NATIONAL BOARD FOR HIGHER MATHEMATICS

MASTER'S AND DOCTORAL SCHOLARSHIP WRITTEN TEST

SATURDAY, 20TH JANUARY 2024, 10:30 A.M. TO 1:30 P.M.

For official use only: ‡ **M-Score** = ‡ **D-Score** =

‡ $\boldsymbol{m_1}$ = ‡ $\boldsymbol{m_2}$ = ‡ $\boldsymbol{d_1}$ = ‡ $\boldsymbol{d_2}$ =

Instructions to Candidates

- APPLICATION NUMBER: ‡ ROLL NUMBER: ‡
- NAME (in full in BLOCK letters): ‡
- SCHOLARSHIP TYPE (circle one and only one of the three options): Master's / Doctoral / Both
- This test has 40 problems distributed over four sections. Each problem carries 4 marks. SOLVE AS MANY AS YOU CAN. The problems in each section are arranged rather randomly.
- This test booklet must have 8 pages (this cover page with instructions and 7 pages of problems). Make sure right at the outset that you have all 8 pages and all 40 problems in your booklet.
- MODE OF ANSWERING: Enter only your final answer in the answer box provided. It is neither necessary nor is there provision of space to indicate the steps taken to reach the final answer.

  ‡ **Only the final answer, written legibly and unambiguously in the answer box, will be marked.**
- MARKING SCHEME: The marking scheme for each section is described at the beginning of that section. **There is negative marking for the TRUE OR FALSE TYPE problems.** There is no negative marking for the SHORT ANSWER TYPE problems.
- M-SCORE AND D-SCORE: If m_1, m_2, d_1, and d_2 denote your "raw" scores (net of any negative marks) in the four sections of this test respectively, your M-Score will be $m_1+m_2+d_1+d_2$; and your D-Score will be $m_1 + m_2 + 3(d_1 + d_2)/2$. The maximum possible M-Score is 160 and the maximum possible D-Score is 190.
- NOTATION AND TERMINOLOGY: The problems make free use of standard notation and terminology. You too are allowed the use of standard notation. For example, answers of the form $e+\sqrt{2}$ and $2\pi/19$ are acceptable; both $3/4$ and 0.75 are acceptable.
- DEVICES: Use of plain pencils, pens, and erasers is allowed. Mobile phones and calculators are not allowed inside the exam hall. More generally, any device (e.g., a smart watch) that can be used for communication or calculation or storage is prohibited. Invigilators have the right to impound (for the duration of the test) any device that arouses their suspicion.
- ROUGH WORK: For rough work, you may use the sheets separately provided, in addition to the blank pages in the test booklet. You must:
  - Write your name and roll number on each such sheet (or set of sheets if stapled).
  - Return all these sheets to the invigilator along with this test booklet at the end of the test.
- Do **not** seek clarification from the invigilator or anyone else about any problem. In the unlikely event that there is a mistake in any problem, appropriate allowances will be made while marking.

Part M, Section 1: PROBLEMS 1 TO 20, (SHORT ANSWER TYPE, MAXIMUM RAW SCORE: 80)

Marking Scheme: $+4$ for every complete and correct response; no negative marking: 0 if either the answer box is left empty, or the response is incorrect, or the response is incomplete.

(1) Let x be the solution to the ODE $\dot{x} = \cos(x),\ x(0) = 0$. How many real numbers $y_0 \neq 0$ are there such that the solution y to the ODE $\dot{y} = \cos(y),\ y(0) = y_0$, satisfies $x(1) = y(1)$? If there are infinitely many such $y_0 \in \mathbb{R}\backslash\{0\}$, then write "infinity" in the answer box.

‡ 0

(2) In how many different ways can one put 5 identical balls in 10 non-identical boxes?

‡ 2002 (Accept ${}^{14}C_5$ and all equivalent notations for the latter.)

(3) Write down (or otherwise describe in case the solution set is infinite) all the **integer** solutions (x, y, z) of the system of equations:

$$\begin{aligned} x + yz &= 2024 \\ xy + z &= 2023. \end{aligned}$$

‡ $(2024, 0, 2023),\ (674, 2, 675)$ (Accept alternative presentations provided the correct triples of x, y, and z are evident.)

