NBHM MASTERS SCHOLARSHIP WRITTEN TEST BOOKLET

19 OCTOBER 2019

TO BE RETURNED AT THE END OF THE EXAM

- Enter your roll number here: ‡
- Enter your name in block letters here: ‡
- This "test booklet" consists of 3 pages of questions and this "cover page" (total 4 pages).
- There are 30 questions. Answer all of them: there is no "choice".
- TIME ALLOWED: 150 minutes (two and a half hours).
- ABOUT THE QUESTIONS:
  - They are of two types: (a) SHORT ANSWER QUESTION OR SAQ, in which you are asked to come up with the answer yourself (no options are provided), and (b) MULTIPLE CHOICE QUESTION OR MCQ, in which you are asked to choose from among four options that are listed.
  - The questions are arranged randomly. They are not sorted by topic, or by type, or by level of difficulty.
- ABOUT THE ANSWERS: In response to either type of question (SAQ/MCQ), you are required to fill in only your final answer in the box provided for it. This box has the following appearance: ‡ . There is no need or space provided to indicate the steps taken to reach the final answer.

  Only your final answer, written legibly and unambiguously in the box, is considered for marking.

- MARKING: Each question carries 4 marks.
  - There is no penalty for incorrect answers. In other words, there is no negative marking.
  - On any SAQ, you get all 4 marks if your answer is correct and 0 marks otherwise. In contrast, on some of MCQs, it is possible to get "partial credit": that is, you could also get 1, 2, or 3 marks. The exact formula to determine the marks is a bit cumbersome,[†] but the main thing to keep in mind is this:

    If you choose any of the incorrect options, then you get no marks at all.

- NOTATION AND TERMINOLOGY: The questions make free use of standard notation and terminology.
  - You too are allowed their use to express your answers.
  - Numbers are denoted in the standard decimal notation. In particular, this clarifies how a phrase such as "two digit integers" is to be understood. The standard symbols are to be interpreted in the context of the question (e.g., 1 could stand for the multiplicative identity of a field).
  - The final answer to an SAQ is almost always a number, and in many cases an integer. The numbers in the answers that are entered in the boxes should be in simplified form: e.g., 48 and not 6×8. Answers of the form $e + \sqrt{2}$ and $2\pi/19$ are acceptable. Both $3/4$ and 0.75 are acceptable.
- DEVICES: Use of plain pencils, pens, and erasers is allowed. Mobile phones are prohibited. So are calculators. More generally, any device (e.g. a smart watch) that can be used for communication or calculation or storage is prohibited. Invigilators have the right to impound any device that arouses their suspicion.
- ROUGH WORK: For rough work, you may use the sheets separately provided. You must:
  - Write your name and roll number on each such sheet.
  - Return all these sheets to the invigilator along with this test booklet at the end of the test.

[†] Denoting by c the number of correct options chosen, by i the number of incorrect options chosen, and by C the total number of correct options, the marks obtained is given by the following formula: 0 if $i > 0$, and the nearest integer to $4c/C$ otherwise. It is guaranteed that there is at least one correct option (possibly "none of the above"): in other words, $C \geq 1$, so $4c/C$ makes sense.

For example, if you choose two of the options and both happen to be correct while there are totally three correct options, then you earn 3 marks (3 being the closest integer to $8/3$).

(1) Let A_n be the real $n \times n$ matrix ($n \geq 2$) whose entry in position (i, j) is $i - j$. What is the rank of A_n as a function of n? | ‡ |

(2) Let $p(x)$ be the polynomial left as remainder when $x^{2019} - 1$ is divided by $x^6 + 1$. What is the remainder left when $p(x)$ is divided by $x - 3$? | ‡ |

(3) Let S be the set of all (unordered) pairs of distinct two digit integers (in the usual decimal notation). If a member $\{a, b\}$ of S is picked at random, what is the probability that $a + b$ is even? | ‡ |

(4) For n a positive integer, let $f_n(x)$ be the continuous function $1/(1+nx)$ with domain the positive real numbers. Let $f(x)$ be the pointwise limit of the sequence $\{f_n(x)\}_{n\geq 1}$ of functions. On which of the following intervals is the convergence $f_n \to f$ uniform? Choose all the correct options: | ‡ |

(a) $(0,1)$
(b) $(1,2)$
(c) $(2,\infty)$
(d) None of the above.

