NATIONAL BOARD FOR HIGHER MATHEMATICS

MASTER'S AND DOCTORAL SCHOLARSHIP WRITTEN TEST

SATURDAY, 25TH JANUARY 2025, 10:30 A.M. TO 1:30 P.M.

For official use only: | ‡ **M-Score** = | ‡ **D-Score** =

‡ m_1 =	‡ m_2 =	‡ d_1 =	‡ d_2 =

Instructions to Candidates

- APPLICATION NUMBER: ‡ ROLL NUMBER: ‡
- NAME (in full in BLOCK letters): ‡
- SCHOLARSHIP TYPE (circle one and only one of the three options): Master's / Doctoral / Both
- This test has 40 problems distributed over four sections. Each problem carries 4 marks. SOLVE AS MANY AS YOU CAN.
- *This test booklet must have* 8 *pages* (this cover page with instructions and 7 pages of problems). *Make sure right at the outset that you have all* 8 *pages and all* 40 *problems in your booklet.*
- MODE OF ANSWERING: Enter only your final answer in the *answer box* provided. It is neither necessary nor is there provision of space to indicate the steps taken to reach the final answer.

  ‡ **Only the final answer, written legibly and unambiguously in the answer box, will be marked.**
- MARKING SCHEME: The marking scheme for each section is described at the beginning of that section. ***There is negative marking for the* TRUE OR FALSE TYPE *problems.*** There is no negative marking for the SHORT ANSWER TYPE problems.
- M-SCORE AND D-SCORE: If m_1, m_2, d_1, and d_2 denote your "raw" scores (net of any negative marks) in the four sections of this test respectively, your M-Score will be $m_1+m_2+d_1+d_2$; and your D-Score will be $m_1 + m_2 + 3(d_1 + d_2)/2$. The maximum possible M-Score is 160 and the maximum possible D-Score is 190.
- NOTATION AND TERMINOLOGY: The problems make use of standard notation and terminology. You too are allowed the use of standard notation. For example, answers of the form $e + \sqrt{2}$ and $2\pi/19$ are acceptable; both $3/4$ and 0.75 are acceptable.
- DEVICES: Use of plain pencils, pens, and erasers is allowed. Mobile phones and calculators are not allowed inside the exam hall. More generally, any device (e.g., a smart watch) that can be used for communication or calculation or storage is prohibited. Invigilators have the right to impound (for the duration of the test) any device that arouses their suspicion.
- ROUGH WORK: For rough work, you may use the sheets separately provided, in addition to the blank pages in the test booklet. You must:
  - Write your name and roll number on each such sheet (or set of sheets if stapled).
  - Return all these sheets to the invigilator along with this test booklet at the end of the test.
- Do **not** seek clarification from the invigilator or anyone else about any problem. In the unlikely event that there is a mistake in any problem, appropriate allowances will be made while marking.

Part M, Section 1: QUESTIONS 1 TO 20, (*Short Answer Type*, MAXIMUM RAW SCORE: 80)

Marking Scheme: $+4$ for every complete and correct response; no negative marking: 0 if either the question is left unanswered or the response is incorrect or the response is incomplete.

(1) Let X be the set of all continuous real-valued functions on the interval $I = [0, 1]$. Equip X with the metric $d : X \times X \to \mathbb{R}$ defined by

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Consider the sequence $(f_n)_{n\geqslant 1}$ in X where

$$f_n(x) = \begin{cases} n, & \text{if } 0 \leqslant x \leqslant \frac{1}{n^2}, \\ \frac{1}{\sqrt{x}}, & \text{if } \frac{1}{n^2} \leqslant x \leqslant 1. \end{cases}$$

On the metric space (X, d), which of the following is/are true? (**Note.** It suffices to state just the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) The sequence $(f_n)_{n\geqslant 1}$ is not Cauchy.
(b) The sequence $(f_n)_{n\geqslant 1}$ is Cauchy.
(c) The sequence $(f_n)_{n\geqslant 1}$ is Cauchy but does not converge in X.
(d) The sequence $(f_n)_{n\geqslant 1}$ converges in X.

