NATIONAL BOARD FOR HIGHER MATHEMATICS

MASTER'S AND DOCTORAL SCHOLARSHIP WRITTEN TEST

SATURDAY, 24TH JANUARY 2026, 10:30 A.M. TO 1:30 P.M.

For official use only: ‡ **M-Score** = ‡ **D-Score** =

‡ $\boldsymbol{m_1}$ = ‡ $\boldsymbol{m_2}$ = ‡ $\boldsymbol{d_1}$ = ‡ $\boldsymbol{d_2}$ =

Instructions to Candidates

- APPLICATION NUMBER: ‡ ROLL NUMBER: ‡
- NAME (in full in BLOCK letters): ‡
- SCHOLARSHIP TYPE (circle one and only one of the three options): Master's / Doctoral / Both
- This test has 40 problems distributed over four sections. Each problem carries 4 marks. SOLVE AS MANY AS YOU CAN.
- *This test booklet must have* 8 *pages* (this cover page with instructions and 7 pages of problems). *Make sure right at the outset that you have all* 8 *pages and all* 40 *problems in your booklet.*
- MODE OF ANSWERING: Enter only your final answer in the *answer box* provided. It is neither necessary nor is there provision of space to indicate the steps taken to reach the final answer.

  ‡ **Only the final answer, written legibly and unambiguously in the answer box, will be marked.**
- MARKING SCHEME: The marking scheme for each section is described at the beginning of that section. ***There is negative marking for the* TRUE OR FALSE TYPE *problems.*** There is no negative marking for the SHORT ANSWER TYPE problems.
- M-SCORE AND D-SCORE: If m_1, m_2, d_1, and d_2 denote your "raw" scores (net of any negative marks) in the four sections of this test respectively, your M-Score will be $m_1+m_2+d_1+d_2$; and your D-Score will be $m_1 + m_2 + 3(d_1 + d_2)/2$. The maximum possible M-Score is 160 and the maximum possible D-Score is 190.
- NOTATION AND TERMINOLOGY: The problems make use of standard notation and terminology. You too are allowed the use of standard notation. For example, answers of the form $e + \sqrt{2}$ and $2\pi/19$ are acceptable; both $3/4$ and 0.75 are acceptable.
- DEVICES: Use of plain pencils, pens, and erasers is allowed. Mobile phones and calculators are not allowed inside the exam hall. More generally, any device (e.g., a smart watch) that can be used for communication or calculation or storage is prohibited. Invigilators have the right to impound (for the duration of the test) any device that arouses their suspicion.
- ROUGH WORK: For rough work, you may use the sheets separately provided, in addition to the blank pages in the test booklet. You must:
  - Write your name and roll number on each such sheet (or set of sheets if stapled).
  - Return all these sheets to the invigilator along with this test booklet at the end of the test.
- Do **not** seek clarification from the invigilator or anyone else about any problem. In the unlikely event that there is a mistake in any problem, appropriate allowances will be made while marking.

Part M, Section 1: QUESTIONS 1 TO 20, (*Short Answer Type*, MAXIMUM RAW SCORE: 80)

Marking Scheme: $+4$ for every complete and correct response; no negative marking: 0 if either the question is left unanswered or the response is incorrect or the response is incomplete.

(1) A number is selected at random from the set $\{1, 2, \ldots, 63\}$. What is the probability that it is coprime to 63?

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(2) A professor graded 5 exam papers and returned them randomly to 5 students. Assume that these are the same students whose exam papers were graded. What is the number of ways in which no student receives their own exam paper?

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(3) Let $\mathbb{F}_7$ denote the field with 7 elements. The cardinality of the ring

$$\frac{\mathbb{F}_7[x]}{\langle x^3 + 2x^2 + 6x + 5,\ x^3 + 3x^2 + 6x + 4\rangle}$$

is equal to

‡

(4) Describe all those positive integers n such that the decimal expansion of the rational number m/n—where m is a positive integer such that m and n are coprime—is eventually constant (like $123.425555\ldots$). (**Note.** To distinguish between the set of **positive** integers and the set of natural numbers, use $\mathbb{Z}_+$ to denote the former and $\mathbb{N}$ to denote the latter, *if* needed.)

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(5) You are given that the equation $3^x = x^9$ has two real solutions. Determine whether it has any integer solutions and, if it does, then give all integer solutions. If there are no integer solutions, then write 'no integer solutions' in the answer box.