(5) Put $\theta := \pi/2019$ and let $\mathbb{N}$ denote the set of positive integers. Which of the following subsets of the real line is compact? Choose all the correct options: | ‡ |

(a) $\{\frac{\sin n\theta}{n} \mid n \in \mathbb{N}\}$
(b) $\{\frac{\cos n\theta}{n} \mid n \in \mathbb{N}\}$
(c) $\{\frac{\tan n\theta}{n} \mid n \in \mathbb{N}\}$
(d) None of the above.

(6) The smallest (positive) integer with exactly 20 divisors (including 1 and itself) is: | ‡ |
(E.g., 10 has exactly four divisors, namely, 1, 2, 5, and 10.)

(7) How many group homomorphisms are there from $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ to $\mathbb{Z}/18\mathbb{Z}$? | ‡ |
Here $\mathbb{Z}/n\mathbb{Z}$ denotes the cyclic group of order n, and $A \times B$ the cartesian product of A and B.

(8) Let, for t a real number, $\lfloor t \rfloor$ denote the largest integer not larger than t.

$$\text{Compute} \quad \int_{0.75}^{100.5} f(t)\,dt \qquad \text{for} \quad f(t) := t - \lfloor t \rfloor - \frac{1}{2}.$$

| ‡ |

(9) Let M denote the real 6×6 matrix all of whose off-diagonal entries are -1 and all of whose diagonal entries are 5. List out the eigenvalues of M (each eigenvalue must be written as many times as its multiplicity):

| ‡ |

(10) Let S be the set of all 2×3 real matrices each of whose entries is 1, 0, or -1. (There are 3^6 matrices in S.) Recall that the column space of a matrix M in S is the subspace of $\mathbb{R}^2$ (the vector space of 2×1 real matrices) spanned by the three columns of M. For two elements M and M' in S, let us write $M \sim M'$ if M and M' have the same column space. Note that $\sim$ is an equivalence relation. How many equivalence classes are there in S? | ‡ |

(11) Given that $f(x, y) = u(x, y) + iv(x, y)$ is an entire function of $z = x + iy$ such that $f(0) = -1$, $\partial u/\partial x = (e^y + e^{-y})\cos x$, and $\partial u/\partial y = (e^y - e^{-y})\sin x$, what is $f(\pi/3)$? | ‡ |

(12) Let $A := \mathbb{Z}/6\mathbb{Z}$ be the group of residue classes modulo 6 of integers. Let G be the group of bijections (as a set) of A, the multiplication being composition. Let H be the subgroup of G consisting of those bijections σ such that $\sigma(x+2) = \sigma(x)+2$ for all x in A. What is the index of H in G? ‡

(13) Let $S := \{x \text{ an integer} \mid 99 < x < 1000, x \equiv 8 \bmod 20, \text{ and } x \equiv 3 \bmod 15\}$. The sum of the elements of S is: ‡ .

(14) Let A be 4×5 real matrix. Consider the system $A\mathbf{x} = \mathbf{b}$ of linear equations where $\mathbf{x}$ is a 5×1 column matrix of indeterminates and $\mathbf{b}$ is some fixed 4×1 column matrix with real entries. Given that

- A is row equivalent to the matrix R below (which means that the rows of A are all linear combinations of the rows of R and vice versa), and
- $\mathbf{c}$ and $\mathbf{d}$ below are both solutions to $A\mathbf{x} = \mathbf{b}$,

what is the value of y? ‡

$$R = \begin{pmatrix} 1 & -2 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} y \\ 3 \\ 4 \\ 5 \\ 5 \end{pmatrix}$$

(15) Let n be the least positive integer such that $\sum_{2 \le k \le n} \frac{1}{k} \ge 5$. Choose the correct option: ‡

(a) $n \le 32$
(b) $32 < n \le 96$
(c) $96 < n \le 729$
(d) $729 < n$

(16) What are the maximum and minimum values in the region $\{(x, y) \mid x^2 + y^2 \le 1, x + y \le 0\}$ of the function $f(x, y) = x^2 + y$? maximum = ‡ , minimum = ‡ .