(4) Write down the set of all real numbers r such that the series

$$\sum_{n=1}^{\infty} \left(\sqrt{n^r + 1} - \sqrt{n^r}\right)$$

converges.

‡ $(2, \infty)$ (Accept $r > 2$.)

(5) In a metric space (X, d), take three distinct points a_1, a_2, and a_3. Consider $f,\ g : X \to \mathbb{R}$ defined by $f(x) = (d(x, a_1) + d(x, a_2))^{-1}$ and $g(x) = (d(x, a_1) + d(x, a_2) + d(x, a_3))^{-1}$. Which of the following is/are true? (**Note.** It suffices to state the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) f is uniformly continuous but g may not be uniformly continuous.
(b) g is uniformly continuous but f may not be uniformly continuous.
(c) Both f and g are uniformly continuous.
(d) Neither f nor g may be uniformly continuous.

‡ (c)

(6) With $a > 0$, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^{10} \sin\left(x^{-a}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Write down all the values of a for which f is twice continuously differentiable on $\mathbb{R}$.

‡ $(0, 4)$ (Accept $0 < a < 4$, obviously, but also accept $a < 4$, since $a > 0$ has been stated in the problem itself.)

(7) For a positive integer n, denote by $\mathbb{Z}/n\mathbb{Z}$ the group of integers modulo n and by S_n the symmetric group on n letters. The number of n's for which we have an injective homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to S_7 is equal to $\boxed{\ddagger\, 9^*}$

*(In case a candidate does not report the answer "9" but claims to have the values of n: if **all nine** correct values of n – i.e., $n = 1, 2, 3, 4, 5, 6, 7, 10, 12$ – are reported, then (and only then) accept the latter response.)

(8) Let E be an n-dimensional inner-product space over $\mathbb{R}$, let $\langle \cdot , \cdot \rangle$ denote the inner product, and let $A = \{v_1, \cdots, v_m\} \subseteq E$ be a subset of unit vectors satisfying $\langle v_i, v_j \rangle = -1$ whenever v_i and v_j are not orthogonal. Denote by r the number of pairs $(i, j), i \neq j$, such that $\langle v_i, v_j \rangle \neq 0$. Which of the following is/are true? (**Note.** It suffices to state the letter corresponding to a statement. If more than one statement is true, then all such must be identified.)

(a) r can take the value n even if A is linearly independent.
(b) r must be at most n.
(c) r must be at most $n/2$.
(d) $r \neq n$ if A is linearly independent.

$\boxed{\ddagger\, \text{(d)}}$

(9) For which values of $n \in \mathbb{Z}$ does the polynomial $x^4 + nx + 3$ have a rational root?

$\boxed{\ddagger\, \pm 4, \pm 28}$

(10) Let $f, g : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}\backslash\{0, 1\}$ be given by

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{x-1}{x}.$$

What is the order of the group under composition generated by f and g?

$\boxed{\ddagger\, 6}$

(11) Let a, b, c be real numbers satisfying

$$a + b + c = 1,$$
$$a^2 + b^2 + c^2 = 1.$$

What is the difference between the maximum and minimum possible values of c?

$\boxed{\ddagger\, 4/3}$

(12) How many monic quadratic polynomials $p(x) = x^2 + bx + c$ are there such that the zeros of $p(x)$ are integers and the coefficients $1, b, c$ are in arithmetic progression? $\boxed{\ddagger\, 2}$

(13) Three distinct points are chosen uniformly at random from the vertices of a cube in $\mathbb{R}^3$. What is the probability that they form an equilateral triangle? $\boxed{\ddagger\, 1/7}$

(14) Let $\mathscr{A} = \{e_1, e_2, \ldots, e_{n+1}\}$ be a collection of $(n+1)$ vectors in $\mathbb{R}^n$ satisfying $\langle e_i, e_j \rangle < 0$ for $i \neq j$, where $\langle \cdot , \cdot \rangle$ is some inner product. How many subsets of $\mathscr{A}$ form a basis of $\mathbb{R}^n$?

