Lecture notes

Some basics of AdS/CFT

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Abstract: These notes[1] are based on my evening lecture presented in Student Talks on Trending Topics in Theory (ST^4) 2017 at CMI, Chennai.

[1] A more detailed version of these notes based on a series of informal lectures given at IMSc, Chennai is under preperation. The write up can be found in my webpage soon(?).

Contents

1 Why bother?

The AdS/CFT correspondence is around for two decades without any concrete proof. You may ask why we should still bother even after 20 years or how does it even qualify as a "Trending Topic"! Here is a very quick motivation/justification. According to INSPIRE the total number of citations of Maldacena's original paper [1] = 12,727 (on May 8, 2017). You may still complain "So what? it's 20 years old!". Well, for last couple of years it has been receiving around 800 citations per year (see figure 1). If we consider only in the weekdays ($5 \times 52 = 260$) arXiv remains active (which is clearly an over estimation) that paper gets $800/260 = 3.08$ citations[1] per day! So AdS/CFT is still an *extremely* active field of research – a "Trending Topic in Theory". `(QED.)`

[1]This is probably the most concrete computation of this lecture. Also this section of motivation/justification is my only original contribution to these notes.

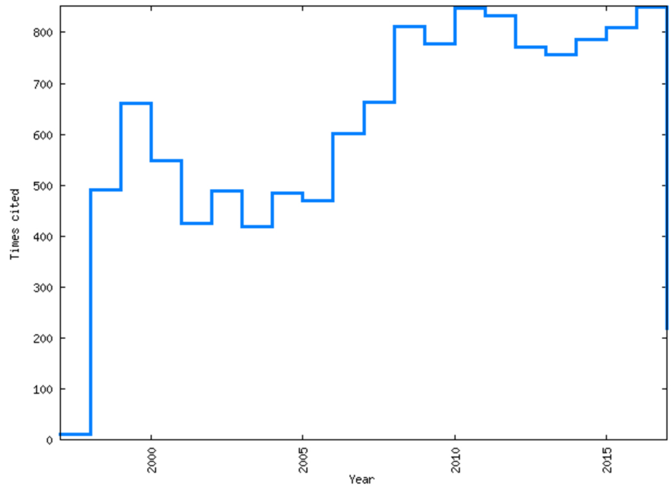


Figure 1: Citation summary of Maldacena's original paper.

I think this is good enough motivation to study or at least to be familiar with the AdS/CFT correspondence.

Apart from that there are two main "traditional motivations" to study AdS/CFT.

1. It provides the *only* description or rather definition of non-perturbative quantum gravity, although in a particular background.

2. It works like an amazing machine which converts some classical gravity results into some useful quantities in very strongly coupled field theories. This is the *only* analytic technique to study strongly coupled field theories.

Let me now state the modern version of Maldacena's conjecture very crudely

The statement of AdS/CFT correspondence

String theory in asymptotically AdS space-time ≡ A Quantum Field Theory on its boundary

This statement is a very 'coarse grained' version of the original conjecture. The main aim of this lecture is to introduce you (or review, in case you are already familiar) to Maldacena's original conjecture.

Plan of the lecture

☞ Dualities – definition, characteristics and few examples.

☞ How come $AdS_{d+1} = CFT_d$ or gauge = gravity?!

☞ "Derivation" of AdS/CFT – the famous decoupling argument.

☞ Dictionary & different regimes

Disclaimer & homework

Before all that I want to make it clear (which is obvious too!) that I *don't* claim originality of these notes. Many of these ideas and descriptions are shamelessly copied from the available literature. Also, effort has been made to keep these notes free from any errors but I'll be responsible for all typos or conceptual disputes, if any. Below is the only homework problem of this lecture.

✍ Please read these notes carefully and let me know if you find any type of mistakes.

2 Dualities in QFTs & string theory

Duality means equivalence between two seemingly different theories. This is actually a very old concept in physics. In this section I shall talk about dualities in quantum field theories and also in string theory. Before going into the details here are few *typical* characteristics of dualities.

❐ Characteristics of dualities

Two sides (theories) of a duality are typically related by following maps.

- Degrees of freedom or the Lagrangian need *not* be same.
- Global symmetries *coincide*.
- Equation of motion $\Longleftrightarrow$ Bianchi identity
- Weak coupling $\Longleftrightarrow$ Strong coupling

The last one typically holds but not always true. When it holds, one calls that a strong-weak duality. We will see that AdS/CFT is a famous example of that, as we go along.

2.1 Quantum field theories

Here is a list of dualities from quantum field theories and/or statistical mechanics.

1. Maxwell duality (1861 or 1931?)
2. Kramers-Wannier duality (1975)
3. Bosonization (1975)
4. Montonen-Olive duality (1977)
5. Seiberg-Witten duality (1994)

I will elaborate on one from the list *viz.* the Maxwell duality and comment on other dualities only briefly.

❐ Maxwell duality

The oldest example of duality goes back to Maxwell. The famous equations due to Maxwell for electric field $\vec{E}$, magnetic field $\vec{B}$, charge density ρ and electric current $\vec{J}$ are given by

$$\left.\begin{aligned} \nabla.\vec{E} &= \rho \\ \nabla\times\mathbf{B} &= \vec{J} + \frac{\partial \vec{E}}{\partial t} \end{aligned}\right\} \qquad \Longleftrightarrow \qquad \partial_\mu F^{\mu\nu} = J^\nu \tag{2.1}$$

$$\left.\begin{aligned}\nabla.\vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}\end{aligned}\right\} \qquad \Longleftrightarrow \qquad \partial_\mu \widetilde{F}^{\mu\nu} = 0 \tag{2.2}$$

where $\widetilde{F}^{\mu\nu} := \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the Hodge dual to $F^{\mu\nu}$. The equations (2.2) are independent of sources whereas (2.1) depend on sources. We refer to (2.1) as "Maxwell equations" and call (2.2) as *Binachi identities.*

Observe, for $\rho = 0$ and $\vec{J} = 0$, $\vec{E} \leftrightarrow -\vec{B}$ is a symmetry. Actually $\vec{E} \leftrightarrow -\vec{B}$ interchanges $F^{\mu\nu} \leftrightarrow \widetilde{F}^{\mu\nu}$ that amounts to interchanging

Dynamical "Maxwell equations" $\longleftrightarrow$ *Geometric* Binachi identities.

