NATIONAL BOARD FOR HIGHER MATHEMATICS

DOCTORAL SCHOLARSHIP SCHEME 2020

WRITTEN TEST, SATURDAY 25TH JANUARY 2020

- Roll number: ‡ Application number: ‡
- Name in full in BLOCK letters: ‡
- This **test booklet** consists of 6 pages of questions and this cover page (**total** 7 **pages**).
- There are 36 questions distributed over 3 sections. Answer all of them.
- TIME ALLOWED: 180 minutes (three hours).
- QUESTIONS in each section are arranged rather randomly. They are not sorted by topic or level of difficulty.
- ABOUT THE ANSWERS: Each time a response is demanded, fill in only your final answer in the box provided for it. This box has the following appearance: ‡ . It is neither necessary nor is there provision of space to indicate the steps taken to reach the final answer.

  Only your final answer, written legibly and unambiguously in the box, is considered for marking.

- MARKING: Each question carries 4 marks. There is ***negative marking*** in Section C (but not in Section A and B). The marking scheme for each section is described in more detail at the beginning of that section.
- NOTATION AND TERMINOLOGY: The questions make free use of standard notation and terminology. You too are allowed the use of standard notation. For example, answers of the form $e+\sqrt{2}$ and $2\pi/19$ are acceptable; both $3/4$ and 0.75 are acceptable.
- DEVICES: Use of plain pencils, pens, and erasers is allowed. Mobile phones are prohibited. So are calculators. More generally, any device (e.g. a smart watch) that can be used for communication or calculation or storage is prohibited. Invigilators have the right to impound any device that arouses their suspicion.
- ROUGH WORK: For rough work, you may use the sheets separately provided. You must:
  - Write your name and roll number on each such sheet (or set of sheets if stapled).
  - Return all these sheets to the invigilator along with this test booklet at the end of the test.

SECTION A (QUESTIONS 1 TO 18)

There are 18 questions in this section. Each question carries 4 marks and demands a short answer (or short answers). The answers must be written only in the boxes provided for them. There is no possibility of partial credit in this section: either you get all 4 marks allotted to a question or none at all.

(1) Find rational numbers a, b, and c such that $(1+\sqrt[3]{2})^{-1} = a + b\sqrt[3]{2} + c\sqrt[3]{2}^2$:

$a =$ ‡ $b =$ ‡ $c =$ ‡

(2) Let u and v be the real and imaginary parts respectively of the function $f(z) = 1/(z^2 - 6z + 8)$ of a complex variable $z = x + iy$. Let C be the simple closed curve $|z| = 3$ oriented in the counter clockwise direction. Evaluate the following integral:

$$\oint_C u\,dy + v\,dx = \text{‡}$$

(3) A point is moving along the curve $y = x^2$ with unit speed. What is the magnitude of its acceleration at the point $(1/2, 1/4)$? ‡

(4) Evaluate $\int_{-\infty}^{\infty}(1+2x^4)e^{-x^2}\,dx$: ‡

(5) Let $p(x)$ be the minimal polynomial of $\sqrt{2}+\sqrt{-2}$ over the field $\mathbb{Q}$ of rational numbers. Evaluate $p(\sqrt{2})$? ‡

(6) Find the volume of the tetrahedron in $\mathbb{R}^3$ bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$, and the tangent plane at the point $(4,5,5)$ to the sphere $(x-3)^2 + (y-3)^2 + (z-3)^2 = 9$. ‡

(7) From the collection of all permutation matrices of size 10×10, one such matrix is randomly picked. What is the expected value of its trace? ‡ (A permutation matrix is one that has precisely one non-zero entry in each column and in each row, that non-zero entry being 1.)

(8) You are given 20 identical balls and 5 bins that are coloured differently (so that any two of the bins can be distinguished from each other). In how many ways can the balls be distributed into the bins in such a way that each bin has at least two balls? ‡

(9) Let G be the symmetric group S_5 of permutations of five symbols. Consider the set $\mathscr{S}$ of subgroups of G that are isomorphic to the non-cyclic group of order 4. Let us call two subgroups H and K belonging to $\mathscr{S}$ as *equivalent* if they are conjugate (that is, there exists $g \in G$ such that $gHg^{-1} = K$). How many equivalence classes are there in $\mathscr{S}$? ‡