‡

(2) Does $\lim_{n\to\infty} \int_0^1 \frac{n\, x^n}{1 + x}\, dx$ exist? If it does not exist, then write 'limit does not exist' in the answer box; if the limit exists, then compute the limit and enter it in the answer box.

‡

(3) For what non-zero reals k does the series

$$\sum_{n=2}^{\infty} \frac{1}{n^{1+((\log\log n)/(k \log n))}}$$

converge?

‡

(4) The prime elements of the ring $\mathbb{Z}[i] = \{a + ib |\ a, b \in \mathbb{Z}\}$ are called Gaussian primes. Which of the following are Gaussian primes? (**Note.** It suffices to state just the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) $5 + 0i$
(b) $0 + 7i$
(c) $3 + 5i$
(d) $4 + 5i$

‡

(5) Given a metric d on $\mathbb{R}^2$, the closed unit ball around $(0,0)$ with respect to d is the set $\{(x,y) \in \mathbb{R}^2 | \, d((x,y),(0,0)) \leqslant 1\}$. If d is induced by a norm, then which of the following shapes can the closed unit ball around $(0,0)$ with respect to d take? (**Note.** It suffices to state just the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) A rectangle
(b) An ellipse
(c) An equilateral triangle
(d) A parallelogram that is not a rectangle

‡

(6) Let S_{11} be the symmetric group on 11 letters. How many subgroups of order 11 are there in S_{11}?

‡

(7) For $n \geqslant 1$, let S_n denote the set of all permutations of $\{1, 2, \dots, n\}$. Let p_n denote the probability of the event that a randomly chosen permutation does not fix any integer in its original position. Find $\lim_{n\to\infty} p_n$.

‡

(8) Let A be an $n \times n$ matrix with real entries and having rank 1, where $n \geqslant 2$. Find $\det(A + I)$ in terms of A.

‡

(9) On $\mathscr{C}[-1,1]$, the set of all continuous real-valued functions on the interval $I = [-1,1]$, consider the following two metrics d_1 and d_2 given by

$$d_1(f,g) = \max_{t\in I} |f(t) - g(t)| \quad \text{and} \quad d_2(f,g) = \frac{1}{2}\int_{-1}^{1} |f(s) - g(s)|\, ds.$$

Let $A \subset \mathscr{C}[-1,1]$ be the set of all continuous functions $f : I \to I$. Which of the following is/are true? (**Note.** It suffices to state just the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) A is a bounded subset of both $(\mathscr{C}[-1,1],\, d_1)$ and $(\mathscr{C}[-1,1],\, d_2)$.
(b) $\sup_{f,g\in A} d_2(f,g) < \sup_{F,G\in A} d_1(F,G)$.
(c) A is a closed subset of $(\mathscr{C}[-1,1],\, d_1)$.
(d) A is a closed subset of $(\mathscr{C}[-1,1],\, d_2)$.

‡

(10) How many 4-digit numbers divisible by 15 can you form using only the digits 2, 3, 5, and 7, with repetition of digits allowed?

‡

(11) Let $\mathscr{P}(4)$ be the vector space of real polynomials of degree less than or equal to 4. Given a basis $B = \{p_0, \dots, p_4\}$ of $\mathscr{P}(4)$, let S_B be the set of degrees of p_i. What are the possible cardinalities of S_B?

‡

(12) What is the value of the triple (a,b,c) if $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^3 - 2x^2 + 3, & \text{if } x \geqslant 1, \\ ax^2 + bx + c, & \text{if } x < 1, \end{cases}$$

is twice differentiable?

