

NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 25, 2010

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 7 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- \mathbb{N} denotes the set of natural numbers, \mathbb{Z} - the integers, \mathbb{Q} - the rationals, \mathbb{R} - the reals and \mathbb{C} - the field of complex numbers. \mathbb{R}^n denotes the n -dimensional Euclidean space.
The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
The symbol I will denote the identity matrix of appropriate order.
The derivative of a function f will be denoted by f' .
All logarithms, unless specified otherwise, are to the base e .
- **Calculators are not allowed.**

Section 1: Algebra

1.1 Let $a, b \in \mathbb{R}$ and assume that $x = 1$ is a root of the polynomial

$$p(x) = x^4 + ax^3 + bx^2 + ax + 1.$$

Find the range of values of a for which p has a complex root which is not real.

1.2 Let $GL_n(\mathbb{R})$ denote the group of all $n \times n$ matrices with real entries (with respect to matrix multiplication) which are invertible. Pick out the normal subgroups from the following:

- The subgroup of all real orthogonal matrices.
- The subgroup of all invertible diagonal matrices.
- The subgroup of all matrices with determinant equal to unity.

1.3 Pick out the true statements:

a. The set

$$\left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}$$

is a group with respect to matrix multiplication.

b. The set

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

is a commutative ring with identity with respect to matrix addition and matrix multiplication.

c. The set

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

is a field with respect to matrix addition and matrix multiplication.

1.4 Let $\mathcal{C}[0, 1]$ denote the ring of all continuous real-valued functions on $[0, 1]$ with respect to pointwise addition and pointwise multiplication. Pick out the true statements:

- $\mathcal{C}[0, 1]$ is an integral domain.
- Let $a \in [0, 1]$. Set

$$\mathcal{I} = \{f \in \mathcal{C}[0, 1] \mid f(a) = 0\}.$$

Then \mathcal{I} is an ideal in $\mathcal{C}[0, 1]$.

- If \mathcal{I} is any proper ideal in $\mathcal{C}[0, 1]$, then there exists at least one point $a \in [0, 1]$ such that $f(a) = 0$ for all $f \in \mathcal{I}$.

1.5 Let V be the real vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to, 3, provided with the standard basis $\{1, x, x^2, x^3\}$. If

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3,$$

define

$$T(p)(x) = a_0 + a_1(x + 1) + a_2(x + 1)^2 + a_3(x + 1)^3.$$

Write down the matrix representing the linear transformation T with respect to this basis.

1.6 Let V be the real vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to, 5. Let W be the subspace defined by

$$W = \{p \in V \mid p(1) = p'(2) = 0\}.$$

What is the dimension of W ?

1.7 Let A be a non-zero 2×2 matrix with real entries. Pick out the true statements:

- a. If $A^2 = A$, then A is diagonalizable.
- b. If $A^2 = 0$, then A is diagonalizable.
- c. If A is invertible, then

$$A = (\operatorname{tr}(A))I - (\det(A))A^{-1}$$

where $\operatorname{tr}(A)$ and $\det(A)$ denote the trace and determinant of A respectively.

1.8 Let A be an $n \times n$ matrix with real entries. Pick out the true statements:

- a. There exists a real symmetric $n \times n$ matrix B such that $B^2 = A^*A$.
- b. If A is symmetric, there exists a real symmetric $n \times n$ matrix B such that $B^2 = A$.
- c. If A is symmetric, there exists a real symmetric $n \times n$ matrix B such that $B^3 = A$.

1.9 Let $S = \{\lambda_1, \dots, \lambda_n\}$ be an ordered set of n real numbers, not all equal, but not all necessarily distinct. Pick out the true statements:

- a. There exists an $n \times n$ matrix with complex entries, which is *not* self-adjoint, whose set of eigenvalues is given by S .
- b. There exists an $n \times n$ self-adjoint, non-diagonal matrix with complex entries whose set of eigenvalues is given by S .
- c. There exists an $n \times n$ symmetric, non-diagonal matrix with real entries whose set of eigenvalues is given by S .