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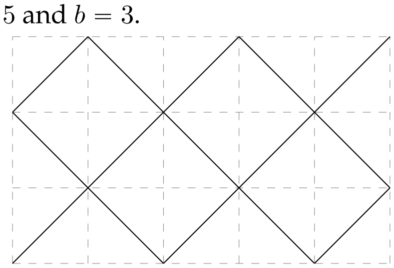
(6) Usually we play carrom on a square board. However, there are games (like billiards) played on a rectangular board. Assume you are playing carrom on a rectangular board (with holes at its four corners) of length l and breadth b (so, $l > b$). Further, let l and b be two coprime positive integers. If we shoot the striker from one of the corners at a 45 degree angle, how many times will the striker bounce off an edge of the board before it falls into a hole? (The figure below shows the striker's trajectory for a board with $l = 5$ and $b = 3$.



In this example, the striker bounces 6 times.)

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(7) Let $f : [0,1] \to \mathbb{R}$ be twice differentiable and $f'(0) = f'(1) = 0$. Let $A = |f(1) - f(0)|$ and $B = \sup\{|f''(b) - f''(a)| : a, b \in (0,1)\}$. Which of the following is true for any f with the stated properties? (**Note.** It suffices to state just the letter corresponding to a statement.)

(a) $B \leqslant 4A$.

(b) $B \geqslant 8A$.

(c) There does not exist any relation between A and B.

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(8) Let $j \geqslant k$ be positive integers and let R_k^j denote the number of distinguishable ways in which j distinct balls can be put into k different but indistinguishable urns in such a way that each urn receives at least one ball. Now, let $n > m$ be two positive integers. If we know that

$$R_{m+1}^n = A\left[R_m^{n-1} + \frac{(n-1)!}{2!(n-2)!}R_m^{n-2} + \cdots + \frac{(n-1)!}{m!(n-m)!}R_m^m\right],$$

then find the value of A.

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(9) Let n be an integer $\geqslant 2$, let $P, Q \notin \{0, I_n\}$ be two linear projection maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, where I_n is the identity on $\mathbb{R}^n$, and suppose $Q - P$ is also a projection. Find the value of $\mathrm{trace}(QP + PQ)$ in terms of P.

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(10) Identify explicitly the set $\{\,\mathrm{trace}(A) : A \in M_3(\mathbb{R}),\ A^{\mathsf{T}}A = I_3,\ \det(A) = 1,\ A \neq I_3\,\}$, where $M_3(\mathbb{R})$ is the set of all 3×3 matrices with real entries and I_3 is the 3×3 identity matrix.

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(11) Let A be a 4×4 complex matrix whose minimal polynomial is $x^2 - 1$. The value of $\mathrm{rank}(A + I_4) + \mathrm{rank}(A - I_4)$, where I_4 is the 4×4 identity matrix, is equal to

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(12) Let $\mathbb{D}$ denote the open disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. Let f be any function that is holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ such that $|f(z)| > 2$ for each $z \in \partial\mathbb{D}$ and $f(0) = 1$. Which of the following is/are true? (**Note.** It suffices to state just the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) f is always non-vanishing in $\mathbb{D}$.

(b) f is never non-vanishing in $\mathbb{D}$.

(c) f always has a simple zero in $\mathbb{D}$.

(d) It is possible for f to have a zero of multiplicity at least 2 in $\mathbb{D}$.

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(13) Two series of the form $\sum_{n=-\infty}^{\infty} a_n z^n$ and $\sum_{n=-\infty}^{\infty} b_n z^n$, where $a_n, b_n \in \mathbb{C}$ for all n, are said to be *distinct* if $a_n \neq b_n$ for at least one $n \in \mathbb{Z}$. You are asked to expand the function

$$f(z) = \frac{1}{(z+1)^2} + \frac{1}{(z-2)^3}$$

in a series of the above form that is absolutely convergent at each point z in a specified domain, where this domain is preserved by any rotation about $0 \in \mathbb{C}$. In how many such distinct series can f be expanded?

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(14) Let $P = \{a_1, a_2, \ldots, a_n\}$ be a permutation of $\{1, 2, \ldots, n\}$. For $n \geqslant 3$, we say that P is a zig-zag if $a_1 > a_2 < a_3 > a_4 < \cdots$. Let Z_n denote the number of zig-zags with n elements. For instance, $Z_3 = 2$, given by the zig-zags $2 > 1 < 3$ and $3 > 1 < 2$. Given that $Z_3 = 2, Z_4 = 5$ and $Z_5 = 16$, what is the value of Z_6?

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(15) Let A be a 3×3 matrix with minimal polynomial $x^3 - 2x^2 + x - 2$. Let f be a function on 3×3 matrices given by $f(M) = M^4 - M^2 + I$. What is the determinant of $f(A)$?