$\boxed{\ddagger\, n+1}$

(15) Let R_1 be the anti-clockwise rotation in the plane about $(0,0)$ by an angle α and R_2 be the anti-clockwise rotation in the plane about $(2,0)$ by α. If $\alpha \neq n\pi$ for any integer n, then $R_2 \circ R_1$ is a rotation about the point

‡ $(1, -\tan(\alpha/2))$ (Accept equivalent expressions: e.g., $(1, (\cos\alpha - 1)/\sin\alpha)$ is fine as $\alpha \neq n\pi$.)

(16) Let $\mathbb{D}$ denote the disc $\{z \in \mathbb{C} : |z| < 1\}$ and let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function. Suppose $f|_{\mathbb{D}}$ is holomorphic, $|f(z)| = 1$ for every $z \in \partial\mathbb{D}$, and f vanishes at exactly one point $a \in \mathbb{D}$. Assume that $a \in (-1, 1)$ and that $f(1) = 1$. Find all the functions f having the above properties.

‡ $f(z) = \left(\frac{z-a}{1-az}\right)^n, \quad n = 1, 2, 3, \dots$

(17) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. For each $n \in \mathbb{N}$, consider $a_n := \left(\int_0^1 (f(x))^{4n}\, dx\right)^{1/4n}$. Determine whether or not $\lim_{n\to\infty} a_n$ exists. If it does not exist, then write "limit does not exist" in the answer box; if the limit exists, then compute the limit and enter it in the answer box.

‡ $\sup\{|f(x)| : x \in [0,1]\}$ (Also accept $\max\{|f(x)| : x \in [0,1]\}$ since f is continuous.)

(18) Define $F : (0,\infty) \to \mathbb{R}$ by

$$F(x) := \int_x^{2x} e^{-2t} t^{-1}\, dt.$$

Determine whether or not $\lim_{x\to 0^+} F(x)$ exists. If it does not exist, then write "limit does not exist" in the answer box; if the limit exists, then compute the limit and enter it in the answer box.

‡ $\log_e 2$, (Accept $\log 2$, which is the more common notation, or $\ln(2)$.)

(19) Find the number of permutations $a_1, a_2, \cdots, a_8$ of the sequence $1, 2, \cdots, 8$ satisfying the condition

$$a_1 \leqslant 2a_2 \leqslant \cdots \leqslant 8a_8.$$

Give an **explicit** positive integer for your answer (words or expressions will not be accepted).

‡ 34

(20) Let A be a $n \times n$ matrix with the entry in the i-th row and j-th column given by $1/\max(i,j)$. Find the determinant of A.

‡ $1/(n!)^2$ (Accept equivalent expressions such as $(1 - \frac{1}{2})(\frac{1}{2} - \frac{1}{3}) \cdots (\frac{1}{n-1} - \frac{1}{n})\frac{1}{n}$.)

Part M, Section 2: PROBLEMS 21 TO 25, (TRUE OR FALSE TYPE, MAXIMUM RAW SCORE: 20)

This section has 5 problems. Each problem has a pair of assertions. For each assertion, you are required to determine its truth value and write either "True" or "False" in the corresponding answer box, as the case may be.

Marking scheme: $+2$ for each correct response, but there is negative marking: -2 for each incorrect response; 0 if the answer box is left empty.

(21) Suppose you are given the 4-letter word LUCK and your friend rearranges the letters arbitrarily. Considering all possible rearrangements of the letters in the word LUCK, let us call each such rearrangement a word. Suppose you choose such a word uniformly at random.

(a) The probability that the chosen word has none of the letters L, U, C, and K in their original positions is less than $1/2$. ‡ True

(b) The probability that the chosen word has L in its original position is $1/4$. ‡ True

(22) Let $\mathbb{R}$ be equipped with the standard metric, and let A and B be non-empty subsets of $\mathbb{R}$.

(a) If A and B are closed in $\mathbb{R}$, then $A + B := \{a + b : a \in A \text{ and } b \in B\}$ is also closed. ‡ False

(b) If A and B are dense in $\mathbb{R}$, and B is also open in $\mathbb{R}$, then $A \cap B$ is dense in $\mathbb{R}$. ‡ True

(23) Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and $T : V \to V$ be a linear transformation.