1.10 Let p be a prime number and let \mathbb{Z}_p denote the field of integers modulo p . Find the number of 2×2 invertible matrices with entries from this field.

Section 2: Analysis

2.1 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f' \left(\frac{k}{n} \right).$$

2.2 In each of the following verify whether the series is absolutely convergent, conditionally convergent or divergent:

a.

$$\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{n}{n+1}}$$

b.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \sin \frac{1}{n}$$

c.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{(n+1)(n+2)}$$

2.3 Pick out the uniformly continuous functions over the interval $]0, 1[$:

a. $f(x) = \sin \frac{1}{x}$

b. $f(x) = x \sin \frac{1}{\sqrt{x}}$

c. $f(x) = \exp(-\frac{1}{x^2})$

2.4 In each of the following, verify if the given function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, differentiable but not continuously differentiable or not differentiable at the origin:

a. $f(x) = x \sin \frac{1}{\sqrt{x}}$, if $x \neq 0$ and $f(0) = 0$.

b. $f(x) = |x|^{\frac{3}{2}}$

c. $f(x) = x \sin |x|$

2.5 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(0) = 0$ and $f'(x) = 1/\sqrt{1+x^2}$. Write down the coefficient of x^7 in the Taylor series expansion of f about the origin.

2.6 Pick out the true statements:

a. $|\cos^2 x - \cos^2 y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

b. If $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y|^{\sqrt{2}}$$

for all $x, y \in \mathbb{R}$, then f must be a constant function.

c. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable and such that $|f'(x)| \leq 4/5$ for all $x \in \mathbb{R}$. Then, there exists a unique $x \in \mathbb{R}$ such that $f(x) = x$.

2.7 Sum the following infinite series:

$$\frac{1}{6} + \frac{5}{6.12} + \frac{5.8}{6.12.18} + \frac{5.8.11}{6.12.18.24} + \dots$$

2.8 Let C be the semicircle $z = 2e^{i\theta}$ where θ varies from 0 to π , in the complex plane. Evaluate:

$$\int_C \frac{z+2}{z} dz.$$

2.9 Find the order of the pole and its residue at $z = 0$ of the function

$$f(z) = \frac{\sinh z}{z^4}.$$

2.10 Let C denote the circle $|z| = 3$ in the complex plane, described in the positive (*i.e.* anti-clockwise) sense. Evaluate:

$$\int_C \frac{2z^2 - z - 2}{z - 2} dz.$$

Section 3: Geometry

3.1 Let $f(x, y) = ax + by + c$ where $a, b, c \in \mathbb{R}$ and $c > 0$. Find the largest value of r such that $f(x, y) > 0$ for all pairs (x, y) satisfying $x^2 + y^2 < r^2$.

3.2 Find the value of a such that the lines $3x + y + 2 = 0$, $2x - y + 3 = 0$ and $x + ay - 3 = 0$ are concurrent.

3.3 Find the condition that amongst the pair of lines represented by the equation $ax^2 + 2hxy + by^2 = 0$, the slope of one is twice that of the other.

3.4 Find the condition that the straight line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$.

3.5 Let (x_1, y_1) lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the slope of the normal to the ellipse at this point.

3.6 Find the lengths of the semi-axes of the ellipse

$$5x^2 - 6xy + 5y^2 = 8.$$

3.7 Let L_n denote the perimeter and A_n the area of a regular polygon of n sides, each of whose vertices is at unit distance from its centroid. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{L_n^2}{A_n}.$$

3.8 Find the coordinates of the reflection of the point $(1, -2, 3)$ with respect to the plane $2x - 3y + 2z + 3 = 0$.

3.9 Find the area of the circle formed by the intersection of the plane $x + 2y + 2z - 20 = 0$ with the sphere

$$x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0.$$

3.10 A sphere of radius r passes through the origin and the other points where it meets the coordinate axes are A, B and C . Find the distance of the centroid of the triangle ABC from the origin.