NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 22, 2018

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions that follow

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 8 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Miscellaneous. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- **Calculators are not allowed.**

Notations

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
- If x and y are vectors in $\mathbb{R}^n$, their usual euclidean inner-product in $\mathbb{R}^n$ will be denoted by (x, y).
- The symbol i will stand for a square root of -1 in $\mathbb{C}$, while the other square root will be $-i$.
- The symbol $\mathbb{Z}_n$ will denote the abelian group of integers modulo n.
- The symbol S_n will stand for the set of all permutations of the symbols $\{1, 2, \cdots, n\}$, which is a group under composition.
- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.
- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval.
- We denote by $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$), the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$).
- We denote by $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) the group (under matrix multiplication) of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$).
- The trace of a square matrix A will be denoted tr(A) and the determinant by det(A).
- The adjoint of a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ will be denoted by the symbol A^T. The null-space of A will be denoted by $\ker(A)$ and the range of A will be denoted by $\mathcal{R}(A)$.
- The symbol I will denote the identity matrix of appropriate order.
- All logarithms, unless specified otherwise, are to the base e.
- If X is a set and A is a subset of X, we write $A \subset X$. This **does not** imply that A is a proper subset of X.

Section 1: Algebra

1.1 The polynomial $x^3 - 11x^2 + ax - 36$ has a positive root which is the product of the other two roots. Find the value of a.

1.2 How many group homomorphisms are there from $\mathbb{Z}_8$ into $\mathbb{Z}_{10}$?

1.3 In which of the following examples is H a normal subgroup of G?
a. $G = GL_n(\mathbb{C})$ and H is the subgroup of all upper triangular matrices in G.
b. $G = GL_n(\mathbb{R})$ and H is the subgroup of all matrices in G with positive determinant.
c. $G = GL_n(\mathbb{C})$ and H is the subgroup of all matrices in G with determinant equal to unity.

1.4 In each of the cases in the preceding exercise, where H is a normal subgroup of G, identify the quotient group G/H.

1.5 Which of the following statements are true?
a. Every group of order 11 is cyclic.
b. Every group of order 111 is cyclic.
c. Every group of order 1111 is cyclic.

1.6 Let $\sigma \in S_7$ be the permutation given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 4 & 7 & 6 & 3 \end{pmatrix}.$$

Which of the following permutations are conjugate to σ?
a.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 2 & 7 & 4 \end{pmatrix}.$$

b.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 5 & 6 & 3 & 2 & 7 \end{pmatrix}.$$

1.7 Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. If W is a subspace of $\mathbb{R}^n$, define

$$W^\perp \;=\; \{y \in \mathbb{R}^n \mid (x, y) = 0 \text{ for all } x \in W\}.$$

Which of the following statements are true?
a. $\mathcal{R}(A) \subset (\ker(A^T))^\perp$.
b. $\mathcal{R}(A) = (\ker(A^T))^\perp$.
c. Neither of the above statements need be necessarily true.

1.8 Let $n \geq 3$. Let $\{w_j\}_{j=1}^n$ be a basis for $\mathbb{R}^n$ such that the inner-product $(w_j, w_\ell) = 0$ whenever $j \neq \ell$. Let W be an arbitrary subspace of $\mathbb{R}^n$. Let $1 < k < n$. Which of the following statements are true?
a. There exists $x \in W$ such that $x \neq 0$ and such that $(x, w_j) = 0$ for all $1 \leq j \leq k-1$.
b. If the dimension of W is equal to k, then there exists $x \in W$ such that $x \neq 0$ and such that $(x, w_j) = 0$ for all $1 \leq j \leq k-1$.
c. If the dimension of W is strictly greater than k, then there exists $x \in W$ such that $x \neq 0$ and such that $(x, w_j) = 0$ for all $1 \leq j \leq k-1$.

1.9 Let $A \in \mathbb{M}_n(\mathbb{R})$ be a non-zero singular matrix. Consider the following problem: find $X \in \mathbb{M}_n(\mathbb{R})$ such that

$$(i) AXA \;=\; A,\ (ii) XAX \;=\; X \text{ and } (iii) AX \;=\; XA.$$

Which of the following statements are true?
a. If a solution to the above problem exists, then A is not nilpotent.
b. If A represents a projection, then the above problem admits a solution.
c. If $n = 2$ and if a solution to the above problem exists, then A is diagonalizable over $\mathbb{R}$.

1.10 Let $A \in \mathbb{M}_2(\mathbb{R})$ be a non-zero singular matrix such that $\mathrm{tr}(A) \neq 0$. Which of the following statements are true?
a. For every such matrix A, the problem stated in the preceding exercise need not have a solution.
b. For every such matrix A, the problem stated in the preceding exercise has a solution given by

$$X \;=\; \frac{1}{\mathrm{tr}(A)} A.$$

c. For every such matrix A, the problem stated in the preceding exercise has a solution given by

$$X \;=\; \frac{1}{(\mathrm{tr}(A))^2} A.$$

Section 2: Analysis

2.1 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Which of the following conditions imply that the sequence is Cauchy?

a.
$$|a_n - a_{n+1}| \ \leq \ \frac{1}{n}.$$

b.
$$|a_n - a_{n+1}| \ \leq \ \frac{(n+1)^n}{n^{\frac{3}{2}+n}}.$$

c.
$$|a_n - a_{n+1}| \ \leq \ \frac{n \log n}{e^n}.$$

2.2 Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two real sequences. Which of the following statements are true?
a. If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.
b. If the series $\sum_{n=1}^{\infty} a_n$ is convergent and if the series $\sum_{n=1}^{\infty} b_n$ is absolutely convergent, then the series $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent.
c. If $a_n \geq 0$ for all n and is such that the series $\sum_{n=1}^{\infty} a_n$ is convergent, and if the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded, then the series $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent.