(17) Let A be a real 2×2 matrix such that $A^6 = I$ (where I denotes the identity 2×2 matrix). The total number of possibilities for the characteristic polynomial of A is: ‡

(18) Find the shortest distance between the origin in $\mathbb{R}^3$ and the surface $z^2 - (x-1)(y-1) = 2$. ‡

(19) For r a positive real number let $f(r) := \int_{C_r} \frac{\sin z}{z} dz$, where C_r is the contour $re^{i\theta}$, $0 \le \theta \le \pi$. What is $\lim_{r \to 0} \frac{f(r)}{r}$? ‡

(20) A degree 3 polynomial $f(x)$ with real coefficients satisfies $f(1) = 2$, $f'(2) = 2$, $f''(2) = 2$, and $f'''(2) = 12$, where $f'(x)$, $f''(x)$, and $f'''(x)$ are the first, second, and third derivatives of $f(x)$ respectively. What is $f(2)$? ‡

(21) Consider the function $f(z) = z + 2z^2 + 3z^3 + \cdots = \sum_{n \ge 0} nz^n$ defined on the open disk $\{z \mid |z| < 1\}$. Choose the correct option: ‡

(a) f is not injective but attains every complex value at least once.
(b) f is injective but does not attain every complex value.
(c) f is injective and attains every complex value.
(d) None of the above.

(22) Find the area of the region $\{(x, y) \mid 0 \le x, 0 \le y, x^{2/3} + y^{2/3} \le 1\}$. ‡

(23) The number of solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers, of the equation $x^2 + 16 = y^2$ is ‡ .

(24) Evaluate $\lim_{n\to\infty} (1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3})^n$: ‡

(25) Let $\mathbf{k}$ be the field with exactly 7 elements. Let $\mathfrak{M}$ be the set of all 2×2 matrices with entries in $\mathbf{k}$. How many elements of $\mathfrak{M}$ are similar to the following matrix? ‡

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(26) For the function $f(x)$ on the real line $\mathbb{R}$ defined below, which of the following statements about f is true? Choose all the correct options: ‡

$$f(x) := \sum_{n\geq 1} \frac{\sin(x/n)}{n}$$

(a) f is continuous but not uniformly continuous on $\mathbb{R}$.
(b) f is uniformly continuous on $\mathbb{R}$.
(c) f is differentiable on $\mathbb{R}$.
(d) f is an increasing function on $\mathbb{R}$.

(27) Let f be a continuous function from the real line $\mathbb{R}$ to the closed interval $[1,3]$ such that:

- $f^{-1}(1)$ and $f^{-1}(3)$ are singletons, and
- $f^{-1}(x)$ consists of exactly two real numbers for every x in $(1,2) \cup (2,3)$.

Which of the following can be the cardinality of $f^{-1}(2)$? Choose all the correct options: ‡

(a) 1
(b) 2
(c) countable infinity
(d) uncountable infinity

(28) Which of the following functions is uniformly continuous on the given domain? Choose all the correct options: ‡

(a) $1/x^2$ on $[1,\infty)$.
(b) $1/x$ on $(0,\infty)$.
(c) $x \sin x$ on the real line $\mathbb{R}$.
(d) $\tan^{-1} x$ on the real line $\mathbb{R}$.

(29) *A* 3×3 real symmetric matrix M admits $(1,2,3)^{\text{transpose}}$ and $(1,1,-1)^{\text{transpose}}$ as eigenvectors. The transpose of which of the following is *surely* an eigenvector for M? Choose all the correct options: ‡

(a) $(1,-1,0)$
(b) $(-5,1,1)$
(c) $(3,2,1)$
(d) none of the above

(30) An insect is moving along the curve $r = |\cos\theta|$ such that $\theta = \pi t/6$, where t is time measured in seconds. What is the distance travelled by the insect in the time interval between $t = 1$ and $t = 2$? ‡