STUDYING QUANTUM STABILITY OF BLACK HOLES:

Vardarajan Suneeta,
IISER PUNE.

Recall:

Classical stability of Schwarzschild black holes
(Regge, Wheeler, Price, Vishveshwara, Chandrasekhar ...)

Q/ Suppose the Schwarzschild black hole is perturbed at some initial time by a perturbation of compact support, will this perturbation remain bounded at all times as it evolves?

$$\text{Let } (g^P)_{ij} = (g_S)_{ij} + \epsilon h_{ij}$$

$(g^P)_{ij}$: Perturbed metric; $(g_S)_{ij}$: Schwarzschild metric; h_{ij}: perturbation

ϵ "small".

We want perturbed spacetime to also satisfy vacuum Einstein eqns. ("on-shell" perturbation):

$$\Rightarrow \quad R_{\mu\nu}(g^P) = 0$$

$$R^S_{\mu\nu} + \delta R_{\mu\nu} = 0 \quad \text{(keeping terms linear in } \epsilon \text{ only)}$$

$$\Rightarrow \delta R_{\mu\nu} = 0.$$

↳ reduces to a second order linear ODE with appropriate ansatz.

⇒ It was found that there are no perturbations that grow in time.

Generalized to Schwarzschild-Tangherlini black holes (Gibbons/Hartnoll, Ishibashi/Kodama)

"Off-shell" or quantum stability of a black hole:

* Euclidean path integral approach to quantum gravity.

$$Z = \int [\mathcal{D}g]\, e^{-S[g]}$$

$S[g]$ → Einstein action + boundary terms.

On-shell geometries obey $\delta S = 0$ (Einstein eqn.) } $(g_{cl})_{ij}$ ← solution.

Consider an "off-shell" perturbation of $(g_{cl})_{ij}$.

$$g^{P}_{ij} = (g_{cl})_{ij} + \epsilon h_{ij}$$

Put appropriate boundary/asymptotic conditions on metric.

Q: Does the perturbation decrease the action?
If yes, then the on-shell geometry is unstable under this perturbation.

[Gross/Perry/Yaffe (GPY)] Euclidean Schwarzschild metric has an unstable mode.
(The perturbed metric is asymptotically flat & approaches $R^3 \times S^1$)

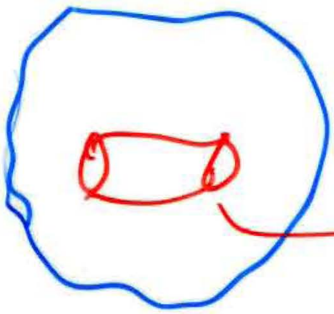
* This mode is evidence of the "off-shell" or "quantum" instability of the Euclidean Schwarzschild black hole in semiclassical gravity.

* (Recall): This 'quantum' instability is the same as the classical Gregory-Laflamme instability of (uncharged) black p-brane ($g_{Sch} + g_{flat}$)
$\underbrace{}_{4\,dim}$ $\underbrace{}_{p\,dim}$

* There is a related notion of 'off-shell' stability in string theory. Relies on the conjecture that many off-shell processes in string field theory are approximated by renormalization group flows (RG flows).

Let us recall the RG flows in string theory . . . . .

RG flows in string theory sigma models



$\to (M, g)$ Target space

$\to (\Sigma, \gamma)$ world-sheet

$X : \Sigma \to M$

Sigma model action:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{\gamma}\, \gamma_{\alpha\beta}\, g_{ij}(X)(\partial_\alpha X^i)(\partial_\beta X^j) + \cdots$$

$g_{ij}(X)$: Target space metric. "Coupling constant" in the sigma model action.

Under an RG transformation, $g_{ij}(X)$ 'flows':

$$\frac{\partial g_{ij}}{\partial \tau} = -\alpha' R_{ij} + O(\alpha'^2)$$

$\tau \to$ RG flow parameter

* One is interested in a flow of metrics mod diffeos.