2.3 Find the largest interval in $\mathbb{R}$ in which the following series converges:

$$(x+1) - \frac{(x+1)^2}{4} + \frac{(x+1)^3}{9} - \frac{(x+1)^4}{16} + \cdots .$$

2.4 Which of the following functions are continuous at every point of their respective domains of definition?

a.
$$f(x) \ = \ \lim_{n \to \infty} \frac{x^2 e^{nx} + x}{e^{nx} + 1}, \ \ x \in \mathbb{R}.$$

b.
$$f(x) \ = \ \lim_{n \to \infty} \frac{x^2 e^{nx} + \cos x}{e^{nx} + 1}, \ \ x \in \mathbb{R}.$$

c.
$$f(x) \ = \ \lim_{n \to \infty} \frac{1}{n} \log(e^n + x^n), \ \ x \geq 0.$$

2.5 Which of the following functions are uniformly continuous over their respective domains of definition?

a.
$$f(x) \ = \ \sum_{n=1}^{\infty} \frac{g(x-n)}{2^n}, \ \ x \in \mathbb{R},$$

where $g : \mathbb{R} \to \mathbb{R}$ is a bounded and uniformly continuous function.

b.
$$f(x) \ = \ \cos x \cos \frac{\pi}{x}, \ \ x \in]0,1[.$$

c.
$$f(x) \ = \ \sin x \sin \frac{\pi}{x}, \ \ x \in]0,1[.$$

2.6 Which of the following functions are differentiable at $x = 0$?
a. $f(x) = \sin(|x|x)$.
b.

$$f(x) = \begin{cases} \sin(x^2), & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

c.

$$f(x) = \begin{cases} \sin(|x|), & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

2.7 Let $n \in \mathbb{N}$ and let $x \in \mathbb{R}\backslash\{0\}$. Find a closed form expression (in terms of n and x only) for the following sum:

$$\sum_{k=1}^{n} ke^{kx}.$$

2.8 Fill in the blank below with the correct symbol ($<, \leq, >$ or $\geq$):

$$\tan^{-1} x \quad \cdots \quad \frac{x}{1+x^2}, \quad \text{for all } x > 0.$$

2.9 What is the coefficient of x^7 in the Taylor series expansion, around $x = 0$, of the function $f(x) = \sin^{-1} x$?

2.10 Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx.$$

Section 3: Miscellaneous

3.1 Let X be a non-empty set and let $A \subset X$. Define

$$\chi_A(x) \;=\; \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

If E and F are subsets of X, find a subset G of X such that

$$|\chi_E - \chi_F| \;=\; \chi_G.$$

3.2 If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are arbitrary points in the plane, define the metric

$$d(P, Q) \;=\; \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Let $P = (2, \frac{1}{2})$. Let $\mathcal{S} = [0, 1] \times [0, 1]$. Which of the following statements are true?

a. There does not exist any point $Q \in \mathcal{S}$ such that

$$d(P, Q) \;=\; \min\{d(P, X) \mid X \in \mathcal{S}\}.$$

b. There exists a unique point $Q \in \mathcal{S}$ such that

$$d(P, Q) \;=\; \min\{d(P, X) \mid X \in \mathcal{S}\}.$$

c. There exist infinitely many points $Q \in \mathcal{S}$ such that

$$d(P, Q) \;=\; \min\{d(P, X) \mid X \in \mathcal{S}\}.$$

3.3 Let $n \in \mathbb{N}$ be fixed. For $0 \leq k \leq n$, let $C_k = \binom{n}{k}$. Evaluate:

$$\sum_{\substack{0 \leq k \leq n \\ k \text{ even}}} \frac{C_k}{k+1}.$$

3.4 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions defined on $\mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$. Let $\varepsilon > 0$. Define

$$E_n(\varepsilon) \;=\; \{x \in \mathbb{R} \mid |f_n(x) - f(x)| \geq \varepsilon\},$$

and

$$D \;=\; \{x \in \mathbb{R} \mid \text{the sequence } \{f_n(x)\}_{n=1}^{\infty} \text{ does not converge to } f(x)\}.$$

Express D in terms of the sets of the form $E_n(\varepsilon)$.

3.5 Evaluate:

$$(\sqrt{3} + i)^{14} + (\sqrt{3} - i)^{14}.$$

3.6 Let $0 < \theta < 2\pi$. Let

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

If B is the closed unit disc in the plane, what is the area of $T(B)$?

3.7 Let A be the point $(0, 4)$ in the plane. Let B be an arbitrary point on the x-axis. Let L be the mid-point of the line segment AB. Let the perpendicular bisector of this line segment meet the y-axis at M. Let N be the midpoint of the line segment LM. Find the locus of N (as B moves on the x-axis).

3.8 A plane in $\mathbb{R}^3$ meets the coordinate axes at points A, B and C respectively. If the centroid of the triangle ΔABC is the point $(1, \frac{1}{2}, -\frac{1}{3})$, find the equation of the plane.

3.9 What is the highest power of 3 dividing 1000! ?

3.10 A sports contingent consists of $m = k + \ell$ members, of which k members form the team and ℓ members are reserves. If $n > m$ candidates are available for selection, we can first choose the m members and then choose the team members amongst them, or, we can first choose the team and then choose the reserves. Write down the combinatorial identity which states that both ways give the same number of ways of forming the contingent.