Here I'll choose a "trivial" example to illustrate the electro-magnetic duality. Let's consider a bunch of photons (A_μ) and they don't interact with any sources (e.g, electrons). That's the reason I call it a "trivial" theory.

$$Z = \int \mathcal{D}A_\mu e^{-i\frac{1}{4g^2}\int F_{\mu\nu}^2}\, \delta(\partial_\mu A^\mu) \tag{2.3}$$

The delta function ensures we are dealing with only *physical* of degrees of freedom. This is known as a gauge choice. Now let's perform a change of variable : $A_\mu \to F_{\mu\nu}$ i.e, our integration variable will be $F_{\mu\nu}$ instead of A_μ. This implies a Jacobian[2] which is not important for the dynamics of the system and can be taken out of the integral.

$$Z = \text{“Jacobian”} \times \int \mathcal{D}F_{\mu\nu}\, e^{-i\frac{1}{4g^2}\int F_{\mu\nu}^2}\, \delta(\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma}) \tag{2.4}$$

I have imposed the Bianchi identities over the path integral. Now our aim is to introduce a Lagrange multiplier C_α into the path integral and integrate out the *dynamical* $F_{\mu\nu}$ to write down a theory for the 'fake variable' C_α. Let's first introduce C_α

[2]The Jacobians are, in general, rather tricky in path integrals. They can be very important which may lead to *anomaly*. But for this particular example it is innocent (believe me!).

$$Z \approx \int \mathcal{D}F_{\mu\nu}\,\mathcal{D}C_\alpha\, e^{-i\int(\frac{1}{4g^2}F_{\mu\nu}^2 + \#\, C_\alpha \epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta})}\delta(\partial_\mu C^\mu) \tag{2.5}$$

The $\delta(\partial_\mu C^\mu)$ is there because if one shifts $C_\alpha \to C_\alpha + \partial_\alpha f$ in the exponent, that does nothing to the integral (extra piece vanishes due to anti-symmetry of $\epsilon^{\alpha\beta\gamma\delta}$). Now I integrate by parts and drop the boundary term[3] assuming 'nice' boundary condition.

$$Z \approx \int \mathcal{D}F_{\mu\nu}\mathcal{D}C_\alpha\, e^{-i\int(\frac{1}{4g^2}F_{\mu\nu}^2 + \#\,\partial_\beta C_\alpha\, \epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta})}\delta(\partial_\mu C^\mu) \tag{2.6}$$

Notice that due to anti-symmetry of $\epsilon^{\alpha\beta\gamma\delta}$ the term $\partial_\beta C_\alpha$ has to be anti-symmetric in its indices. Therefore one can define this as the *field strength* for the new "gauge field" C_α : $G_{\alpha\beta} := \partial_\alpha C_\beta - \partial_\beta C_\alpha$.

The above action is a just quadratic in $F_{\mu\nu}$ and therefore we can easily integrate out $F_{\mu\nu}$ to obtain

$$Z \approx \int \mathcal{D}C_\alpha\, e^{-\frac{ig^2}{\#}\int G_{\mu\nu}^2}\delta(\partial_\mu C^\mu) \tag{2.7}$$

This is almost the same $U(1)$ theory but of a completely different "gauge field" C_μ. Few remarks in order.

- This is an example of *self duality* where a Maxwell theory goes to another Maxwell theory. But the coupling is inverted $g \to \frac{1}{g}$. This is also an example of strong-weak duality.
- A_μ and C_μ are completely different degrees of freedom. They are related by extremely involved relation.
- Degrees of freedom need not be the same in both sides of a duality. But in this case a gauge field goes to another.

❒ Kramers-Wannier duality

It relates the partition function of a two-dimensional square-lattice Ising model at a low temperature to that of another Ising model at a high temperature. Using this duality Kramers and Wannier predicted the exact location of the critical point of 2D Ising model before Onsagar could solve that model exactly in 1944!

[3] This step is very non-trivial. We are assuming very particular boundary conditions. By 'nice' I mean the field C_α and/or its derivative dies down at the boundary. But some non-trivial boundary condition can give rise to more interesting physics - topological theories.

❐ Bosonization

In 1+1 dimensions one can map an interacting fermionic system to a system of bosons. *E.g.,* massive Thirring model is dual to sine-Gordan model. This is also an example of strong-weak duality. This duality was uncovered independently by particle physicists (Coleman & Mandelstam) and condensed matter physicists (Mattis & Luther) in 1975.

❐ Montonen-Olive duality

This is generalization of Maxwell duality with magnetic charge and current but in $\mathcal{N} = 4$ SYM. This is again a strong-weak duality and it relates 'elementary particles' of one side to 'monopoles' of the other side.

❐ Seiberg-Witten duality

Similar to Montonen-Olive duality but it is for IR effective theory of $\mathcal{N} = 2$ SUSY theory in $D = 4$. Unlike $\mathcal{N} = 4$ SYM this theory not conformally invariant in general *i.e.* its beta function runs – more interesting dynamics.

2.2 String theory

In string theory there are mainly two important dualities.

1. T-duality (1982?)
2. S-duality (1994)

I'll discuss T-duality in some detail since it's very simple and elegant. Another reason is this is an example of duality which is *not* strong-weak type.

❐ T-duality

Einstein changed our view of space and time by marrying them. The notion of 'space' and 'time' became rather observer dependent. T-duality goes one step further to completely change our notion of space-time itself. It shows how different objects or probes perceive space-time quite differently. In that sense the notion of space-time itself is an 'emergent' concept.

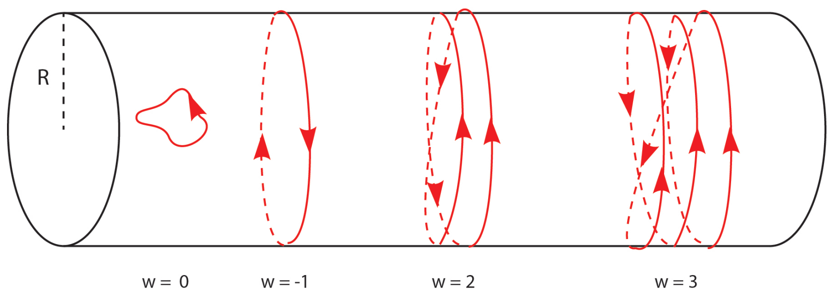


Figure 2: Closed strings wrapping a compact direction in different 'windings'.

Let's see how T-duality works in string theory. Consider a flat 1+1 dimensional spacetime (higher dimensional generalization is straight forward). This is just a plane sheet of paper. Let's compactify the spatial direction to make it an infinite cylinder with radius R.