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(16) Let A_6 denote the group of all even permutations of 6 symbols. How many elements of order 5 are there in A_6?

‡

(17) Let $(a_n)_{n \geqslant 1}$ be the sequence of real numbers satisfying $a_1 = 1$ and $a_{n+1} = a_n + (1/a_n)$ for all $n \geqslant 1$. Determine the value of $\lfloor a_{100} \rfloor$, the greatest integer $\leqslant a_{100}$.

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(18) For which of the following functions $f : (-1, 1) \to \mathbb{R}$ does there exist a function $F : (-1, 1) \to \mathbb{R}$ such that F is differentiable on $(-1, 1)$ and $F'(x) = f(x)$ for all $x \in (-1, 1)$? (**Note.** It suffices to state just the letter corresponding to an option. If more than one option is valid, then all such must be identified.)

(a) $f(x) = |x|$.

(b) $f(x) = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x \geqslant 0. \end{cases}$

(c) $f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$

(d) $f(x) = \begin{cases} \cos(1/x), & \text{if } x \neq 0, \\ 1, & \text{otherwise.} \end{cases}$

‡

(19) For which real numbers x do the vectors: $(1, x, x, x), (x, 1, x, x), (x, x, 1, x), (x, x, x, 1)$ **not form** a basis of $\mathbb{R}^4$?

‡

(20) Given $x \in \mathbb{R}$, let $\sqrt[3]{x}$ denote the unique **real** number y such that $y^3 = x$ (i.e., the unique real cube root of x). Let $f : [-1, 1] \to \mathbb{R}$ be an even function that satisfies

$$f(x) = f(\sqrt[3]{x}) \quad \forall x \in [-1, 1].$$

Assume that f is continuous at $x = 1$. What can you say about f? Give the most specific answer possible from the information given.

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Part M, Section 2: QUESTIONS 21 TO 25, (*True or False* TYPE, MAXIMUM RAW SCORE: 20)

This section has 5 questions. Each question has a pair of assertions. For each assertion, you are required to determine its truth value and write either "True" or "False" in the corresponding answer box, as the case may be.

Marking scheme: $+2$ for each correct response, but *there is negative marking*: -2 for each incorrect response; 0 if the answer box is left empty.

(21) Consider the ODE $\ddot{x} + (\dot{x})^2 = 0$ in $(0, \infty)$.

(a) It has two linearly independent solutions in $(0, \infty)$. ‡

(b) It has at most two linearly independent solutions in $(0, \infty)$. ‡

(22) Given a sequence $(f_n)_{n \geqslant 1}$ of functions $f_n : [0, 1] \to [-1, 1]$, define the function h on $[0, 1]$ by

$$h(x) = \sup\{f_1(x),\ f_2(x),\ f_3(x), \ldots\}.$$

(a) If every f_n is continuous, then so is h. ‡

(b) If every f_n is non-negative and differentiable, then so is h. ‡

(23) Let $\mathbb{F}$ be a field and fix an integer $n \geqslant 2$. Let A and B be $n \times n$ matrices with entries in $\mathbb{F}$. Let I_n denote the $n \times n$ identity matrix.

(a) Whenever $(I_n - AB)$ is invertible, $(I_n - BA)$ is necessarily invertible. ‡

(b) Whenever $(I_n - AB)$ and $(I_n - BA)$ are both invertible, $(I_n - BA)^{-1}$ can be expressed as an $\mathbb{F}$-linear combination of monomials involving I_n, A, B, and $(I_n - AB)^{-1}$. ‡

(24) Consider the real line $\mathbb{R}$ equipped with the standard Euclidean metric d and let $\mathbb{Z}_+$ denote the set of all positive integers.

(a) Let S be the subset of $\mathbb{R}$ defined by $S = \bigcup_{n \in \mathbb{Z}_+} I_n$ where $I_n = [n,\, n + n^{-2}]$. The function $f : S \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous. ‡

(b) There exists a metric D on $\mathbb{R}$ such that the Cauchy sequences of $(\mathbb{R},\, d)$ and $(\mathbb{R},\, D)$ are identical and the identity map from $(\mathbb{R},\, d)$ to $(\mathbb{R},\, D)$ is continuous but not uniformly continuous. ‡

(25) Let d denote the Euclidean distance on $\mathbb{R}^3$. Assume $f : \mathbb{R}^3 \to \mathbb{R}^3$ is any isometry of the metric space $(\mathbb{R}^3, d)$ that is not the identity map.