‡

(13) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with $f(x+iy) = u(x,y) + iv(x,y)$ (where u and v are real-valued). Further, let

$$\frac{\partial u}{\partial x} = x^2 - y^2, \qquad \frac{\partial u}{\partial y} = -2xy,$$

and $f(0) = -1$. Evaluate $f(2)$. ‡

(14) Which of the following subsets of $\mathbb{N}$, where $\mathbb{N} := \{0, 1, 2, \dots\}$, are closed under multiplication? (**Note.** It suffices to state just the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) $\{a^2 + b^2 \mid a, b \in \mathbb{N}\}$
(b) $\{a^3 + b^3 \mid a, b \in \mathbb{N}\}$
(c) $\{a^2 + b^2 + c^2 \mid a, b, c \in \mathbb{N}\}$
(d) $\{a^2 + b^2 + c^2 + d^2 \mid a, b, c, d \in \mathbb{N}\}$

‡

(15) What is the number of homomorphisms from the symmetric group S_7 to the alternating group A_8 (i.e., the subgroup of all even permutations in the symmetric group S_8)? ‡

(16) Consider the finite sequence $A_1 = (2, 2^2, 2^3, \dots, 2^n)$ with n terms, $n \geqslant 3$. We form the new finite sequence A_2 with $(n-1)$ terms by taking the averages of consecutive terms in A_n, i.e.,

$$A_2 = \left(\frac{2+2^2}{2}, \frac{2^2+2^3}{2}, \dots, \frac{2^{n-1}+2^n}{2}\right).$$

Similarly, define A_{i+1} by taking averages of consecutive elements of the sequence A_i for $i \geqslant 2$. Repeat this process until you have only one term. What is its value?

‡

(17) Find the number of 2×2 real invertible matrices whose entries are 1, 2, 3, or 4, allowing repetition of these numbers for different entries. ‡

(18) Let x, y, z be integers and let $1 < x < y < z$. You are given that $(x-1)(y-1)(z-1) \mid (xyz-1)$. You are also given that there is more than one integer value of $(xyz-1)/\big((x-1)(y-1)(z-1)\big)$ for x, y, z as above. Find all integer values of $(xyz-1)/\big((x-1)(y-1)(z-1)\big)$ for x, y, z as stated: i.e., $x, y, z \in \mathbb{Z}$ and $1 < x < y < z$. ‡

(19) Let $V = M_n(\mathbb{R})$ be the real vector space of $n \times n$ matrices with real entries. Let $A \in V$ and consider the linear map $T(X) = AX$, for all $X \in V$. The trace of T in terms of A is given by

‡

(20) Let a_n be a sequence defined recursively as $a_0 = 2$, $a_1 = 5$, and $a_{n+2} = 5a_{n+1} - 4a_n$ for all $n \geqslant 0$. Find a formula (i.e., a closed-form expression in terms of n) for a_n. ‡

Part M, Section 2: QUESTIONS 21 TO 25, (*True or False Type*, MAXIMUM RAW SCORE: 20)

This section has 5 questions. Each question has a pair of assertions. For each assertion, you are required to determine its truth value and write either "True" or "False" in the corresponding answer box, as the case may be.

Marking scheme: $+2$ for each correct response, but *there is negative marking*: -2 for each incorrect response; 0 if the answer box is left empty.

(21) Let x be a non-zero solution to the ODE $\ddot{x} - \sin^2(t)x^3 = 0$ with initial condition $x(0) = x_0$.

(a) For $x_0 = 0$, there are infinitely many solutions in any neighbourhood of $t = 0$.

‡

(b) Any nonzero solution can vanish at most once in $[0, \infty)$.

‡

(22) It is known that in an arithmetic progression $\{a, a + d, a + 2d, \ldots\}$, where $a, d \in \mathbb{Z}_+$, there are infinitely many primes, provided that $\gcd(a, d) = 1$.

(a) For a prime number p, let $m_p = \max\{m \in \mathbb{Z}_+ \backslash \{1\} :$ each of the numbers $p - 1,\ p + 1,$ and $p + 2$ has at least m distinct prime factors$\}$. Then, there exists an N such that $m_p < N$ for all primes p.

‡

(b) Let $\gcd(a, d) = 1$. Then, there are infinitely many numbers in the arithmetic progression $\{a, a + d, a + 2d, \ldots\}$ having exactly two prime factors.

‡

(23) Let $i := \sqrt{-1}$ denote a square root of -1.

(a) All subrings of $\mathbb{Q}[i]$ are unique factorization domains.

‡

(b) All subrings of $\mathbb{Q}$ are exactly of the form $\mathbb{Z}[\frac{1}{n}]$ for some non-zero integer n.

