NATIONAL BOARD FOR HIGHER MATHEMATICS
DOCTORAL SCHOLARSHIP SCHEME 2021
WRITTEN TEST, SUNDAY 11TH APRIL 2021

- Roll number: ‡ Application number: ‡
- Name in full in BLOCK letters: ‡
- There are 39 questions on this test distributed over two sections. Answer as many as you can.
- This **test booklet must have** 7 **pages** (6 pages of questions and this cover page with instructions). Make sure that your copy is correctly printed and has all 7 pages and all 39 questions.
- TIME ALLOWED: 180 minutes (three hours).
- QUESTIONS in each section are arranged rather randomly. They are not sorted by topic.
- MODE OF ANSWERING: Fill in only your final answer in the box provided for it. This box has the following appearance: ‡ . It is neither necessary nor is there provision of space to indicate the steps taken to reach the final answer.

  Only your final answer, written legibly and unambiguously in the box, will count.

- MARKING: The marking scheme for each section is described at the beginning of that section. There is **negative marking** in Section B (but not in Section A).
- NOTATION AND TERMINOLOGY: The questions make free use of standard notation and terminology. You too are allowed the use of standard notation in expressing your answers. For example, answers of the form $e + \sqrt{2}$ and $2\pi/19$ are acceptable; both $3/4$ and 0.75 are acceptable.
- DEVICES: Use of plain pencils, pens, and erasers is allowed. Mobile phones are prohibited in the exam hall. So are calculators. More generally, any device (e.g. a smart watch) that can be used for communication or calculation or storage is prohibited. Invigilators have the right to impound any device that arouses their suspicion (for the duration of the test).
- ROUGH WORK: For rough work, you may use the sheets separately provided, in addition to the blank pages in your test booklet. You must:
  - Write your name and roll number on each such sheet (or set of sheets if stapled).
  - Return all these sheets to the invigilator along with this test booklet at the end of the test.
- Do **not** seek clarification from the invigilators about any question. In the unlikely event that there is a mistake in any question, appropriate allowance will be made while marking.

SECTION A (QUESTIONS 1 TO 30) **Short Answer** TYPE

There are 30 questions in this section. Each question carries 2 marks and demands a short answer. The answers must be written only in the boxes provided for them. There is no negative marking in this section. In other words, there is no penalty for incorrect answers.

(1) A smooth function y of a single variable satisfies the differential equation $y'' - 2y' + y = 4$ and the initial conditions $y(0) = 5$, $y'(0) = 2$. Find $y(1)$. ‡

(2) Given that the following Laurent series expansion is valid in the annulus $1 < |z| < 2$, what is a_3?

$$\frac{1}{(z-1)(z-2)} = \sum_{n=-\infty}^{\infty} a_n z^n$$

‡

(3) Let $\mathbb{R}^3$ be the real vector space consisting of 1×3 real matrices. Let A be the 3×3 real matrix with the property that the linear transformation $x \mapsto xA$ from $\mathbb{R}^3$ to itself projects every vector x in $\mathbb{R}^3$ orthogonally onto the line in the direction of $(1, 0, 1)$. (Here xA denotes the usual matrix product of the matrices x and A.) What is the sum of the entries of A? ‡

(4) Let G denote the group of rational numbers $\mathbb{Q}$ with respect to addition. What is the cardinality of the automorphism group of G? Choose the correct option from among the following four:
(a) 1 (that is, the only automorphism is the identity map) ‡
(b) finite but not 1
(c) countable (but not finite)
(d) uncountable

(5) Given a positive real number x, a sequence $\{a_n(x)\}_{n\geq 1}$ is defined as follows:

$$a_1(x) := x \quad \text{and} \quad a_n(x) := x^{a_{n-1}(x)} \text{ recursively for all } n \geq 2.$$

Determine the largest value of x for which $\lim_{n\to\infty} a_n(x)$ exists. ‡

(6) Among 3×3 invertible matrices with entries in the finite field $\mathbb{Z}/3\mathbb{Z}$ containing 3 elements, how many are similar to the following matrix? ‡