* Neglecting $O(\alpha'^2)$ terms & higher, this flow is called Ricci flow (used to prove Poincaré conjecture)

Fixed points of RG flow satisfy (neglecting $O(\alpha'^2)$)

$$R_{ij} = 0$$

$\to$ solutions are "on-shell" geometries. Sigma model is a conformal field theory

Conjecture in string theory: (Banks, Vafa, ——)

* Many off-shell processes in string field theory (mediated by tachyonic operators) are approximated by solutions to RG flows. (evidence for this)

⇒ (In)stability of an on-shell geometry under RG flow should be indicative of its off-shell (in)stability.

Argument: In many cases, in the CFT describing the on-shell geometry (fixed pt. of RG flow) it is possible to construct operators that are:

(a) Relevant perturbations of the fixed point (causing RG flow)

and as well

(b) Tachyonic, triggering off-shell processes.

⇒ Tachyonic "off-shell" instability related to instability under RG flow.

Proposal :

We investigate the "quantum" or off-shell stability of Schwarzschild - Tangherlini black holes by studying their stability under string theory RG (Ricci) flow.

We will show that :

* This problem is closely related to the quantum (in)stability in the path integral approach.

* The problem is mathematically well-posed.

* Techniques used in classical stability analysis turn out to be very useful.

The details: ... (w/ Suvankar Dutta)

Schwarzschild - Tangherlini black hole metrics are D-dim. spacetimes given by

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\tilde{s}_d^2$$

ds_d^2 : Standard metric on S^d.

$$f(r) = \left[1 - \left(\frac{\alpha}{r}\right)^{d-1}\right]$$

We are interested in stability under the flow:

$$\frac{\partial g_{ij}}{\partial \tau} = -\alpha' R_{ij} \qquad \text{(Ricci flow)}$$

Schwarzschild - Tangherlini spacetimes obey $R_{ij} = 0$, fixed points of flow.

Q: If $g^p_{ij} = (g_{ST})_{ij} + \epsilon\, h_{ij}$ → perturbation

(g^p_{ij}: Perturbed metric; $(g_{ST})_{ij}$: Schwarzschild-Tangherlini metric)

then does the perturbation remain bounded for all τ? If yes, then $(g_{ST})_{ij}$ is (linearly) stable under this (Ricci) flow.

Q. Which type of perturbations should we consider?

* Ricci flow does not make sense (is not a wellposed PDE) when considered as a flow through all Lorentzian spacetimes.

* Makes sense as a flow through static space-times which can be Wick-rotated.
(In fact, this is assumed in the derivation of Ricci flow as an RG flow in string theory).

We therefore consider static perturbations of the Schwarzschild-Tangherlini space-times.

Q. Is there "gauge freedom" in the problem?

Under diffeos. generated by a vector field V, Ricci flow becomes:

$$\frac{\partial \tilde{g}_{ij}}{\partial \tau} = -\alpha' \left(\tilde{R}_{ij} + \tilde{\nabla}_i V_j + \tilde{\nabla}_j V_i \right)$$

We can choose V appropriately to fix gauge & simplify the computation.

Let $g'_{ij} = (g_{ST})_{ij} + \epsilon h_{ij}$

$h_{ij} := h_{ij}(r, \hat{x}, \tau)$ $\hat{x}$: coord. on S^d

In a linearized approximation, flow of h_{ij} after choosing V appropriately is:

$$\frac{\partial h_{ij}}{\partial \tau} = \frac{\alpha'}{2}\left[-(\Delta_L h)_{ij}\right]$$

Δ_L: "Lichnerowicz Laplacian

$$(\Delta_L h)_{ij} = -\Delta h_{ij} + 2R^k{}_{ij\ell}\, h^\ell{}_k + R_i{}^\ell h_{j\ell} + R^\ell{}_j h_{i\ell}$$

(all curvature tensors are those of $(g_{ST})_{ij}$.)

Suppose we now look for modes of the form:

$$h_{ij}(r, \hat{x}, \tau) = R(r, \tilde{x})\, e^{\Omega \tau}$$

Then $\Omega > 0 \Rightarrow$ unstable (growing) perturbations.