‡

(24) In what follows, V will denote a subset of $\mathbb{R}$.

(a) If V can be endowed with the structure of a non-zero vector space over $\mathbb{R}$, then V must necessarily be an unbounded set.

‡

(b) Let $V = (0, +\infty)$ and define $x \boxplus y := xy$ (i.e., the usual multiplication in $\mathbb{R}$) for each $x, y \in V$. There does not exist any scalar multiplication between the reals and the elements of V that will make V a vector space over $\mathbb{R}$ with vector addition given by $\boxplus$.

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(25) Let A be a non-zero 2×2 complex matrix such that $\exp(A) = I_2$, where I_2 is the 2×2 identity matrix.

(a) A is diagonalizable.

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(b) $\mathrm{Trace}(A)$ is zero.

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Part D, Section 1: QUESTIONS 26 TO 35, *Short Answer Type*, MAXIMUM RAW SCORE: 40
YOUR RAW SCORE ON THIS SECTION COUNTS 1.5 TIMES TOWARDS YOUR D-SCORE.

Marking Scheme: $+4$ for every complete and correct response; no negative marking: 0 if either the question is left unanswered or the response is incorrect or the response is incomplete.

(26) Let $(x(t), y(t)) \in \mathbb{R}^2$ be a non-constant solution to the ODE

$$\dot{x} = \frac{x^3}{1+y^2}, \quad \dot{y} = \frac{y^3}{1+x^2}, \quad t > 0,$$

and assume that the limits $\lim_{t\to 0^+} x(t) =: x(0)$ and $\lim_{t\to 0^+} y(t) =: y(0)$ exist. Let $T = \{t \geqslant 0 : (x(t), y(t)) = (x(0), y(0))\}$. What is the cardinality of T? ‡

(27) Suppose 40 identical empty boxes are arranged in a line. You are given 30 balls of identical size and of 5 different colours. In how many ways can you put these balls into the given boxes so that each box contains at most one ball and all balls of the same colour are consecutively placed in line? ‡

(28) It is known that

$$\lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right) = \gamma,$$

where γ is the Euler–Mascheroni constant. For a real number x, suppose $\{x\}$ denotes the fractional part of x, i.e., $\{x\} \in [0, 1)$ and $x - \{x\} \in \mathbb{Z}$. In terms of γ, evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left\{ \frac{n}{j} \right\}.$$

‡

(29) Let $R = \mathbb{Z}/n\mathbb{Z}$ be the commutative ring of integers modulo n and consider $p(x) = x^2 + x + 1$ and $q(x) = x^4 + 2x^3 + x^2 + 2025x + 2024$ from $R[x]$. The number of integers n, where $n \geqslant 10$, such that $p(x)$ divides $q(x)$ in $R[x]$ is equal to ‡

(30) Let D be a domain in the complex plane, let $a \in D$, and let $f : D \to \mathbb{C}$ be a holomorphic function. Suppose there exists a number $r > 0$ such that the disc $D(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ is contained in D and such that $|f(a)| \leqslant |f(z)|$ for every $z \in D(a, r)$. What more can you say about f? Give the most specific answer possible from the information given. ‡

(31) Let X be a path connected, locally path connected, and semi-locally simply connected topological space. If the fundamental group of X is $\mathbb{Z}/6\mathbb{Z}$, then how many equivalence classes of path connected covering spaces does X have? ‡

(32) Let $\Pi(x - \lambda_i)^{n_i}, n_i \in \mathbb{Z}_+$, be the minimal polynomial of an $n \times n$ matrix A with real entries, $n \geqslant 2$. Find the minimal polynomial of the matrix $\begin{pmatrix} A & I \\ 0 & A \end{pmatrix}$. ‡

(33) Consider the real space $L^2[-1,1]$ with the standard inner product and the subspace $M = \text{span}\{t^{2k} : k = 0, 1, 2, \ldots\}$. Find $M^\perp$, i.e., the orthogonal complement to M in $L^2[-1,1]$.