First consider a particle (or a field) of mass m moving on this cylinder. Its momentum ($\vec{p}$) has two orthogonal components : along the circle (p_θ) and along the non-compact direction ($p_\perp$). But along the compact direction p_θ has to obey the periodicity condition $e^{i\, p_\theta (2\pi R)} = 1$ $i.e.$ $p_\theta = \frac{n}{R}$ where $n \in \mathbb{Z}$. Thus the total momentum

$$\begin{aligned} \vec{p} &= p_\theta\, \boldsymbol{e}_\theta + p_\perp\, \boldsymbol{e}_\perp \\ &= \frac{n}{R}\, \boldsymbol{e}_\theta + p_\perp\, \boldsymbol{e}_\perp \end{aligned} \tag{2.8}$$

$\boldsymbol{e}_\theta$, $\boldsymbol{e}_\perp$ are the corresponding unit vectors. The energy is given by

$$\begin{aligned} E^2 &= p^2 + m^2 \\ &= (p_\perp^2 + m^2) + \frac{n^2}{R^2} \end{aligned} \tag{2.9}$$

Notice that $E \to \infty$, if one take $R \to 0$ unless $n = 0$. Physically this means when the compact direction is very small the particle can not 'sense' or probe that direction and effectively 'lives' only in the non-compact dimension.

Q. What happens if we replace the particle with a closed string?

A. Unlike the particle it can wind around (fig. 2) the cylinder! Therefore there will be an extra contribution to the energy from these winding modes.

$$\begin{aligned} E^2 &= p^2 + M^2 + (\text{“winding energy”})^2 \\ &= \left(p_\perp^2 + \frac{\mathbb{N}}{\alpha'}\right) + \frac{n^2}{R^2} + (\text{“winding energy”})^2 \end{aligned} \tag{2.10}$$

where α' is the string tension and $\mathbb{N}$ indicates the ‘level’ of the tower of closed string states[4]. Winding energy (E_w) of a string which wraps the cylinder w times is given by

$$\begin{aligned} E_w &= \text{length of the string} \times \text{string tension} \\ &= w \times (2\pi R) \times \frac{1}{2\pi\alpha'} \\ &= \frac{wR}{\alpha'} \end{aligned} \tag{2.11}$$

Thus the total energy becomes

$$E^2 = \left(p_\perp^2 + \frac{\mathbb{N}}{\alpha'}\right) + \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} \tag{2.12}$$

Now if we take $R \to 0$ the momentum modes along the compact direction become very ‘heavy’ as before but at the same time the winding modes become very ‘light’! On the other hand if we take $R \to \infty$ momentum modes play the role of ‘light’ modes and winding modes become ‘heavy’. Clearly there is a duality at work here and to be precise if we make the following transformations

$$\begin{aligned} R &\to \frac{\alpha'}{R} \\ (n, w) &\to (w, n) \end{aligned}$$

expression for energy remains unaltered. $R := \sqrt{\alpha'} = l_s$ is called the *self-dual* radius. Physics for $R < \alpha'$ is identical to physics with $R > \alpha'$. I just want to point out the following characteristics of T-duality.

1. It is intrinsically stringy – there is no field theoretic analog to this. Strings perceive the spacetime quite differently compared to point particles.

2. This is *not* a strong-weak duality.

[4]If you are not familiar with string spectrum you can happily ignore this comment. The terms inside the parentheses don't play any role in the point I will try to make.

❒ S-duality

This is actually a strong coupling- weak coupling duality[5] in string theory. S-duality in string theory was first proposed Ashoke Sen in 1994. This duality maps one string theory with coupling g_s to another string theory with coupling $\frac{1}{g_s}$. For example, type IIB string theory with the coupling constant g_s is equivalent via S-duality to the same string theory with the coupling constant $\frac{1}{g_s}$. Similarly, type I string theory and the SO(32) heterotic string theory are dual to each other.

2.3 Gauge/string duality

This duality mixes the above two frameworks *viz.* QFTs and string theory. It was proposed by Juan Maldacena in 1997.

3 How come $AdS_{d+1} = CFT_d$ or gauge = gravity?!

At first sight the equivalence of gravity with gauge theory in one lower spacetime dimensions might seem 'extremely crazy' mainly for following reasons.

1. Two theories don't even live in same number of spacetime dimensions.
2. One is gauge theory without gravity and other one *is* a gravity theory.

I will take you back to few (three, to be precise) influential discoveries of theoretical physics in last few decades to make this duality look more plausible or rather 'less crazy'.

❒ Open string-closed string duality

Closed string spectrum : Graviton + infinite tower massive modes.

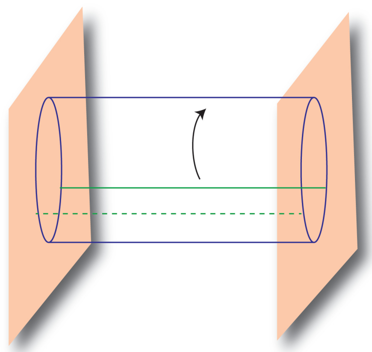
Open string spectrum : Gauge field + infinite tower of massive modes.

Thus if we are interested only in low energy physics, closed string has "gravity" in it where as open string contains Yang-Mills "gauge fields". Now let's look at the following process in fig. 3 closely. One can look at it in two completely different but still equivalent ways namely a closed string is being exchanged between the D-branes or an open

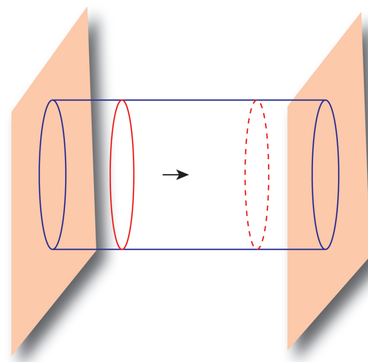
[5] EM duality, Kramers-Wannier duality, Montonen-Olive duality, Seiberg-Witten duality- all are examples of S-duality in QFTs.

string is running in a loop between them. Roughly it means,

$$\text{Closed string tree} = \text{Open string loop}.$$



(a) Open string running in a loop



(b) Closed string is exchanged between D-branes

Figure 3: Open string-closed string duality

Therefore one would expect, at least in some particular sense, there should be an equivalence between gauge theory and gravity.