& satisfy $\frac{\alpha'}{2}\left[-(\Delta_L h)_{ij}\right] = \Omega h_{ij}$

They correspond to negative eigenvalues of Δ_L. These are the type of instabilities found by Gross/Perry/Yaffe.

(connection between 'quantum' instability & instability under Ricci flow)

Q// Is $(g_{ST})_{ij}$ stable under the flow

$$\frac{\partial h_{ij}}{\partial \tau} = -\frac{\alpha'}{2}(\Delta_L h)_{ij} \quad ?$$

Strategy:

Break up

$$h_{ij} = h_{ij}^{TT} + \frac{H}{d+2} g_{ij} + \nabla_i Y_j + \nabla_j Y_i - \frac{\nabla_k Y_k}{(d+2)} g_{ij}$$

g_{ij} : Background metric $(g_{ST})_{ij}$

H : $g_{ij} h^{ij}$ = Trace of perturbation.

h_{ij}^{TT} : Transverse traceless perturbation ($\nabla^i h_{ij}^{TT} = 0$, $g^{ij} h_{ij}^{TT} = 0$)

* Analogous to classical stability computations, 'pure divergence' perturbations are "gauge".

* Unlike classical stability computations, trace of perturbation is not gauge.

We need to study flows of H, h_{ij}^{TT} & Y separately.

Flow of trace H:

Flow of trace of an arbitrary static perturbation $h_{ij}(r, \tilde{x}, \tau)$ is:

$$\frac{\partial H}{\partial \tau} = \frac{\alpha'}{2} \Delta H .$$

Ansatz: $H(r, \hat{x}, \tau) = R(r)\, r^{-d/2} \tilde{H}(x^a)\, e^{\Omega_T \tau}$

where $\tilde{\Delta} \tilde{H} = \tilde{\lambda}_T \tilde{H}$

($\tilde{H}$: eigenfn. of $\tilde{\Delta}$ on S^d)

Solutions with $\Omega_T > 0$ which are normalizable $\Rightarrow$ unstable modes.

Result: We find that there are no unstable modes iff $\tilde{\lambda}_T < \frac{d(d-2)}{4}$ ($\tilde{\lambda}_T$ is the eigenvalue of $\tilde{\Delta}$ on S^d)

For S^d, eigenspectrum is known, & always satisfies the above condition. (it is negative)

$\Rightarrow$ No instability in flow of trace!

Note on boundary condns. Treating horizon also as boundary, Dirichlet/Neumann b.c.s at horizon and as $r \to \infty$.

Flow of (h^{TT}_{ij})

Nice property of Δ_L:
When background metric is $(g_{ST})_{ij}$ (or any Einstein metric, in fact):

$$-(\Delta_L h^{TT})_{ij} \text{ is also } TT.$$

So $\dfrac{\partial h^{TT}_{ij}}{\partial \tau}$ decouples from the flow of H and Y.

This problem is computationally difficult.
We use:

(Kodama/Sasaki): Any covariant second order DE on such a spacetime can be decomposed into eqns. for perturbations h^{TT}_{ij} which are scalar, vector and tensor on S^d respectively. So we can study the three cases separately.

We will study "tensor" perturbations first. (transform as tensor of rank 2 on S^d).

These are given by:

$$h_{ra} = h_{ta} = 0$$

where a is any spacetime index.

For $\frac{\partial h_{ij}^{TT}}{\partial t}$ where h_{ij}^{TT} is a "tensor" on S^d, use the ansatz:

$$h_{ij}^{TT} = \phi(r)\, \tilde{h}^{TT}_{\alpha\beta}(\tilde{x})\, r^{\frac{(4-d)}{2}} e^{\Omega t}$$

$\tilde{h}^{TT}_{\alpha\beta}$ is given by $(\tilde{\Delta}_L \tilde{h}_{TT})_{\alpha\beta} = \tilde{\lambda} \tilde{h}^{TT}_{\alpha\beta}$

h_{ij}^{TT} normalizable with $\Omega > 0$
$\Rightarrow$ unstable mode.

