(1) Let A_n be the real $n \times n$ matrix ($n \geq 2$) whose entry in position (i, j) is $i - j$. What is the rank of A_n as a function of n? **2**

(2) Let $p(x)$ be the polynomial left as remainder when $x^{2019} - 1$ is divided by $x^6 + 1$. What is the remainder left when $p(x)$ is divided by $x - 3$? **26**

(3) Let S be the set of all (unordered) pairs of distinct two digit integers (in the usual decimal notation). If a member $\{a, b\}$ of S is picked at random, what is the probability that $a + b$ is even? **44/89**

(4) For n a positive integer, let $f_n(x)$ be the continuous function $1/(1 + nx)$ with domain the positive real numbers. Let $f(x)$ be the pointwise limit of the sequence $\{f_n(x)\}_{n \geq 1}$ of functions. On which of the following intervals is the convergence $f_n \to f$ uniform? Choose all the correct options: **(b), (c)**

(a) $(0, 1)$
(b) $(1, 2)$
(c) $(2, \infty)$
(d) None of the above.

(5) Put $\theta := \pi/2019$ and let $\mathbb{N}$ denote the set of positive integers. Which of the following subsets of the real line is compact? Choose all the correct options: **(a), (c)**

(a) $\{\frac{\sin n\theta}{n} \mid n \in \mathbb{N}\}$
(b) $\{\frac{\cos n\theta}{n} \mid n \in \mathbb{N}\}$
(c) $\{\frac{\tan n\theta}{n} \mid n \in \mathbb{N}\}$
(d) None of the above.

(6) The smallest (positive) integer with exactly 20 divisors (including 1 and itself) is: **240**
(E.g., 10 has exactly four divisors, namely, 1, 2, 5, and 10.)

(7) How many group homomorphisms are there from $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ to $\mathbb{Z}/18\mathbb{Z}$? **54**
Here $\mathbb{Z}/n\mathbb{Z}$ denotes the cyclic group of order n, and $A \times B$ the cartesian product of A and B.

(8) Let, for t a real number, $\lfloor t \rfloor$ denote the largest integer not larger than t.

$$\text{Compute} \quad \int_{0.75}^{100.5} f(t)\,dt \quad \text{for} \quad f(t) := t - \lfloor t \rfloor - \frac{1}{2}.$$

−1/32 or −0.03125

(9) Let M denote the real 6×6 matrix all of whose off-diagonal entries are -1 and all of whose diagonal entries are 5. List out the eigenvalues of M (each eigenvalue must be written as many times as its multiplicity):

6, 6, 6, 6, 6, 0

(10) Let S be the set of all 2×3 real matrices each of whose entries is 1, 0, or -1. (There are 3^6 matrices in S.) Recall that the column space of a matrix M in S is the subspace of $\mathbb{R}^2$ (the vector space of 2×1 real matrices) spanned by the three columns of M. For two elements M and M' in S, let us write $M \sim M'$ if M and M' have the same column space. Note that $\sim$ is an equivalence relation. How many equivalence classes are there in S? **6**

(11) Given that $f(x, y) = u(x, y) + i\,v(x, y)$ is an entire function of $z = x + i\,y$ such that $f(0) = -1$,
$\partial u/\partial x = (e^y + e^{-y})\cos x$, and $\partial u/\partial y = (e^y - e^{-y})\sin x$, what is $f(\pi/3)$? **$\sqrt{3} - 1$**

(12) Let $A := \mathbb{Z}/6\mathbb{Z}$ be the group of residue classes modulo 6 of integers. Let G be the group of bijections (as a set) of A, the multiplication being composition. Let H be the subgroup of G consisting of those bijections σ such that $\sigma(x+2) = \sigma(x)+2$ for all x in A. What is the index of H in G? $\boxed{\mathbf{40}}$

(13) Let $S := \{x \text{ an integer} \mid 99 < x < 1000, x \equiv 8 \bmod 20, \text{ and } x \equiv 3 \bmod 15\}$. The sum of the elements of S is: $\boxed{\mathbf{7920}}$.

(14) Let A be 4×5 real matrix. Consider the system $A\mathbf{x} = \mathbf{b}$ of linear equations where $\mathbf{x}$ is a 5×1 column matrix of indeterminates and $\mathbf{b}$ is some fixed 4×1 column matrix with real entries. Given that

- A is row equivalent to the matrix R below (which means that the rows of A are all linear combinations of the rows of R and vice versa), and
- $\mathbf{c}$ and $\mathbf{d}$ below are both solutions to $A\mathbf{x} = \mathbf{b}$,

what is the value of y? $\boxed{\mathbf{7}}$

$$R = \begin{pmatrix} 1 & -2 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} y \\ 3 \\ 4 \\ 5 \\ 5 \end{pmatrix}$$

(15) Let n be the least positive integer such that $\sum_{2 \le k \le n} \frac{1}{k} \ge 5$. Choose the correct option: $\boxed{\mathbf{(c)}}$

(a) $n \le 32$
(b) $32 < n \le 96$
(c) $96 < n \le 729$
(d) $729 < n$

(16) What are the maximum and minimum values in the region $\{(x,y) \mid x^2 + y^2 \le 1, x + y \le 0\}$ of the function $f(x,y) = x^2 + y$? maximum = $\boxed{(\mathbf{1}+\sqrt{\mathbf{2}})/\mathbf{2} \text{ or } \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}}$ minimum = $\boxed{\mathbf{-1}}$.