❒ Large-N gauge theories

It is established that strong nuclear force is described by QCD which is nothing but a Yang-Mills theory with gauge group SU(3). Here three indicates the number of colors. The Yang-Mills coupling undergoes dynamical transmutation and QCD doesn't have a free parameter to play with – QCD is very difficult. What will happen if one works with infinite number of colors instead of only three? This was the question 't Hooft asked in seventies. Actually the theory simplifies[6] a lot! 't Hooft introduced a parameter N which is the number of colors and it plays the role of a free parameter now. The $N \to \infty$ limit is similar to taking $\hbar \to 0$ *i.e,* 'classical' limit of QCD.

[6]This is in the same spirit in statistical mechanics. When fluctuations are important one way to handle them is to work with a lot of such fluctuating variables. 3-body problem is very difficult but a box of gas with huge number of molecules is easier to handle! Same is true with dimensionality. In lower dimensions there are lot of fluctuations. Mean field theory is easier because one works in infinite dimensions.

For this discussion I shall consider pure SU(N) YM *i.e.* no 'quarks'. This will be enough for the point I want to make. But adding 'quarks' is a very easy extension. Anyway the 'gluons' are adjoint valued elements of SU(N) and the Lagrangian

$$\mathcal{L} = -\frac{1}{4\,g_{YM}^2} \int F^a_{\mu\nu b} F^{\mu\nu b}_a \tag{3.1}$$

Note that $\mathcal{L}$ is a Lorentz scalar since μ, ν indices are contracted and is Singlet under SU(N) since a, b indices are contracted. From the Lagrangian it is clear that in this theory

$$\text{Propagator} \sim g_{YM}^2$$
$$\text{Interaction vertices} \sim \frac{1}{g_{YM}^2}$$

We will follow the *double-line* notation what 't Hooft introduced to make the counting easy – replacing each gluon propagator by a quark-antiquark pair (see fig. 4).

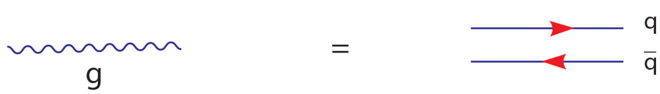


Figure 4: The gluon propagator $\sim g_{YM}^2$

In this notation the 3-point and 4-point functions look as follows.

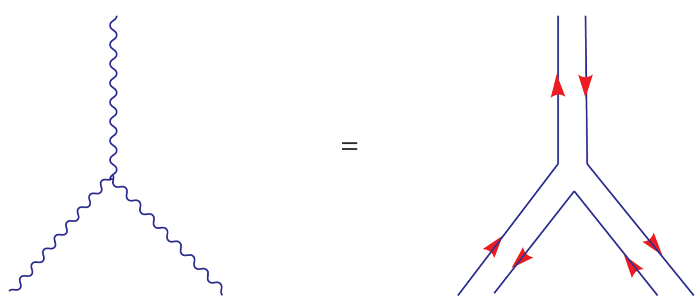


Figure 5: 3-pt vertex $\sim \frac{1}{g_{YM}^2}$

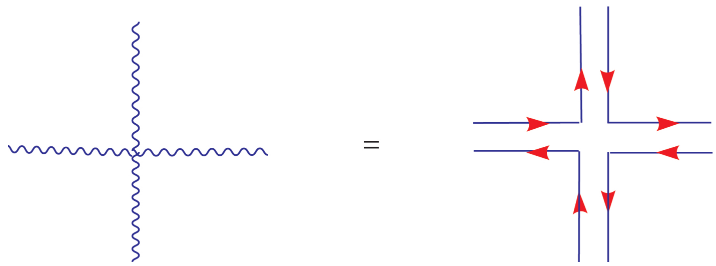
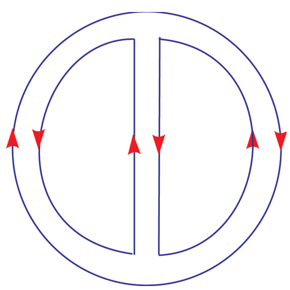
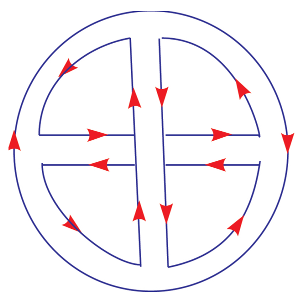


Figure 6: 4-pt vertex $\sim \frac{1}{g_{YM}^2}$

Suppose we are interested in vacuum-to-vacuum amplitudes (see fig. 7). Our aim is to see how the diagrams scale with N. For that we just need to count the number of propagators and vertices. We know how they scale with the coupling g_{YM}. On top of that whenever we have a *color loop* (color index is summed over) that should correspond to a factor of N since there are total N colors.



(a) Planar diagram : $\mathcal{O}(N^2)$



(b) Non-planar diagram : $\mathcal{O}(N^0)$

Figure 7: Vacuum-to-vacuum amplitude in double-line notation

For fig. 7a :

$\#$ of propagators $= 3$
$\#$ of vertices $= 2$
$\#$ of loops $= 3$

$\therefore$ It scales as $\sim (g_{YM}^2)^3 \frac{1}{(g_{YM}^2)^2} N^3 = (g_{YM}^2 N)\, N^2 \equiv \lambda\, \boldsymbol{N^2}$

For fig. 7b :

$\#$ of propagators $= 6$
$\#$ of vertices $= 4$
$\#$ of loops $= 2$

$\therefore$ It scales as $\sim (g^2_{YM})^6 \frac{1}{(g^2_{YM})^4} N^2 = (g^2_{YM}N)^2\, N^0 \equiv \lambda^2\, \boldsymbol{N^0}$

I have defined[7] a new effective coupling $\lambda := g^2_{YM}N$ and have extracted the N-dependence. If we keep λ to a fixed value as $N \to \infty$, the fig. 7a contributes at $\mathcal{O}(N^2)$ whereas fig. 7b contributes at $\mathcal{O}(N^0)$. Notice that the fig. 7a can be drawn on a plane or a sphere and is called *planar* diagram. On the other hand fig. 7b can not be drawn on a plane – one requires a torus. This is a *non-planar* diagram.

Therefore at large N and fixed (but small λ) one can schematically write down SU(N) YM vacuum-to-vacuum amplitude as following.