(a) $\mathbb{F} = \mathbb{R}$ and $\mathrm{Trace}(T|_W) = 0$ for all T-invariant subspaces $W \subseteq V$ imply that T is nilpotent. ‡ False

(b) $\mathbb{F} = \mathbb{C}$ and $\mathrm{Trace}(T|_W) = 0$ for all T-invariant subspaces $W \subseteq V$ imply that T is nilpotent. ‡ True

(24) Consider the following statements about the relation between inner products on $\mathbb{R}^n$, $n \geqslant 2$, and real symmetric matrices.

(a) There exists a symmetric non-invertible $n \times n$ matrix A such that $v^T Aw$ is an inner product. ‡ False

(b) There exists a symmetric invertible $n \times n$ matrix A such that $v^T Aw$ is not an inner product. ‡ True

(25) Let $f : (0, +\infty) \to \mathbb{R}$ be a function that satisfies

$$f(x) = f\left(\frac{x}{2} + 1\right) \qquad \forall x \in (0, +\infty).$$

Then,

(a) there exists a point $p \in (0, +\infty)$ such that if f is continuous at p, then f is continuous. ‡ True

(b) f is necessarily continuous. ‡ False

Part D, Section 1: PROBLEMS 26 TO 35, SHORT ANSWER TYPE, MAXIMUM RAW SCORE: 40
YOUR RAW SCORE ON THIS SECTION COUNTS 1.5 TIMES TOWARDS YOUR D-SCORE.

Marking Scheme: $+4$ for every complete and correct response; no negative marking: 0 if either the answer box is left empty, or the response is incorrect, or the response is incomplete.

(26) Let m be the smallest positive integer such that $m^{1/3} = n + l$, where n is a positive integer and $0 < l < 1/2024$. What is the value of n?
$\boxed{\ddagger\ 26}$

(27) Let $A(t) = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}$. Consider the ODE $\dot{x} = A(t)x$ with $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find the value of

$$\lim_{t\to\infty} \sqrt{x_1^2(t) + x_2^2(t)}.$$

$\boxed{\ddagger\ \infty}$

(28) For $a \geqslant 0$, consider the normed space $(X_a, \|\cdot\|_a)$, where X_a is the space of all continuous functions $f : [0, \infty) \to \mathbb{R}$ satisfying

$$\|f\|_a := \sup_{x \geqslant 0} e^{ax}|f(x)| < \infty.$$

Let $0 \leqslant a < b$, and define the linear operator $S : X_a \to X_a$ by

$$Sf(x) = \int_0^x e^{-b(x-y)} f(y)\, dy.$$

What is the operator norm of S?
$\boxed{\ddagger\ (b-a)^{-1}}$

(29) Let X be a Banach space and $T_n : X \to X$ be a sequence of bounded linear operators. Consider the set

$$A = \{x \in X \,:\, T_n(x) \text{ does not converge to } 0\}.$$

If A is known to be non-empty, then find $\overline{A}$.
$\boxed{\ddagger\ X}$

(30) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that f is bounded on the set

$$S := \{re^{i\theta} : r > 0 \text{ and } \pi/4 \leqslant \theta \leqslant 7\pi/4\} \cup \{0\},$$

and that $\mathrm{Re}\big(f(z)\big) > 0$ for each $z \in \mathbb{C} \setminus S$. What more can you say about f? Give the most specific answer possible from the information given.

$\boxed{\ddagger\ f \text{ is a constant } c \text{ with } \mathrm{Re}(c) > 0 \text{ (Accept “} f \text{ is a constant,” since positivity of the real part is given in the problem.)}}$

(31) Let a be some positive real number. Consider the function $f : (0, a) \to S^1$ defined as $f(t) = e^{2\pi i t}$. Identify all the values of a for which f a surjective local homeomorphism, but not a covering map.
$\boxed{\ddagger\ (1, \infty) \text{ (Accept } a > 1\text{, obviously.)}}$

(32) Let A be a non-empty subset of $\mathbb{N}$. For $1 \leqslant p < \infty$, let $l^p(A)$ denote the Banach space

$$\Big\{(x_j)_{j\in A} : x_j \in \mathbb{R} \ \ \forall j \in A \text{ and } \sum\nolimits_{j\in A} |x_j|^p < \infty\Big\}$$

with the norm $\|(x_j)_{j\in A}\| := \big(\sum_{j\in A} |x_j|^p\big)^{1/p}$. Assume that $l^2(A)$ is isomorphic as a Banach space to a closed subspace of the Banach space $l^1(A)$. What can you say about the set A? Give the most specific answer possible from the information given. $\boxed{\ddagger\ A \text{ is a finite set}}$

