even-point conformal blocks



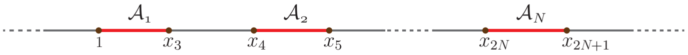
Higher point block from bulk

Odd-point conformal blocks



EE : An Application

EE for excited states



- EE from Rényi entropy : $S_{\mathcal{A}}^{(n)} = \frac{1}{1-n} \log \, \mathrm{tr}_{\mathcal{A}} \left(\rho_{\mathcal{A}}\right)^n \; ; \; n \to 1$

- Effectively need to compute (for $n \to 1$)

$$\begin{aligned} G_n(x_i, \bar{x}_i) &= \langle \Psi | \sigma(1) \bar{\sigma}(x_3) \sigma(x_4) \bar{\sigma}(x_5) \sigma(x_6) \bar{\sigma}(x_7) \ldots \sigma(x_{2N}) \bar{\sigma}(x_{2N+1}) | \Psi \rangle \\ &= \langle 0 | \; \Psi(\infty) \; \sigma(1) \bar{\sigma}(x_3) \; \prod_{i=4,6,\cdots}^{2N} \sigma(x_i) \bar{\sigma}(x_{i+1}) \; \Psi(0) \; |0\rangle \end{aligned}$$

- Dimensions of the twist and anti-twist operators

$$h_\sigma = h_{\bar{\sigma}} = \frac{c}{24} \left(n - \frac{1}{n} \right)$$

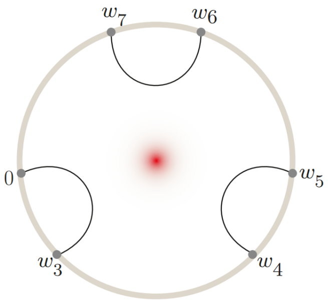
EE for excited states

- In the limit $n \to 1$

$\sigma, \bar{\sigma}$: Light operators
Ψ : Heavy operator

$$S_{\mathcal{A}} = \lim_{n\to 1} S_{\mathcal{A}}^{(n)} = \frac{c}{3} \min_i \left\{ \sum_{\widetilde{\boldsymbol{\Omega}}_i \mapsto \{(p,q)\}} \log \frac{(x_p^{\alpha} - x_q^{\alpha})}{\alpha (x_p x_q)^{\frac{\alpha-1}{2}}} \right\}.$$

with, $\alpha = \sqrt{1 - 24h_H/c}$



Conclusions & Outlook

Summary

- Higher point conformal blocks are tractable in the heavy-light limit.
- These conformal blocks can be reproduced precisely from the dual gravity picture.
- This is applied to find entanglement entropy of disjoint intervals in heavy states.
- This conformal block can be rewritten in terms of geodesic lengths (bulk locality?)

Outlook

Applications

1. Tripartite information
2. Mutual information in local quenches
3. Scrambling, chaos, . . .

Extensions

1. Higher spin holography
2. One-loop corrections
3. Higher dimensions, . . .

Thank you