An Introduction to Automatic Sequences*

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Let us consider two famous problems in Number Theory.

1. Let us write $\sqrt{2}$ in the base 10 as

$$\sqrt{2} = 1.414213562373...{}_{10} = \sum_{k=0}^{\infty} a_k 10^{-k}, \text{ say,}$$

and consider the real number α which, in the base 11, has the same digits as $\sqrt{2}$ in the base 10, that is to say

$$\alpha = 1.414213562373...{}_{11} = \sum_{k=0}^{\infty} a_k 11^{-k}.$$

Is it true that α is transcendental?

2. What is the distribution of the the sequence of the fractional parts of $(3/2)^n$? Is it true that those values are "nicely" distributed in the interval $[0\,,\,1)$?

Not only those two questions are unsolved, but we may be far from seeing their solution. However, those questions have similar formulations in the frame of function fields (in a naïve approach, let us say that we perform addition without carry over). Not only both our questions can be solved in this setting, but moreover the tools which are used for the proofs , *finite automata,* play an important rôle in number theory and discrete mathematics; finally, they essentially rely on basic concepts in graph theory and linear algebra.

Project for a series of lectures at IMSc during the special year 2011

1 Introduction

1. a First example : the Thue - Siegel - Morse sequence ; definition, distribution of the digits, automatic setting.

1. b Second example : the least non zero-digit ; definition, distribution of the digits, automatic setting.

1. c Third example : the largest non-zero digit ; definition, distribution of the digits, automatic settings.

2 Finite q-automata

2. a Definition of a finite q-automaton.

2. b Graph and transition matrix associated to a q-automaton.

3 Markov chains and stochastic matrices

The content will depend of what the students already know and what they have to know about those important notions at the end of their third year.

4 Automatic sequences

4. a Definition. Right-automata versus left-automata.

4. b Fixed points of substitutions.

4. c Densities of automatic sequences.

4. d The Cobham theorem.

5 Function generated by an automatic sequence

5. a Definition, examples.

5. b The Chritol - Kamae - Mendès-France - Rauzy theorem .

5. c Applications.