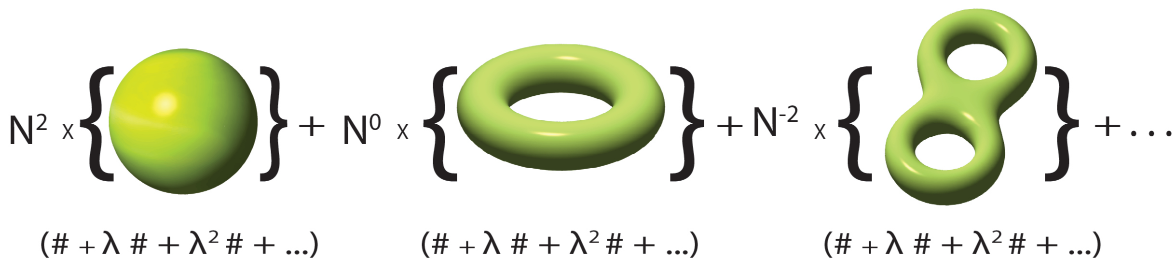


Figure 8: Large N expansion of SU(N) gauge theory

Notice that at large N one needs to consider only the planar (or sphere) diagrams but there are infinitely many such terms since it is a perturbative expansion in λ. At this point someone familiar with string perturbation theory can easily compare this with perturbative string amplitude which looks as follows

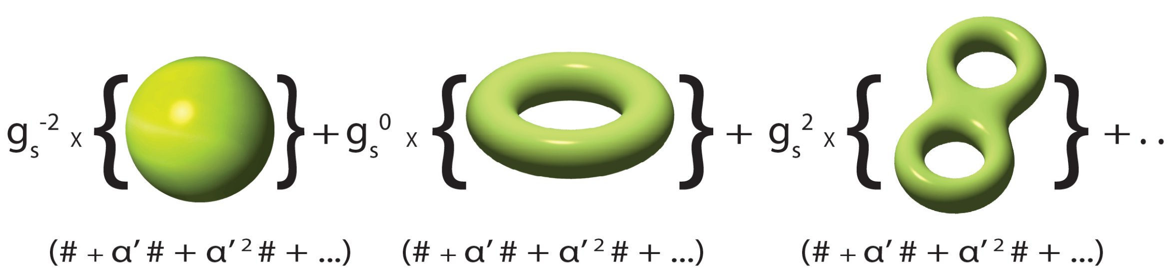


Figure 9: Perturbative expansion of closed strings

and (s)he will be tempted to formally identify

[7]This λ is called 't Hooft coupling since 't Hooft introduced this quantity. Also keeping $\lambda := g^2_{YM}N$ *fixed*, with $N \to \infty$ is known as 't Hooft scaling limit for the same reason.

$$\boxed{\begin{array}{c} g_s \Leftrightarrow \dfrac{1}{N} \\ \alpha' \Leftrightarrow \lambda \end{array}}$$

Thus gauge theory at large N and string theory have similar perturbative expansions – it's not very hard to imagine that they can be related to each other.

❒ **Holographic principle**

The holographic principle, originally proposed by 't Hooft, states that the total information contained in a volume of space corresponds to exactly same amount of information tiled on the boundary of that space. Later Susskind worked on this principle in string theory context. We don't really need the details. The expression for Bekenstein-Hawking entropy of black hole

$$S_{BH} = \frac{\text{Horizon area}}{4G}$$

will ring a bell. Roughly it says 'volume' is equivalent to 'its boundary'. The key point here is : if a $d+1$ dimensional gravity theory is dual to a d dimensional field theory living on its boundary – it shouldn't be so surprising!

4 The decoupling limit

Hopefully it is clear by now that one would expect some relationship between gauge theory and gravity (string theory). After the 'discovery' of D-branes in mid-nineties there was a rapid development in this direction. In 1997 Maldacena conjectured a duality between $\mathcal{N} = 4$ SYM and Type IIB string theory in $AdS_5 \times S^5$. His argument to reach this conjecture is famously known as *the decoupling argument.*

4.1 Different descriptions of same physics

Before discussing about the decoupling argument let's discuss about some simple examples of describing same physical phenomenon using two complementary point of views[8]. This discussion will come in handy in describing Maldacena's decoupling limit.

[8] In this section I heavily follow Pedro Viera's PhD thesis and MAGOO [2].

❐ QED

Q. Let's start with a very basic example from QED *viz.* how do we describe an electron's motion in presence of a proton?

A. We can treat this problem perturbatively and sum up all possible Feynman diagrams. Here are few of them.

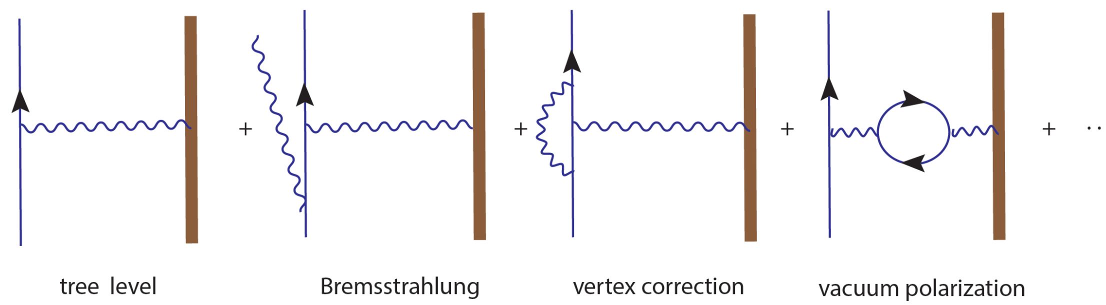


Figure 10: Electron's motion near proton

The first diagram in position space gives the standard Coulomb potential $V(r) \sim -\frac{1}{r}$. The other diagrams are corrections to this "classical potential". There will be infinitely many diagrams as one goes to higher loops. The more number of diagrams one considers the more accurate the description will be. Effectively the extra diagrams change the form of the potential, $V(r) = -\frac{\alpha}{r}[1 + \#\alpha\frac{e^{-2m_e r}}{(m_e r)^{3/2}} + \ldots]$

There are two different ways of describing the same phenomenon.

- **<u>Picture I</u> :** The electron and the proton are in vacuum and they are interacting via exchanging photons (see fig. 10). Then sum all such Feynman diagrams.

- **<u>Picture II</u> :** Another way of describing the same problem is the following. There is no proton but the electron is moving in a background potential $V(r) = -\frac{\alpha}{r}[1 + \#\alpha\frac{e^{-2m_e r}}{(m_e r)^{3/2}} + \ldots]$ (see fig. 11).

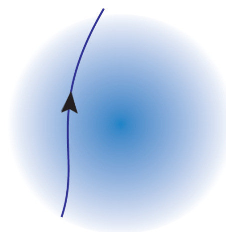


Figure 11: Electron moving in a 'background' field

❒ String theory

Q. What can be the analogous picture in string theory?

A. One should replace the electron by an 'elementary' closed string and the 'heavy' proton by a heavy and extended object available in the theory – D-brane!