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(7) For any integer n, let I_n denote the ideal $\{m \in \mathbb{Z} \mid m^r \in n\mathbb{Z} \text{ for some } r \geq 1\}$. What is the cardinality of the quotient ring $\mathbb{Z}/(I_{63} \cap I_{84})$? ‡

(8) Let A be an invertible 3×3 real matrix such that A and A^2 have the same characteristic polynomial. What are the possible traces of such a matrix A? ‡

(9) Let v be a fixed non-zero vector of an n-dimensional real vector space V. Let $\mathscr{S}(v)$ be the subspace of the vector space of linear operators on V consisting of those operators that admit v as an eigenvector. What is the dimension of $\mathscr{S}(v)$ as a real vector space (in terms of n)? ‡

(10) Find the largest positive real number δ such that, for all real numbers x and y, we have: $|\cos x - \cos y| < \sqrt{2}$ whenever $|x - y| < \delta$. ‡

(11) A particle is at the origin of the Cartesian coordinate plane to begin with. At the end of every second, it jumps one unit either to the East or to the West or to the North or to the South (from wherever it is at the beginning of the second) with equal probability. What is the probability that the particle is back at the origin after six seconds? ‡

(12) As $f(x)$ varies over all continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ with the property that $f(0) = 10$ and $f(1) = 0$, find the infimum of $\int_0^1 \sqrt{1 + f'(x)^2}\, dx$. ‡

(13) A particle is moving on the x-axis such that

$$\frac{dx}{dt} = (x-1)(x+2)(x-3).$$

Here x denotes the x-coordinate of the particle and t denotes time. The particle is so positioned initially that it does not wander off to infinity. Which point of equilibrium will it be close to after a sufficiently long time? ‡

(14) Let G be the group of homeomorphisms of the real line $\mathbb{R}$ with its usual topology, the group operation being composition. Consider the elements $f(x) = 2x$ and $g(x) = 8x$ of G. Suppose $h(x)$ in G is such that it conjugates f to g, that is, $(h \circ f \circ h^{-1})(x) = g(x)$, and further satisfies $h(1) = 5$. What is $h(2)$? ‡

(15) An element a of a ring R is said to be an *idempotent* if $a^2 = a$. Note that 0 and 1 are idempotents. How many idempotents (including 0 and 1) are there in the ring $\mathbb{Z}/120\mathbb{Z}$? ‡

(16) Consider the Taylor expansion of the function $\dfrac{1}{1+x^3}$ in powers of $x - \dfrac{1}{2}$:

$$\frac{1}{1+x^3} = \sum_{n \geq 0} a_n (x - \frac{1}{2})^n$$

What is the radius of convergence of this series? ‡

(17) Let n be a positive integer and $\mathscr{P}$ the real vector space of polynomials with real coefficients and degree at most n. Let T be the linear operator on $\mathscr{P}$ defined by $Tf(x) = xf'(x) - f(x-1)$, where $f'(x)$ is the derivative of $f(x)$. What is the trace of T? ‡

(18) Let D be the disc in the complex plane centred at the point $\pi/4$ and of radius r. Let D' be the image of this disk under the map $z \mapsto \sin z$. Evaluate the following limit: ‡

$$\lim_{r \to 0} \frac{\text{Area}(D')}{\text{Area}(D)}$$

(19) A prime p is such that $1/p$ when represented in octal (base 8) notation equals $0.\overline{00331}$ (meaning that the digits 0, 0, 3, 3, 1 get repeated ad infinitum). What is the order of 2 in the group of units of the ring of integers modulo p? ‡

(20) Let $M_5(\mathbb{C})$ be the set of 5×5 complex matrices and let A be a fixed element of $M_5(\mathbb{C})$. Consider the linear transformation $T_A : M_5(\mathbb{C}) \to M_5(\mathbb{C})$ given by $X \mapsto AX$, where AX is the usual matrix product of elements in $M_5(\mathbb{C})$. If A has rank 2, what is the rank of T_A? ‡