(10) Let M be a 7×6 real matrix. The entries of M in the positions $(1,3)$, $(1,4)$, $(3,3)$, $(3,4)$, and $(5,4)$ are changed to obtain another 7×6 real matrix $\widetilde{M}$. Suppose that the rank of $\widetilde{M}$ is 4. What could be the rank of M? List all possibilities: ‡

(11) What are the maximum and minimum values of $x + y$ in the region $S = \{(x, y) : x^2 + 4y^2 \leq 1\}$?

maximum = ‡ minimum = ‡

(12) Let k be the field obtained by adjoining to the field $\mathbb{Q}$ of rational numbers the roots of the polynomial $x^4 - 2$. Let k' be the field obtained by adjoining to k the roots of the polynomial $x^4 + 2$. What is the degree of k' over k? ‡

(13) Evaluate the (absolute value of the) surface integral $\left|\int_S \overline{F}.d\overline{A}\right|$ of the vector field $\overline{F}(x, y, z) := (e^y, 0, e^x)$ on the surface

$$S := \{(x, y, z) \mid x^2 + y^2 = 25,\ 0 \leq z \leq 2,\ x \geq 0,\ y \geq 0\}.$$

‡

(14) Let k be a field with five elements. Let V be the k-vector space of 5×1 matrices with entries in k. Let S be a subset of V such that $u^t v = 0$ for all u and v in S: here u^t denotes the transpose of u and $u^t v$ the usual matrix product. What is the maximum possible cardinality of S? ‡

(15) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that the real part of $f''(z)$ is strictly positive for all $z \in \mathbb{C}$. What is the maximum possible number of solutions of the equation $f(z) = az + b$, as a and b vary over complex numbers? ‡

(16) What is the expected minimum number of tosses of a fair coin required to get both heads and tails each at least once? ‡

(17) How many real solutions does the equation $f(x) = 0$ have, where $f(x)$ is defined as follows? ‡

$$f(x) := \sum_{i=1}^{2020} \frac{i^2}{(x - i)}$$

(18) Let $SL_2(\mathbb{Z})$ denote the group (under usual matrix multiplication) of 2×2 matrices with integer entries and determinant 1. Let H be the subgroup of $SL_2(\mathbb{Z})$ consisting of those matrices such that:

- the diagonal entries are all equivalent to 1 mod 3.
- the off-diagonal entries are all divisible by 3.

What is the index of H in $SL_2(\mathbb{Z})$? ‡

SECTION B (QUESTIONS 19–22)

There are 4 questions in this section. Each of these questions has 4 parts and each part carries 1 mark. Partial credit is possible: for example, if you answer correctly only one part of a question but leave the other parts blank (or answer them incorrectly), you earn 1 mark on that question. There is no negative marking: in other words, there is no penalty for wrong answers.

(19) For each of the following numbers q in turn, consider a field k of order q. In each case, determine the number of elements α in k such that the smallest subfield of k containing α is k itself.

(a) 2^4 ‡

(b) 3^5 ‡

(c) 5^{10} ‡

(d) 7^{12} ‡

(20) Let B_r denote the closed disk $\{z \in \mathbb{C} : |z| \leq r\}$. State whether ∞ is a removable singularity (RS), pole (P), or essential singularity (ES) in each of the following cases. There may be more than one possibility in each case.

(a) f is a non-constant polynomial in z. ‡

(b) $f(z) = \frac{p(z)}{q(z)}$, where p, q are are non-zero polynomials of the same degree. ‡

(c) f is an entire function for which $f^{-1}(B_1)$ is bounded. ‡

(d) f is an entire function for which $f^{-1}(B_r)$ is bounded for all $r > 0$. ‡

(21) Let $R := \mathbb{Z}/2020\mathbb{Z}$ be the quotient ring of the integers $\mathbb{Z}$ by the ideal $2020\mathbb{Z}$.

(a) What is the number of ideals in R? ‡

(b) What is the number of units in R? ‡

(c) What is the number of elements r in R such that $r^n = 1$ for some integer $n \geq 1$? ‡

(d) What is the number of elements r in R such that $r^n = 0$ for some integer $n \geq 1$? ‡

(22) Let X be a three element set. For each of the following numbers n, determine the number of distinct homeomorphism classes of topologies on X with exactly n open subsets (including the empty set and the whole set). Write that number in the box.