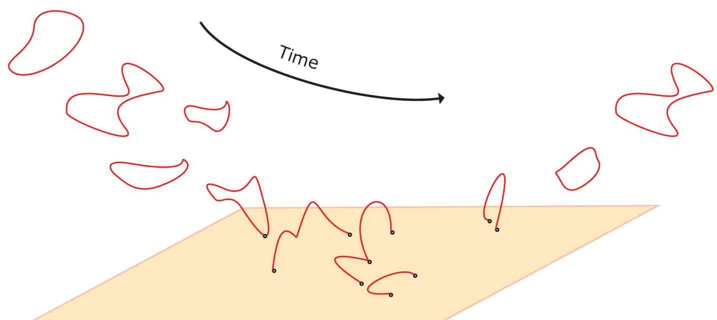


Figure 12: A closed string moving near a D-brane.

Again we have two different ways of looking at this phenomenon.

- **Picture I :** First approach would be analogous to summing over Feynman diagrams *i.e.*, studying the scattering of a closed string with a D-brane perturbatively. The string can split into many closed strings or can become an open string on the D-brane and then can further split into many open strings on that brane (see fig. 12). Some of the open strings can join the end points on the D-brane and leave the brane as closed strings.

  In the world sheet picture it is easier to keep track of the factors of couplings (similar to counting loops in QED). The number of handle indicates string split-

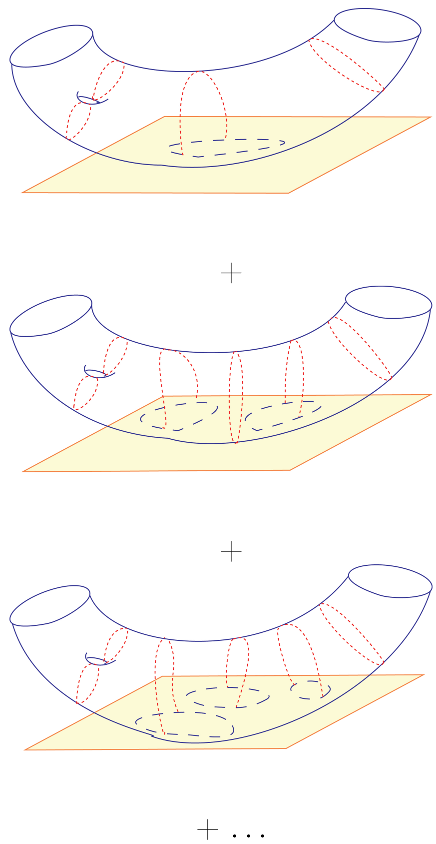


Figure 13: Worldsheet picture of closed string - D-brane interaction

ting and number of boundary of the worldsheet signifies the interaction with the D-brane (see Fig 13).

- **Picture II :** Here is another equivalent description of the same phenomenon. One can forget about the existence of the D-brane and replace all intermediate effects (Feynman diagrams) by an effective background (see fig. 14) in which the closed string moves. In this picture we are considering D-brane as a source of closed strings. The 'coherent state' of large number of closed strings effectively changes the background near the brane.

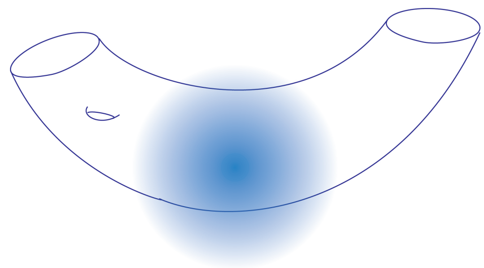


Figure 14: A closed string moving near a D-brane.

Picture I : Holds true only in the perturbative regime *i.e,* low energy action of open strings on the brane. This is described by SYM theories.

Picture II : D-brane is the source of closed strings. Since the closed strings change the background this should have some gravitational description.

4.2 Maldacena's Argument

In his original paper [1], Maldacena started with a stack of N $D3$ branes. One can again describe the system in two alternative ways – (i) by open string dynamics or (ii) by closed string dynamics[9].

[9]Look at fig. 13. If you try to look along a D-brane you can 'see' the worldsheet of open string fluctuating. Now if you look perpendicular to the D-brane you can 'see' closed string(s) being emitted (or absorbed, depending on the direction you are looking from) by the D-brane. Try this, it's fun!

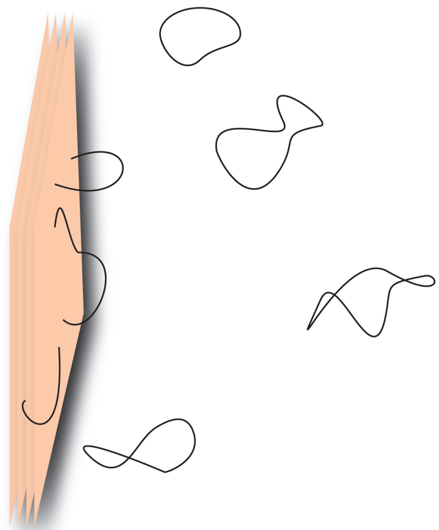


Figure 15: A stack of N D3-branes

Picture I

Let's see how one would describe the low energy dynamics of this system from open string perspective. The stack of N branes are described by $\mathcal{N} = 4$ U(N) SYM theory plus higher derivative terms. These higher derivative interactions come due to integrating out all massive open string modes. These are all suppressed by increasing powers of α'. Similarly away from the branes (we call it 'bulk') the physics should be described by 10D low energy string theory (type IIB super-string theory since D3 brane appears in IIB theory) which is known as type IIB super-gravity. Again there will be higher derivative interactions suppressed by different powers of α'. And these two theories can interact. So schematically the action for the total system looks as following.

$$S = \underbrace{S_{branes}}_{\text{(SYM + higher derivatives)}} + \underbrace{S_{bulk}}_{\text{(10D SUGRA + higher derivatives)}} + S_{int} \tag{4.1}$$

The 'bulk' and the 'branes' interact gravitationally. Maldacena's main aim was to turning off this interaction by tuning some coupling and to decouple the theories. Notice that if we take $\alpha' \to 0$ keeping g_s and N fixed, it is equivalent to taking Newton's constant $G_N \to 0$ because $\sqrt{G_N} \sim g_s\alpha'^2$. But α' is a dimensionful quantity, therefore it can not be taken to zero. The correct way to take the limit is to make α' smaller compared to the energy (or inverse length) scale one is looking at *i.e.*,

$$\alpha'|\vec{k}|^2 \ll 1 \quad \text{or,} \quad \frac{|\vec{x} - \vec{x}'|^2}{\alpha'} \gg 1$$

Taking such a limit amounts to turning off all the interactions and all higher derivative terms since they come with positive powers of G_N or α'. Thus we are left with 4D SYM and 10D super-gravity which are not talking to each other.

$$\boxed{\text{Picture I}: \quad \mathcal{N} = 4 \text{ SYM in 4 dimensions} \oplus \text{Super-gravity in 10 dimensions}}$$

Picture II

Q. Following same chain of arguments we want to see the stack of D3 branes a gravitational solution or we should ask, what background do the stack of branes produce?