(a) If f is an orientation preserving isometry that fixes the origin, then the set $\{x \in \mathbb{R}^3 | f(x) = x\}$ is a one dimensional subspace of $\mathbb{R}^3$. ‡

(b) A composition of reflections with respect to at most 4 distinct affine planes in $\mathbb{R}^3$ is sufficient to express f. ‡

Part D, Section 1: QUESTIONS 26 TO 35, *Short Answer Type*, MAXIMUM RAW SCORE: 40
YOUR RAW SCORE ON THIS SECTION COUNTS 1.5 TIMES TOWARDS YOUR D-SCORE.

Marking Scheme: $+4$ for every complete and correct response; no negative marking: 0 if either the question is left unanswered or the response is incorrect or the response is incomplete.

(26) Let $K/\mathbb{Q}$ be a Galois extension with Galois group D_4, the dihedral group with 8 elements. The number of intermediate fields $\mathbb{Q} \subsetneq L \subsetneq K$ such that $[L : \mathbb{Q}] = 4$ is equal to

‡

(27) State the value of

$$\sum_{k=0}^{2026} (-1)^k \binom{2026}{k} (2026 - k)^{2026}.$$

‡

(28) Let $\mathbb{F}_p$ denote the field with p elements. Which of the following rings are fields? (**Note.** It suffices to state just the letter corresponding to an option. If more than one option is valid, then all such must be identified.)

(a) $\mathbb{F}_7[x, y]/\langle x^2 + 1,\ y + x\rangle$
(b) $\mathbb{F}_7[x]/\langle x^2 + 1\rangle$
(c) $\mathbb{F}_{11}[x]/\langle x^3 + 2x + 5\rangle$
(d) $\mathbb{F}_{11}[x, y]/\langle x^2 + 1,\ y^{11} - y - 2x\rangle$

‡

(29) What is the largest positive integer k such that 5^N and 5^{N+k} have the same last k decimal digits for all N sufficiently large?

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(30) Let $\mathbb{D}$ denote the open disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. You are given a sequence $(x_n)_{n\geqslant 1}$ in $(0, 1)$ such that $\lim_{n\to\infty} x_n = 0$ and a sequence of non-constant functions $(f_n)_{n\geqslant 1}$ such that each f_n is continuous on $\overline{\mathbb{D}}$, holomorphic on $\mathbb{D}$, and maps $\overline{\mathbb{D}}$ into $\{x + iy \in \overline{\mathbb{D}} : x \geqslant 0\}$. For any positive integer p, let $(\cdot)^{1/p}$ denote the **principal** holomorphic branch of the p-th root. Suppose $(p_n)_{n\geqslant 1}$ is an increasing sequence of positive integers such that

$$\lim_{n\to\infty} \big(f_n(0)\big)^{1/p_n} \text{ exists and is not equal to } 1.$$

Which of the following is/are true? (**Note.** It suffices to state just the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) There always exists a sequence of indices $(n_k)_{k\geqslant 1}$ such that $\lim_{k\to\infty} \big(f_{n_k}(x_{n_k})\big)^{1/p_{n_k}} = 1$.
(b) There always exists a sequence of indices $(n_k)_{k\geqslant 1}$ such that $\lim_{k\to\infty} f_{n_k}(x_{n_k})$ exists.
(c) The sequence $(f_n(x_n))_{n\geqslant 1}$ is never convergent.
(d) For any $(x_n)_{n\geqslant 1}$ and $(f_n)_{n\geqslant 1}$ as above, there exist a sequence $(p_n)_{n\geqslant 1}$ (with the above-mentioned properties) and a sequence of indices $(n_k)_{k\geqslant 1}$ such that $\lim_{k\to\infty} \big(f_{n_k}(x_{n_k})\big)^{1/p_{n_k}} = 1$.

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(31) Let $n \geqslant 2$. Let the minimal polynomial of $A \in M_n(\mathbb{C})$, an $n \times n$ matrix with complex entries, be

$$m_A(x) = (x - \lambda)^n$$

for some $\lambda \in \mathbb{C}\backslash\{0\}$. Describe, in terms of A, all $B \in M_n(\mathbb{C})$ that commute with A.

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(32) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Let A be the set consisting of all those $n \in \mathbb{Z}_+$ (i.e., all those positive integers n) such that there exist $a_n, b_n \in [0,1]$ satisfying $b_n - a_n = 1/n$ and $f(b_n) = f(a_n)$. Describe the set A.

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(33) Write down the set of all real numbers a for which the integral

$$\int_0^\infty \frac{|\cos(2x) - \cos(x)|}{x^a}\, dx.$$

exists (i.e., the integral is finite).