(21) Let $\mathbb{Z}$ denote the set of integers. For c and r in $\mathbb{Z}$, define:

$$B(c, r) := \{c + kr \mid k \in \mathbb{Z}\}.$$

As c varies over all integers and r over all positive integers, the sets $B(c, r)$ form a basis for a topology on $\mathbb{Z}$. Does the following limit exist with respect to this topology?

$$\lim_{n \to \infty} (n! - 2)^2$$

If so, then write the value of the limit in the box; if not, write "No". ‡

(22) Let $S := \mathbb{R}[x]$ denote the polynomial ring in one variable over the field $\mathbb{R}$ of real numbers. Find a monic polynomial of least degree in S that is a square root of -4 modulo the ideal generated by $(x^2 + 1)^2$. ‡

(23) Let $M_n(\mathbb{R})$ be the real vector space of $n \times n$ matrices with entries in $\mathbb{R}$. Consider the subset M of $M_n(\mathbb{R})$ consisting of matrices having the property that the entries in every row add up to zero and the same holds for every column:

$$M := \{A = (a_{ij})_{1 \le i,j \le n} \in M_n(\mathbb{R}) \mid \sum_{j=1}^{n} a_{ij} = 0 \text{ for every } i \text{ and } \sum_{i=1}^{n} a_{ij} = 0 \text{ for every } j\}$$

What is the dimension of M (as a real vector space)? ‡

(24) Consider the entire function $f(z) = z(z - i)$. Put: ‡

$$S := \{\frac{1}{|f(z)|} \mid |z| \ge 2\}$$

At what value(s) of z is the maximum of the set S attained?

(25) Consider the following system of linear equations over the field $\mathbb{Z}/5\mathbb{Z}$. How many solutions does it have? ‡

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & 1 & 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

(26) Let $f(x)$ be an irreducible polynomial over $\mathbb{Q}$ of degree 2021 and let K be the field obtained by adjoining to $\mathbb{Q}$ a root α of $f(x)$. Suppose that the Galois group over $\mathbb{Q}$ of the splitting field of $f(x)$ is abelian. Then how many fields F are there such that $\mathbb{Q} \subseteq F \subseteq K$? (Note: 43×47 is the prime factorisation of 2021.) Choose one of the following four options. ‡

(a) 2
(b) 3
(c) 4
(d) Cannot be determined from the given data.

(27) Find the smallest positive real number k such that, given any finite set $z_1, \ldots, z_n$ of complex numbers, all with strictly positive real and imaginary parts, the following inequality holds:

$$|z_1 + \cdots + z_n| \geq \frac{1}{k} \cdot (|z_1| + \cdots + |z_n|)$$ ‡

(28) A "Fibonacci like" sequence $\{a_n\}_{n\geq 0}$ is defined as follows:

$$a_0 = 1, \quad a_1 = 1, \quad \text{and} \quad a_n = 2a_{n-2} + a_{n-1} \text{ for all } n \geq 2$$

Find a closed form formula for a_n (that works for all $n \geq 0$)? ‡

(29) Find the real number a such that

$$\oint_{|z-i|=1} \frac{dz}{z^2 - z + a} = \pi$$ ‡

(30) Let $\mathscr{L}$ be the set of all $k \times k$ real matrices A such that $\lim_{n\to\infty} A^n$ exists as a $k \times k$ real matrix. (The limit is entry-wise of the sequence A^n of $k \times k$ matrices.) Let $\mathscr{P}$ be the subset of $\mathscr{L}$ consisting of all matrices in $\mathscr{L}$ that are symmetric and positive definite. What is the image of the map $\varphi : \mathscr{P} \to \mathbb{R}$ defined by $\varphi(A) = \text{trace}(A)$? ‡

SECTION B (QUESTIONS 31–39) **True or False**

This section features 20 assertions grouped into 9 questions. For each assertion, you are required to determine its truth value and accordingly write either "True" or "False" in the corresponding box, as the case may be. Each correct response will fetch you 1 mark. But there is negative marking:

Each incorrect response carries a penalty of 1 mark.

