Research Awards Screening Test, 2006

KEY

Section 1: Algebra

1.1 a, b, c
1.2 b, c
1.3 c
1.4 0, 5, 8, 9
1.5 c
1.6 12
1.7 Any matrix of the form: \((a, b \text{ and } c \text{ all non-zero})\)
\[
\begin{bmatrix}
  a & b & c/2 \\
  0 & b & c \\
  0 & 0 & c/3 \\
\end{bmatrix}
\]
1.8 a, b
1.9 b, c
1.10 b, c, d

Section 2: Analysis

2.1 \(C = \cap_{n=1}^{\infty} A_n\)
2.2 \(2n \int_0^x (2x - t)^{n-1} f(t) \, dt + x^n f(x)\)
2.3 a
2.4 b
2.5 b, c
2.6 \(b \frac{\partial g}{\partial x} = a \frac{\partial g}{\partial y}\).
2.7 \(\left(\frac{1}{2x}, -\frac{1}{2y}\right)\).
2.8 \(e^{-\frac{x}{y}}\).
2.9 All integers
2.10 \(\left[\left(\frac{\partial f}{\partial x}(x_0, y_0, z_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0, y_0, z_0)\right)^2 + \left(\frac{\partial f}{\partial z}(x_0, y_0, z_0)\right)^2\right]\frac{1}{\pi}\)
Section 3: Topology

3.1 (i) continuous at all irrationals, (ii) continuous only at \( t = 1 \)
3.2 \( f(a) = g(a) \)
3.3 \( A \) and \( B \) have the same cardinality
3.4 (i) \( f(D) \) is necessarily an interval; (ii) \([a,b]\)
3.5 (i) \( X_5 \) is connected if and only if \( \alpha \leq \frac{3}{4} \). (ii) When not connected, it has 3 components
3.6 \( X_2 \) and \( X_4 \) are homeomorphic
3.7 Compact sets are \( X_2 \) and \( X_3 \)
3.8 Locally compact sets are \( X_1, X_2 \) and \( X_3 \)
3.9 Complete metric spaces are \( X_1, X_2 \) and \( X_3 \)
3.10 \( X_2 \)

Section 4: Applied Mathematics

4.1 \( \frac{3}{4}\sqrt{\pi} \)
4.2 0
4.3 \( \frac{12\pi}{5}a^6 \)
4.4 3
4.5 b, c
4.6 c = 2
4.7 elliptic in the region \( \{(x,y) \in \mathbb{R}^2 : y < 0\} \)
4.8 \( \pi/5 \)
4.9 a, b, d
4.10 A linear functional in 3 variables with coefficients 6, -6 and -4; example:
\[ 6w_1 - 6w_2 - 4w_3 \]

Section 5: Miscellaneous

5.1 \( m \) and \( n \) are coprime
5.2 0
5.3 \( n(n - 1)2^{n-2} \)
5.4 \( 2^{-7} \)
5.5 2
5.6 there is no solution
5.7 \([0,1/2]\) for each \( k \)
5.8 Yes; if \( K = \{a_1, \ldots, a_n\} \), then take \( (x - a_1) \ldots (x - a_n) + 1 \), for example.
5.9 \( 2e \)
5.10 a, c