(a) 3 ‡

(b) 4 ‡

(c) 5 ‡

(d) 7 ‡

SECTION C (QUESTION 23–36)

There are 14 questions in this section. Each question has 4 parts. Each part carries 1 mark. Partial credit is possible: for example, if you answer correctly only one part of a question but leave the other parts blank, you earn 1 mark on that question. But **there is negative marking**, with the penalty being 1 mark for each incorrect response: for example, if you answer two parts of a question correctly, a third part incorrectly, and leave the fourth blank, then you earn 1 mark ("two minus one") on that question.

(23) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x + y$. For each of the following statements, state whether it is true or false.

(a) Image under f of any open set is open. ‡

(b) Image under f of any closed set is closed. ‡

(c) Image under f of any dense set is dense. ‡

(d) Image under f of any discrete set is discrete. ‡

(24) Listed below are four subsets of $\mathbb{C}^2$. For each of them, write "Bounded" or "Unbounded" in the box as the case may be. ($\Re(z)$ denotes the real part of a complex variable z.)

(a) $\{(z, w) \in \mathbb{C}^2 : z^2 + w^2 = 1\}$ ‡

(b) $\{(z, w) \in \mathbb{C}^2 : |\Re(z)|^2 + |\Re(w)|^2 = 1\}$ ‡

(c) $\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ ‡

(d) $\{(z, w) \in \mathbb{C}^2 : |z|^2 - |w|^2 = 1\}$ ‡

(25) For each of the following series, write "convergent" or "divergent" in the box, as the case may be:

(a) $\displaystyle\sum_{n\geq 2} \frac{1}{n \log n}$ ‡

(b) $\displaystyle\sum_{n\geq 2} \frac{\log^2 n}{n^2}$ ‡

(c) $\displaystyle\sum_{n\geq 2} \frac{1}{n \log^2 n}$ ‡

(d) $\displaystyle\sum_{n\geq 2} \frac{\sqrt{n+1} - \sqrt{n}}{n}$ ‡

(26) Let T be a nilpotent linear operator on the vector space $\mathbb{R}^5$ (nilpotent means that $T^n = 0$ for large n). Let d_i denote the dimension of the kernel of T^i. Which of the following can possibly occur as a value of (d_1, d_2, d_3)? Write "Yes" in the box if it can, and "No" if it cannot.

(a) $(1, 2, 3)$ ‡

(b) $(2, 3, 5)$ ‡

(c) $(2, 2, 4)$ ‡

(d) $(2, 4, 5)$ ‡

(27) For n a positive integer, let $\mathbb{Q}/n\mathbb{Z}$ be the quotient of the group of rational numbers $\mathbb{Q}$ (under addition) by the subgroup $n\mathbb{Z}$. For each of the following statements, state whether it is true or false.

(a) Every element of $\mathbb{Q}/n\mathbb{Z}$ is of finite order. ‡

(b) There are only finitely many elements in $\mathbb{Q}/n\mathbb{Z}$ of any given finite order. ‡

(c) Every proper subgroup of $\mathbb{Q}/n\mathbb{Z}$ is finite. ‡

(d) $\mathbb{Q}/2\mathbb{Z}$ and $\mathbb{Q}/5\mathbb{Z}$ are isomorphic as groups. ‡

(28) Let $\mathscr{S}$ be the family of continuous real valued functions on $(0,\infty)$ defined by:

$$\mathscr{S} := \{f : (0,\infty) \to \mathbb{R} \mid f(x) = f(2x)\ \forall x \in (0,\infty)\}.$$

For each of the following statements, state whether it is true or false.

(a) Any element $f \in \mathscr{S}$ is bounded. ‡

(b) Any element $f \in \mathscr{S}$ is uniformly continuous. ‡

(c) Any element $f \in \mathscr{S}$ is differentiable. ‡

(d) Any uniformly bounded sequence in $\mathscr{S}$ has a uniformly converging subsequence. ‡

(29) Let $B_1 := \{z \in \mathbb{C} : |z| \leq 1\}$, and let $C^0(B_1,\mathbb{C})$ be the space of continuous complex-valued functions on B_1 equipped with the uniform convergence topology. Listed below are four subsets of $C^0(B_1,\mathbb{C})$. For each of them, decide whether or not it is dense in $C^0(B_1,\mathbb{C})$. Accordingly write "Dense" or "Not dense" in the box.

(a) Restrictions to B_1 of polynomials in z. ‡

(b) Restrictions to B_1 of polynomials in z and $\overline{z}$. ‡

(c) The set of smooth functions $f : B_1 \to \mathbb{C}$ that vanish on the boundary ∂B_1. ‡

(d) The set of smooth functions $f : B_1 \to \mathbb{C}$ whose normal derivative vanishes along the boundary ∂B_1. ‡

(30) Let p be a prime number, and let $S = [0,1] \cap \{q/p^n \mid q \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\}$. Assume that S has the subspace topology induced from the inclusion $S \subseteq [0,1]$. For each of the following statements, state whether it is true or false.