‡

(34) Let $\mathbb{D}^*$ denote the set $\{z \in \mathbb{C} : |z| < 1,\ z \neq 0\}$. Let $f : \mathbb{D}^* \to \mathbb{C} \setminus \{\pm 10\}$ be a holomorphic map. Which of the following is/are true? (**Note.** It suffices to state just the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) f always has an essential singularity at $z = 0$.

(b) f either has an essential singularity or a pole at $z = 0$.

(c) f can never have an essential singularity at $z = 0$.

(d) f has neither an essential singularity nor a pole at $z = 0$.

‡

(35) Identify explicitly the set $\{\det(A - A^T) \in \mathbb{R} : A \in M_4(\mathbb{R})\}$, where $M_4(\mathbb{R})$ is the real vector space of 4×4 matrices with real entries and A^T denotes the transpose of the matrix A.

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Part D, Section 2: QUESTIONS 36 TO 40, *True or False Type*, MAXIMUM RAW SCORE: 20
YOUR RAW SCORE ON THIS SECTION COUNTS 1.5 TIMES TOWARDS YOUR D-SCORE.

This section has 5 questions. Each question has a pair of assertions. For each assertion, you are required to determine its truth value and respond by writing either "True" or "False" in the corresponding answer box, as the case may be.

Marking scheme: $+2$ for each correct response, but *there is negative marking*: -2 for each incorrect response; 0 if the answer box is left empty.

(36) Let I denote the interval $[0, 1]$.

(a) Let $f : I \to \mathbb{R}$ be continuous a.e.. Then there exists a continuous function $g : I \to \mathbb{R}$ such that $f = g$ a.e.. ‡

(b) Let $g : I \to \mathbb{R}$ be a continuous function. If a function $f : I \to \mathbb{R}$ is such that $f = g$ a.e. then f is continuous a.e.. ‡

(37) Let p be a fixed prime and $\mathbb{F}_p$ be the finite field with p elements.

(a) Suppose $L \supset K \supset \mathbb{F}_p$ be field extensions such that $Gal(L/K) = \mathbb{Z}/m\mathbb{Z}$ and $Gal(K/\mathbb{F}_p) = \mathbb{Z}/n\mathbb{Z}$. Then, $Gal(L/\mathbb{F}_p) = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. ‡

(b) There exist infinitely many Galois extensions of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}$. ‡

(38) Let X be a Banach space.

(a) Given a sequence of unit vectors $(u_n)_{n\geqslant 1}$ in X and $(\lambda_n)_{n\geqslant 1} \subset \mathbb{R}$, $\sum_n \lambda_n u_n$ converges in X if and only if $\sum_n |\lambda_n| < \infty$. ‡

(b) X is finite dimensional if and only if each of its subspaces is closed. ‡

(39) By a rational map f we mean a complex-valued function f of the form $f = p/q$, where $p, q \in \mathbb{C}[z]$ and have no common zeros, defined at all complex numbers other than the zeros of q.

(a) Let $\mathscr{D}$ denote the intersection of two fixed, distinct, intersecting open discs in $\mathbb{C}$. There exists no rational function that maps $\mathscr{D}$ conformally and injectively onto the sector

$$S_a = \{re^{i\theta} \in \mathbb{C} : r > 0 \text{ and } -a < \theta < a\}$$

for some $a \in (0, \pi)$. ‡

(b) Given some (fixed) open disc in $\mathbb{C}$, there exists some rational map p/q with q **non-constant** that maps this disc onto a bounded image. ‡

(40) Let $S^1 \subset \mathbb{C}$ denote the unit circle, which forms a group under the operation $e^{i\theta} \cdot e^{i\gamma} = e^{i(\theta+\gamma)}$ with identity element $1 \in \mathbb{C}$. Define $G := \{a + ib \in S^1 : a, b \in \mathbb{Q}\}$. Note that G itself forms a subgroup of S^1.

(a) The group G is isomorphic to $\mathbb{Q}/\mathbb{Z}$. ‡

(b) For a fixed prime p, the subset $H := \{(a, b) \in G : a = r/p^s \text{ for some } r, s \in \mathbb{Z}\}$ forms a subgroup of G. ‡