(31) Let A be a real $n \times n$ symmetric matrix with distinct eigenvalues (that is, A no repeated eigenvalue). Let B be a real $n \times n$ matrix that commutes with A. Then:

- (a) The eigenvalues of B are real. ‡
- (b) Every eigenvector of A is also an eigenvector for B. ‡
- (c) B is diagonalizable. ‡

(32) For $f : (0, 1) \to \mathbb{R}$ a real valued function on the open interval $(0, 1)$:

- (a) Suppose that f is uniformly continuous. Then the image under f of a Cauchy sequence is a Cauchy sequence. ‡
- (b) Suppose that, for every Cauchy sequence in $(0, 1)$, its image under f is a Cauchy sequence. Then f is uniformly continuous. ‡

(33)
- (a) Every bounded continuous real valued function on $\mathbb{Q}^2$ can be extended to a bounded continuous function on $\mathbb{R}^2$. ‡
- (b) There exists a non-finite subset X of $\mathbb{Q}^2$ such that every real valued continuous function on X is bounded. ‡

(34) Let $C(\mathbb{R})$ and $C[0, 1]$ be the rings of continuous real valued functions on $\mathbb{R}$ and $[0, 1]$ respectively (with pointwise operations).

- (a) Let $\pi : C(\mathbb{R}) \to C[0, 1]$ be the ring homomorphism given by restriction: $\pi(f) := f|_{[0,1]}$ for $f \in C(\mathbb{R})$. The kernel of π is a prime ideal in $C(\mathbb{R})$. ‡
- (b) The ideal I defined as follows is a maximal ideal in $C(\mathbb{R})$:

  $I := \{f \in C(\mathbb{R}) \mid \text{there exists } N \in \mathbb{R} \text{ such that } f(x) = 0 \text{ for all } x > N\}$. ‡

(35) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous function such that the inverse image of any bounded set is bounded. Then:

- (a) The image under f of a closed set is closed. ‡
- (b) The image of f is all of $\mathbb{R}$. (In other words, f is surjective.) ‡

(36) Let a, b, and c be three non-collinear points in the complex plane. Let $f(z)$ be the following function defined on the complex plane:

$$f(z) := |z-a| \cdot |z-b| \cdot |z-c|$$

Let M be the maximum value of $f(z)$ on the (closed) triangle T with vertices a, b, and c. For each of the following two statements, determine whether or not it holds always (i.e., for all possible a, b, c).

(a) $f(\alpha) \geq M$ where α is the centroid of T (the centre of mass of T). ‡

(b) $f(\alpha) \geq M$ where α is the orthocentre of T (intersection of the altitudes of T). ‡

(37) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous function. Define

$$S := \{y \in \mathbb{R} \mid \text{ there exists a sequence } x_n \text{ in } \mathbb{R} \text{ with } \lim_{n\to\infty} x_n = \infty \text{ and } \lim_{n\to\infty} f(x_n) = y\}.$$

Then:

(a) S is closed. ‡

(b) S is connected. ‡

(38) (a) Every infinite group has infinitely many subgroups. ‡

(b) Every countable group has only countably many subgroups. ‡

(c) Every uncountable group has uncountably many subgroups. ‡

(39) (a) $\sum_{n\geq 1} \sqrt{\frac{a_n}{n}}$ converges for every sequence $\{a_n\}_{n\geq 1}$ of positive real numbers such that the series $\sum_{n\geq 1} a_n$ converges. ‡

(b) $\sum_{n\geq 1} \frac{\sqrt{a_n}}{n}$ converges for every sequence $\{a_n\}_{n\geq 1}$ of positive real numbers such that the series $\sum_{n\geq 1} a_n$ converges. ‡