(a) Any bounded function on S uniquely extends to a bounded function on $[0,1]$. ‡

(b) Any continuous function on S uniquely extends to a continuous function on $[0,1]$. ‡

(c) Any uniformly continuous function on S uniquely extends to a uniformly continuous function on $[0,1]$. ‡

(d) Any bounded continuous function on S uniquely extends to a bounded continuous function on $[0,1]$. ‡

(31) Let $X = GL_2(\mathbb{R})$ be the set of all 2×2 invertible real matrices. Consider X as a subset of the topological space $\mathfrak{M}$ of all 2×2 real matrices and let X be given the subspace topology ($\mathfrak{M}$ is identified with $\mathbb{R}^4$ in the standard way and thus becomes a topological space). Which of the following topological spaces is obtained as the image under a continuous surjection from X? In each case, write "Yes" in the box if the space is thus obtained, and "No" otherwise:

(a) the real line $\mathbb{R}$. ‡

(b) the subspace $\{(x,1/x) \mid x \in \mathbb{R}, x \neq 0\}$ of $\mathbb{R}^2$. ‡

(c) the complement in $\mathbb{R}^2$ of the set $\{(x,1/x) \mid x \in \mathbb{R}, x \neq 0\}$. ‡

(d) the closed disk $\{(x,y) \mid x^2 + y^2 \leq 1\}$ of $\mathbb{R}^2$. ‡

(32) For n a positive integer, let $f_n(x) : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x/(1+nx^2)$. For each of the following statements, state whether it is true or false.

(a) The sequence $\{f_n(x)\}$ of functions converges uniformly on $\mathbb{R}$. ‡

(b) The sequence $\{f_n(x)\}$ of functions converges uniformly on $[1,b]$ for any $b > 1$. ‡

(c) The sequence $\{f'_n(x)\}$ of derivatives converges uniformly on $\mathbb{R}$. ‡

(d) The sequence $\{f'_n(x)\}$ of derivatives converges uniformly on $[1,b]$ for any $b > 1$. ‡

(33) For which of the following subspaces X of $\mathbb{R}$ does every continuous surjective map $f : X \to X$ have a fixed point? Write "Yes" in the box if it does, "No" if it does not.

(a) $[1,2]$ ‡

(b) $[1,2] \cup [3,7]$ ‡

(c) $[3,\infty)$ ‡

(d) $[1,2] \cup [3,\infty)$ ‡

(34) Let A be an arbitrary real 5×5 matrix row equivalent to the following matrix:

$$R = \begin{pmatrix} 1 & 0 & 0 & -3 & -1 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(Two matrices are row equivalent if they have the same row space.) For each of the following statements, state whether it is true or false.

(a) The first two rows of A are linearly independent. ‡

(b) The last four rows of A generate a space of dimension at least 2. ‡

(c) The first two columns of A are linearly independent. ‡

(d) The last four columns of A generate a space of dimension 3. ‡

(35) Consider the space $X := \mathbb{R}^{[0,1]}$ of real valued functions on $[0,1]$ given the product topology. Given below are four subsets of X. In each case, determine whether or not it is closed in X. Write "Closed" in the box if it is closed, and "Not closed" otherwise.

(a) The subset consisting of all continuous functions. ‡

(b) The subset consisting of functions that take integer values everywhere. ‡

(c) The subset consisting of all unbounded functions. ‡

(d) The subset consisting of all bounded functions. ‡

(36) Let X be the space of all real polynomials $a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$ of degree at most 5. We may think of X as a topological space via its identification with $\mathbb{R}^6$ given by:

$$a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \quad \leftrightarrow \quad (a_5, a_4, a_3, a_2, a_1, a_0)$$

Which of the following subsets of X is connected? Write "Connected" or "Disconnected" in the box as the case may be.

(a) All polynomials in X that do not vanish at $t = 2$. ‡

(b) All polynomials in X whose derivatives vanish at $t = 3$. ‡

(c) All polynomials in X that vanish at both $t = 4$ and $t = 5$. ‡

(d) All polynomials in X that are increasing (as functions from $\mathbb{R}$ to $\mathbb{R}$). ‡