(17) Let A be a real 2×2 matrix such that $A^6 = I$ (where I denotes the identity 2×2 matrix). The total number of possibilities for the characteristic polynomial of A is: $\boxed{\mathbf{5}}$

(18) The shortest distance from the origin in $\mathbb{R}^3$ to the surface $z^2 - (x-1)(y-1) = 2$ is $\boxed{\sqrt{\mathbf{24}}/\mathbf{3} \text{ or } \sqrt{\mathbf{8/3}}}$.

(19) For r a positive real number let $f(r) := \int_{C_r} \frac{\sin z}{z} dz$, where C_r is the contour $re^{i\theta}$, $0 \le \theta \le \pi$. What is $\lim_{r\to 0} \frac{f(r)}{r}$? $\boxed{\mathbf{-2}}$

(20) A degree 3 polynomial $f(x)$ with real coefficients satisfies $f(1) = 2$, $f'(2) = 2$, $f''(2) = 2$, and $f'''(2) = 12$, where $f'(x)$, $f''(x)$, and $f'''(x)$ are the first, second, and third derivatives of $f(x)$ respectively. What is $f(2)$? $\boxed{\mathbf{5}}$

(21) Consider the function $f(z) = z + 2z^2 + 3z^3 + \cdots = \sum_{n\ge 0} nz^n$ defined on the open disk $\{z \mid |z| < 1\}$. Choose the correct option: $\boxed{\mathbf{(b)}}$

(a) f is not injective but attains every complex value at least once.
(b) f is injective but does not attain every complex value.
(c) f is injective and attains every complex value.
(d) None of the above.

(22) Find the area of the region $\{(x,y) \mid 0 \le x, 0 \le y, x^{2/3} + y^{2/3} \le 1\}$. $\boxed{\mathbf{3\pi/32}}$

(23) The number of solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers, of the equation $x^2 + 16 = y^2$ is **6**.

(24) Evaluate $\lim_{n\to\infty} (1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3})^n$: **1/e**

(25) Let $\mathbf{k}$ be the field with exactly 7 elements. Let $\mathfrak{M}$ be the set of all 2×2 matrices with entries in $\mathbf{k}$. How many elements of $\mathfrak{M}$ are similar to the following matrix? **56**

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(26) For the function $f(x)$ on the real line $\mathbb{R}$ defined below, which of the following statements about f is true? Choose all the correct options: **(b), (c)**

$$f(x) := \sum_{n\geq 1} \frac{\sin(x/n)}{n}$$

(a) f is continuous but not uniformly continuous on $\mathbb{R}$.
(b) f is uniformly continuous on $\mathbb{R}$.
(c) f is differentiable on $\mathbb{R}$.
(d) f is an increasing function on $\mathbb{R}$.

(27) Let f be a continuous function from the real line $\mathbb{R}$ to the closed interval $[1,3]$ such that:

- $f^{-1}(1)$ and $f^{-1}(3)$ are singletons, and
- $f^{-1}(x)$ consists of exactly two real numbers for every x in $(1,2) \cup (2,3)$.

Which of the following can be the cardinality of $f^{-1}(2)$? Choose all the correct options: **(a), (d)**

(a) 1
(b) 2
(c) countable infinity
(d) uncountable infinity

(28) Which of the following functions is uniformly continuous on the given domain? Choose all the correct options: **(a), (d)**

(a) $1/x^2$ on $[1, \infty)$.
(b) $1/x$ on $(0, \infty)$.
(c) $x \sin x$ on the real line $\mathbb{R}$.
(d) $\tan^{-1} x$ on the real line $\mathbb{R}$.

(29) *A* 3×3 real symmetric matrix M admits $(1,2,3)^{\text{transpose}}$ and $(1,1,-1)^{\text{transpose}}$ as eigenvectors. The transpose of which of the following is *surely* an eigenvector for M? Choose all the correct options: **(d)**

(a) $(1,-1,0)$
(b) $(-5,1,1)$
(c) $(3,2,1)$
(d) none of the above

(30) An insect is moving along the curve $r = |\cos\theta|$ such that $\theta = \pi t/6$, where t is time measured in seconds. What is the distance travelled by the insect in the time interval between $t = 1$ and $t = 2$? **$\pi/6$**