A. This "classical" background should be described by low energy effective action of string theory which for this particular case is type IIB super-gravity (Einstein action + "other fields"). Therefore the aim is to look for a solution or a metric for this stack of N D3 branes (fig. 16).

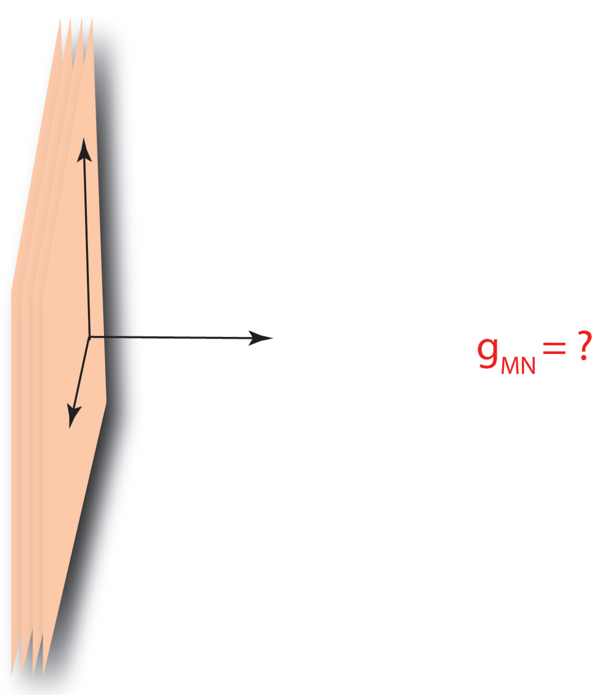


Figure 16: Stack of D3-branes as a gravitational solution

Exploiting the symmetry of the system we start with the following ansatz

$$ds^2 = \eta_{\mu\nu}\frac{dx_\mu dx_\nu}{f(r)} + \widetilde{f}(r)dx^m dx_m \quad \text{where,} \quad r^2 = x^m x_m \tag{4.2}$$

To satisfy Einstein equations the unknown functions in the ansatz have to have the following forms

$$f(r) = \widetilde{f}(r) = \sqrt{1 + \frac{L^4}{r^4}} \quad \text{with,} \ L^4 = g_s N(4\pi\alpha'^2)$$

Once we have the metric there are two obvious 'extreme' limits we can look at in this picture II :

(i) $r \to \infty$
(ii) $r \to 0$

Far away from the branes ($r \to \infty$)

Far away from the branes the geometry has to be flat space $\mathbb{R}^{9,1}$

$$ds^2 = dx^\mu dx_\mu + dx^m dx_m \tag{4.3}$$

Near the branes ($r \to 0$)

$$ds^2 = \underbrace{\frac{r^2}{L^2}\eta_{\mu\nu}dx_\mu dx_\nu + \frac{L^2}{r^2}dr^2}_{AdS_5} + \underbrace{L^2 d\Omega_5^2}_{S^5} \tag{4.4}$$

Let's look more closely. In GR we always talk about observables with respect to particular observers. Let's ask the question what do we mean by "time"?

$$dx^\mu dx_\mu = -dt^2 + d\vec{x}^2$$

This t is just co-ordinate time and it is 'physical' or 'proper' time only for an observer at $r = \infty$. Therefore the natural question arises what is the "time" for an observer at arbitrary r?

The proper time for an observer is related to co-ordinate time as follows.

$$\begin{aligned}\Delta t_{prop} &= \sqrt{g_{tt}}\,\Delta t \\ &= \frac{r}{L}\,\Delta t \end{aligned}\tag{4.5}$$