(33) Let $\mathbb{Z}/13\mathbb{Z}$ be the finite field with 13 elements. What are all the possible orders of the invertible 2×2 matrices M with entries in $\mathbb{Z}/13\mathbb{Z}$ satisfying the equation $M^2 = aI$ for some non-zero $a \in \mathbb{Z}/13\mathbb{Z}$ (where I is the identity matrix)? $\boxed{\ddagger\ 1, 2, 3, 4, 6, 8, 12, 24}$

(34) Let $G := \mathbb{Q}/\mathbb{Z}$ be the quotient group under addition. How many elements in G have order 2024?

‡ 880 or $\phi(2024)$ where ϕ is Euler's totient function (Accept $\phi(2024)$ even if the meaning of ϕ isn't spelled out.)

(35) Let $f : (0, 1) \to [0, \infty)$ be a continuous function and let

$$\Sigma_f(y) := \text{the sum of the lengths of the disjoint intervals whose union is the set } \{x \in (0, 1) : f(x) > y\}, \quad y > 0.$$

Let $1 \leqslant p < \infty$. Give an expression for the integral $\int_0^1 \big(f(x)\big)^p\, dx$ in terms of p and Σ_f such that the **only** involvement of f occurs in Σ_f. $\boxed{\ddagger\ p \displaystyle\int_0^\infty y^{p-1} \Sigma_f(y)\, dy}$

Part D, Section 2: PROBLEMS 36 TO 40, TRUE OR FALSE TYPE, MAXIMUM RAW SCORE: 20
YOUR RAW SCORE ON THIS SECTION COUNTS 1.5 TIMES TOWARDS YOUR D-SCORE.

This section has 5 problems. Each problem has a pair of assertions. For each assertion, you are required to determine its truth value and respond by writing either "True" or "False" in the corresponding answer box, as the case may be.

Marking scheme: $+2$ for each correct response, but there is negative marking: -2 for each incorrect response; 0 if the answer box is left empty.

(36) Let X be a random vector chosen uniformly from $\{0,1\}^3$. Let $D \in \mathbb{R}^{3\times 3}$ be a non-zero matrix. Let $\mathbb{P}(A)$ denote the probability of the event A.

(a) The supremum of $\mathbb{P}(DX = 0)$ as D varies is $1/2$. ‡ True

(b) The supremum of $\mathbb{P}(DX = 0)$ is not attained for any non-zero matrix D as D varies. ‡ False

(37) Let f be a non-constant holomorphic function on the disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For this f, you are given that there exists a holomorphic function g on $\mathbb{D}$ such that $f = e^g$ on $\mathbb{D}$.

(a) There is a unique g for which the above equation is true. ‡ False

(b) For any f as described above, $f(\mathbb{D})$ is necessarily simply connected. ‡ False

(38) Let $f : [0,\infty) \to [0,\infty)$ be a continuous function.

(a) If $\int_0^\infty f(x)\,dx < \infty$, then $\int_0^\infty \big(f(x)\big)^2\,dx < \infty$. ‡ False

(b) If $\int_0^\infty \big(f(x)\big)^2\,dx < \infty$, then $\int_0^\infty f(x)\,dx < \infty$. ‡ False

(39) Let $M(2,\mathbb{R})$ be the set of all 2×2 matrices with real entries. An identification with $\mathbb{R}^4$ gives a natural topology on $M(2,\mathbb{R})$. Then,

(a) the set of all upper triangular matrices viewed as a subspace of $M(2,\mathbb{R})$ is connected. ‡ True

(b) the set of all orthogonal matrices viewed as a subspace of $M(2,\mathbb{R})$ is connected. ‡ False

(40) Let $f : X \to Y$ be a continuous function between two topological spaces, let $x_0 \in X$, and let $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ be the induced homomorphism between fundamental groups. Then:

(a) if f is injective, then f_* is always injective. ‡ False

(b) if f is surjective, then f_* is always surjective. ‡ False