Result: No unstable modes iff
$$\tilde{\lambda} > -\frac{(d^2-10d+8)}{4}$$

We know that on S^d

$$(\tilde{\Delta}_L \tilde{h}^{TT})_{\alpha\beta} = -(\tilde{\Delta} \tilde{h}^{TT})_{\alpha\beta} + 2d\, \tilde{h}^{TT}_{\alpha\beta}$$

$$\& \; (\tilde{\Delta} \tilde{h}^{TT})_{\alpha\beta} = -[k(k+d-1)-2]\tilde{h}_{\alpha\beta}$$

for integer $k \geq 2$

$\Rightarrow$ no unstable modes for h_{ij}^{TT} that satisfy this ansatz, & are "tensor" on S^d.

Going beyond a specific ansatz...

For both trace H & a special class of h_{ij}^{TT} we proved stability of $(g_{ST})_{ij}$. We used an ansatz, for. eg.

$$h_{ij}^{TT} = \phi(r)\, r^{\frac{4-d}{2}}\, \tilde{h}_{\alpha\beta}^{TT}\, e^{\Omega\tau}$$

& concluded there were no normalizable modes with $\Omega > 0$.

$\Rightarrow$ No normalizable eigenmodes of $(-\Delta_L)$ with $\Omega > 0$ satisfying these conditions.

However – generically, operators such as Δ_L have a spectrum with a continuous part on non-compact mfds. Corresponding eigen-tensors will be nonnormalizable.

Q// How general was our choice of ansatz?

Problem: We could still have perturbations of compact support constructed from these nonnormalizable modes. These would grow in τ.

Clearly we can still write the dependence of perturbation on coordinates in S^d by using tensor spherical harmonics.

$$h^{TT}_{ij}(r, \tilde{x}, \tau) = \Phi(r, \tau)\, \tilde{h}^{TT}_{\alpha\beta}(\tilde{x})$$

↑ Tensor spherical harmonics

A more rigorous argument:

We show that

$$\int_{-\infty}^{\infty} |\Phi(r_*, \tau)|^2 \, dr_* \quad \& \quad \int_{-\infty}^{\infty} \left|\frac{\partial \Phi}{\partial r_*}\right|^2 dr_* \quad \text{are}$$

bounded from above by constants.

(r_* is related to r by $dr_* = dr/f$)

Further we can derive the inequality

$$|\Phi(r_*, \tau)|^2 \le \int_{-\infty}^{\infty} |\Phi(r_*, \tau)|^2 \, dr_* + \int_{-\infty}^{\infty} \left|\frac{\partial \Phi}{\partial r_*}\right|^2 dr_*$$

$$\Rightarrow |\Phi(r_*, \tau)|^2 \le C \rightarrow \text{constant.}$$

⇒ No growing perturbations!!

(Analogous to a proof of Wald on classical stability of black hole)

To summarize, we showed: (for Schwarzschild, Tangherlini)

(a) There is no instability under the flow of the trace for an arbitrary static perturbation.

(b) For static h_{ij}^{TT} obeying $h^{TT}_{ra} = h^{TT}_{ta} = 0$ (rank 2 tensor on S^d), no instability.

(c) This analysis generalizes easily to AdS-Schwarzschild black holes.

* These can be viewed as quantum stability results for Schwarzschild-Tangherlini spacetimes in Euclidean path integral or as off-shell stability results in string theory.

For the future ....

* Clearly the unstable mode of Gross/Perry/Yaffe for the Schwarzschild black hole is not in the class of perturbations we studied. We would like to widen our stability analysis & consider perturbations that are 'scalar' & 'vector' on S^d as well.

* There are arguments suggesting a GPY type mode for all Schwarzschild-Tangherlini black holes (Kol/Sorkin). Our goal is to systematically analyze all classes of perturbations & obtain an understanding of the instabilities of these black holes in quantum gravity.

* Stability of noncompact geometries under Ricci flow is of interest in math as well. Classical stability calculations of GR seem more useful for this problem than techniques developed in Ricci flow on compact manifolds.