On dimensional ground the 'proper' energy

$$\Delta E_{prop} = \frac{L}{r}\,\Delta E \tag{4.6}$$

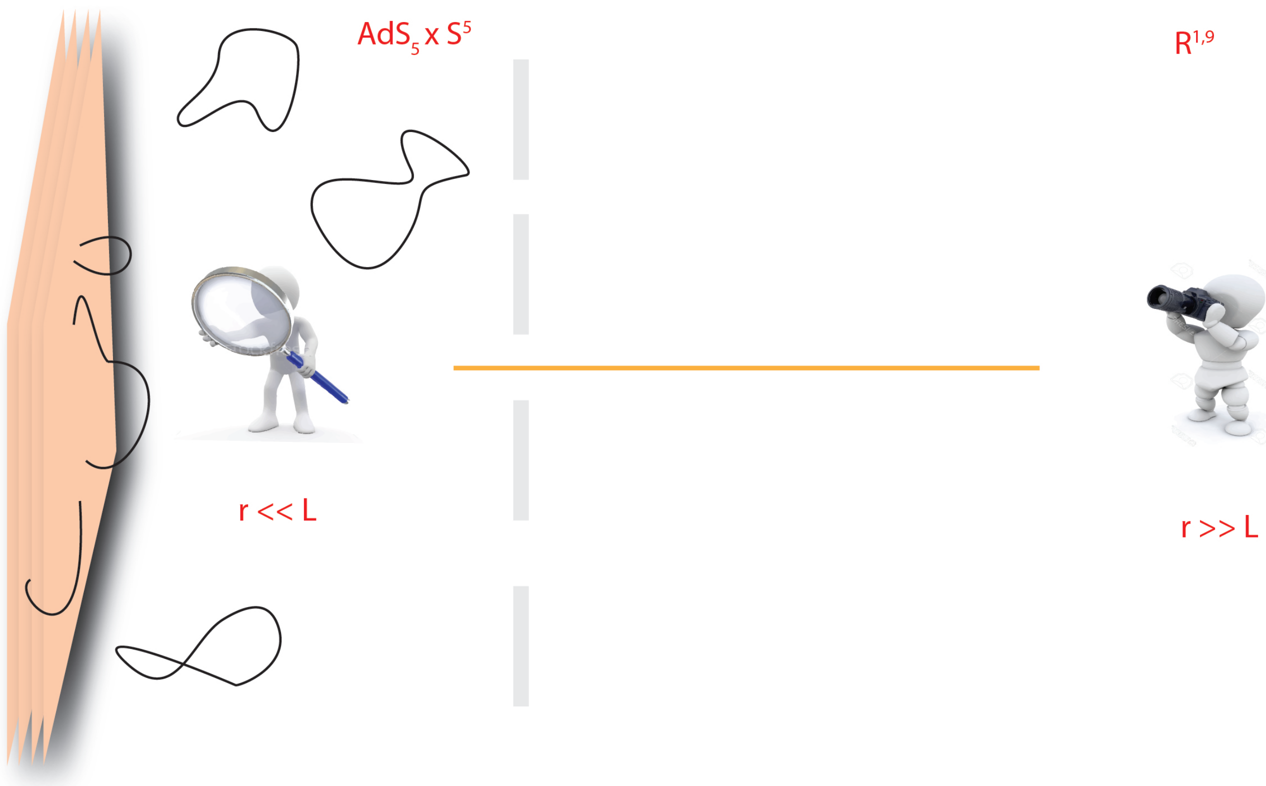


Figure 17: The decoupling limit

Notice that for $r \to 0$ there is an infinite red shift. So even if the near the stack of branes the energy E is arbitrarily large[10] for the observer at $r \to \infty$ it is *finite* due to the redshift.

[10] This is very crucial point. So let me elaborate on it with a simple thought experiment. Suppose A and B are at $r \to \infty$. A is carrying a $10^{100}\,GeV$ 'lamp' (don't worry, it's a thought experiment!). Suddenly A finds the stack of D3 branes very attractive and decides to walk towards it. Due to the red shift factor, to B the lamp energy keeps decreasing (*i.e.,* lamp's frequency gets smaller) as A approaches the stack. When A is very close to the branes, such that the redshift factor is 10^{-109} say, to B the lamp's energy is just $1\,eV$! But for A it is still the $10^{100}\,GeV$ lamp! Therefore arbitrarily large energy near the branes is finite energy for the observer far away. The bottom line is, low energy theory for the observer at infinity includes all possible high energy phenomena near the D-branes – full string theory in $AdS_5 \times S^5$.

Q. What's low energy for the observer at $r \to \infty$?

A. The 'stuff' near him/her are already low energy *i.e,* 10D super-gravity and *anything* near $r \to 0$. Here by *anything* I mean *full string theory* in $AdS_5 \times S^5$.

Picture II : Full type IIB string theory in $AdS_5 \times S^5 \oplus$ Super-gravity in 10 dimensions

Now let's do what Maldacena did in 1997 – "equate" Picture I and Picture II. Here are the detailed steps which he didn't explicitly write down in his original paper.

$$\text{Picture I} = \text{Picture II}$$

$$\mathcal{N} = 4 \text{ SYM in 4D} \oplus \cancel{\text{10D Sugra}} = \text{IIB string theory in } AdS_5 \times S^5 \oplus \cancel{\text{10D Sugra}}$$

$$\boxed{\therefore \; \mathcal{N} = 4 \text{ SYM in 4 dimensions} \equiv \text{Full type IIB string theory in } AdS_5 \times S^5}$$

This completes the "derivation" of the AdS/CFT duality.

5 The dictionary of parameters

There are two dimensionless parameters in the gauge theory namely g_{YM} and N. As we have discussed before it is more convenient to define dimensionless 't Hooft coupling $\lambda \equiv g_{YM}^2 N$. Thus the gauge theory has two independent dimensionless couplings g_{YM} and λ.

On the other hand the string theory in $AdS_5 \times S^5$ has one dimensionless copling g_s and two dimensionful parameters namely the string length $l_s = \sqrt{\alpha'}$ and the AdS radius L. Thus effectively this theory also has two dimensionless parameters g_s and $\frac{L}{l_s}$.

Q. How are these parameters related to each other?

A. They are related as follows.

$$\boxed{\begin{aligned} g_{YM}^2 &= g_s \\ \lambda \equiv g_{YM}^2 N &= \left(\frac{L}{l_s}\right)^4 \end{aligned}}$$

Planar limit : According to the stronger version of the conjecture, the above matching of parameters hold true for all values of the parameters. Things get simplified if one takes $N \to \infty$ keeping $\lambda = fixed$ *i.e,* $g_{YM}^2 \to 0$. This is famous 't Hooft limit. In this limit, as we have seen in the large N gauge theories, only the planar diagrams in $\mathcal{N} = 4$ SYM contribute since the other non-planar diagrams are suppressed by powers of $1/N$. Analogously in the string theory side $g_s \to 0$ and $\frac{L}{l_s}$ remains finite which means that string cannot split and join (*i.e,* no 'handles' in the string world sheet).

Let's say we restrict ourselves in this planar limit where $N \to \infty$ and $\lambda = fixed$. Still there are two possibilities : λ can be *large* or *small*.

Gauge theory	**String theory**
Small λ $\Rightarrow$ Perturbative SYM `(Easy)`	Small λ *i.e,* $L \sim l_s$ $\Rightarrow$ Highly (stringy) quantum theory `(Hard)`
Large λ $\Rightarrow$ Strongly coupled theory `(Hard)`	Large λ i.e, $L \gg l_s$ $\Rightarrow$ Classical SUGRA `(Easy)`

One can easily see that when one side of the duality is computationally easy the other side becomes extremely hard to handle.

Q. Is it good or bad?

A. Depends on one's taste!

- **Bad :** Not only it is very difficult to *prove* the duality but even hard to *check*. This also explains why it is still a conjecture even after 20 years of intense research (see fig. 1).

- **Good :** Extremely powerful! One can calculate very complicated quantities in strongly coupled quantum field theories by computing corresponding quantities in dual classical gravitational theory.

Note : *In this lecture I have tried to motivate the AdS/CFT correspondence and the main focus is on Maldacena's decoupling argument. For the sake of completeness here are some important topics in this context that have not been covered in this lecture.*

1. *Computation of CFT correlators and Wilson loop from gravity theory, (see [3–5])*
2. *Prescription for computing real-time correlators, (see [6])*
3. *Generalization of this conjecture to finite temperature and density (see [7, 8])*

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