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(34) Suppose that an urn contains n_1 red balls and n_2 black balls, where $n_1, n_2 > 0$. A fair 6-faced die is thrown and the number seen on its upturned face, say N, is noted. Next, N balls are drawn from the urn **with** replacement. What is the probability that an equal number of red and black balls are drawn? (**Note.** It suffices to state just the letter corresponding to the expression chosen.)

(a) $\frac{1}{6}\frac{n_1 n_2}{(n_1+n_2)^4} + \frac{1}{6}\frac{n_1^2 n_2^2}{(n_1+n_2)^4} + \frac{1}{6}\frac{n_1^3 n_2^3}{(n_1+n_2)^4}$

(b) $\frac{1}{3}\frac{n_1 n_2}{(n_1+n_2)^2} + \frac{n_1^2 n_2^2}{(n_1+n_2)^4} + \frac{10}{3}\frac{n_1^3 n_2^3}{(n_1+n_2)^6}$

(c) $\frac{1}{6}\frac{n_1 n_2}{(n_1+n_2)^2} + \frac{1}{6}\frac{n_1^2 n_2^2}{(n_1+n_2)^4} + \frac{1}{6}\frac{n_1^3 n_2^3}{(n_1+n_2)^6}$

(d) $\frac{n_1 n_2}{(n_1+n_2)^2} + \frac{n_1^2 n_2^2}{(n_1+n_2)^4} + \frac{10}{3}\frac{n_1^3 n_2^3}{(n_1+n_2)^6}$

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(35) Let $\mathbb{F}_3$ denote the field with 3 elements, let $M_3(\mathbb{F}_3)$ denote the set of all 3 matrices with entries in $\mathbb{F}_3$, and let

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The number of matrices $B \in M_3(\mathbb{F}_3)$ that are similar to J is equal to

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Part D, Section 2: QUESTIONS 36 TO 40, *True or False* TYPE, MAXIMUM RAW SCORE: 20
YOUR RAW SCORE ON THIS SECTION COUNTS 1.5 TIMES TOWARDS YOUR D-SCORE.

This section has 5 questions. Each question has a pair of assertions. For each assertion, you are required to determine its truth value and respond by writing either "True" or "False" in the corresponding answer box, as the case may be.

Marking scheme: $+2$ for each correct response, but *there is negative marking*: -2 for each incorrect response; 0 if the answer box is left empty.

(36) A Banach space $(X, \|\cdot\|)$ is said to be strictly convex if $\|x\| = 1 = \|y\|$, $x \neq y$, implies $\|(x+y)/2\| < 1$.

(a) $(\mathbb{R}^n, \|\cdot\|_1)$, $n \geqslant 2$, where $\|x\|_1 = |x_1| + \cdots + |x_n|$, is not strictly convex. ‡

(b) For every nonzero, bounded linear functional f on a strictly convex Banach space $(X, \|\cdot\|)$, there is at most one $x \in X$ satisfying $\|x\| = 1$ and $f(x) = \|f\|_{op}$. Here $\|\cdot\|_{op}$ denotes the operator norm. ‡

(37) For a ring R, denote as $R[x]$ the polynomial ring over R in the variable x.

(a) The polynomial $x^5 + 5x^2 + 10$ is irreducible in $\mathbb{Q}[x]$. ‡

(b) $x^5 + 2x + 1$ is irreducible in $\mathbb{F}_3[x]$. ‡

(38) Consider linear Diophantine equation of the form $ax + by = c$, where $a, b, c \in \mathbb{Z}$ and the solutions (x, y) are restricted to positive integers.

(a) Given any pair of positive integers m and n, there exists a linear equation $ax + by = c$ such that the only solution in positive integers $(x, y) = (m, n)$. ‡

(b) For any positive integer k, there exists a linear equation $ax + by = c$ such that there are exactly k solutions in positive integers. ‡

(39) Consider the following subspace of $\mathbb{R}^2$

$$X = [0,1] \times \{0\} \bigcup \left(\bigcup_{r \in \mathbb{Q} \cap [0,1]} \{r\} \times [0, 1-r] \right).$$

(a) X deformation retracts to the point $(0, 1)$. ‡

(b) X deformation retracts to the point $(1, 0)$. ‡

(40) Let $f : [0, \infty) \to [0, \infty)$ be a non-negative continuous function, and suppose $f \not\equiv 0$. Define $S_f := \{t \in (0, \infty) : \int_0^\infty (f(x))^t \, dx < \infty\}$.

(a) The set S_f is connected for any f for which $S_f \neq \varnothing$. ‡

(b) There exists a function f with the above-mentioned properties such that S_f is a non-empty finite set. ‡