NATIONAL BOARD FOR HIGHER MATHEMATICS

MASTER'S AND DOCTORAL SCHOLARSHIP WRITTEN TEST

SATURDAY 29 APRIL 2023, 10:30 A.M. TO 1:30 P.M.

For official use only: ‡ **M-Score** = ‡ **D-Score** =

‡ $\mathbf{m_1}$ = ‡ $\mathbf{m_2}$ = ‡ $\mathbf{d_1}$ = ‡ $\mathbf{d_2}$ =

Instructions to Candidates

- APPLICATION NUMBER: ‡ ROLL NUMBER: ‡
- NAME in full in BLOCK letters: ‡
- SCHOLARSHIP TYPE (circle one and only one of the three options): Master's / Doctoral / Both
- This test has 40 questions distributed over four sections. Each question carries 4 marks. ANSWER AS MANY AS YOU CAN. The questions in each section are arranged rather randomly.
- **This test booklet must have 6 pages** (this cover page with instructions and 5 pages of questions). **Make sure right at the outset that you have all 6 pages and all 40 questions in your booklet.**
- MODE OF ANSWERING: Enter only your final answer in the answer box provided. It is neither necessary nor is there provision of space to indicate the steps taken to reach the final answer.

  ‡ **Only the final answer, written legibly and unambiguously in the answer box, will be marked.**
- MARKING SCHEME: The marking scheme for each section is described at the beginning of that section. **There is negative marking on the TRUE OR FALSE TYPE questions.** There is no negative marking on the SHORT ANSWER TYPE questions.
- M-SCORE AND D-SCORE: If m_1, m_2, d_1, and d_2 denote your "raw" scores (net of any negative marks) in the four sections of this test respectively, your M-Score will be $m_1+m_2+d_1+d_2$; and your D-Score will be $m_1+m_2+3(d_1+d_2)/2$. The maximum possible M-Score is 160 and the maximum possible D-Score is 190.
- NOTATION AND TERMINOLOGY: The questions make free use of standard notation and terminology. You too are allowed the use of standard notation. For example, answers of the form $e+\sqrt{2}$ and $2\pi/19$ are acceptable; both $3/4$ and 0.75 are acceptable.
- DEVICES: Use of plain pencils, pens, and erasers is allowed. Mobile phones and calculators are not allowed inside the exam hall. More generally, any device (e.g. a smart watch) that can be used for communication or calculation or storage is prohibited. Invigilators have the right to impound (for the duration of the test) any device that arouses their suspicion.
- ROUGH WORK: For rough work, you may use the sheets separately provided, in addition to the blank pages in the test booklet. You must:
  - Write your name and roll number on each such sheet (or set of sheets if stapled).
  - Return all these sheets to the invigilator along with this test booklet at the end of the test.
- Do **not** seek clarification from the invigilator or anyone else about any question. In the unlikely event that there is a mistake in any question, appropriate allowance will be made while marking.

Part M, Section 1: QUESTIONS 1 TO 20, **Short Answer Type**, MAXIMUM RAW SCORE: 80

Marking Scheme: +4 for every complete and correct response; no negative marking: 0 if either the question is left unanswered or the response is incorrect or the response is incomplete.

(1) There are exactly 2023 elements in a set X. What is the number of ordered 3-tuples (P, Q, R), where P, Q, R are subsets (possibly empty) of X such that no two of them intersect (that is, all three intersections $P \cap Q$, $Q \cap R$, and $R \cap P$ are empty)? $\boxed{\ddagger\ \mathbf{4^{2023}}}$

(2) A complex number λ is called a period of a function $f(z)$ on the complex plane if $f(z+\lambda) = f(z)$ (for all $z \in \mathbb{C}$). The set of periods of any function is a subgroup of the group $\mathbb{C}$ under addition. Find all possible generators of this group when $f(z) = |\sin z| + |\cos z|$. $\boxed{\ddagger\ \pm\pi/\mathbf{2}}$

(3) Let $\mathscr{F}$ be the set of all continuous functions $f : [1,3] \to [-1,1]$ such that $\int_1^3 f(x)\,dx = 0$.
Find: $\boxed{\ddagger\ \log 4 - \log 3 \textbf{ or } 2\log 2 - \log 3}$

$$\sup_{f \in \mathscr{F}} \int_1^3 \frac{f(x)}{x}\,dx$$

(4) A finite set of positive integers is called selfish if it contains its own cardinality as a member. For example, $\{2,5\}$ is selfish but $\{1,2,5\}$ is not. A selfish set is called simple selfish if none of its proper subsets is selfish. For example, $\{2,5\}$ is simple selfish; and $\{1,3,7\}$ is selfish but not simple selfish. How many simple selfish subsets does $\{1,2,3,4,5,6,7,8,9\}$ have? $\boxed{\ddagger\ \mathbf{34}}$

(5) A sequence $\{x_n\}$ of real numbers converges to a real number x: $\lim\limits_{n\to\infty} x_n = x$. What is the maximum possible number of limit points that the sequence $\{\lfloor x_n^2 \rfloor + \lfloor x_n \rfloor\}$ can have, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to the real number y. $\boxed{\ddagger\ \mathbf{2}}$

(6) A tractor is moving with a (non-zero) constant velocity (with its wheels turning normally, without slipping). Its back wheels have twice the radius of its front wheels. Let f be a point on the rim of a front wheel and b a point on the rim of a back wheel. Let a_f and a_b be respectively the magnitudes of the accelerations of the points f and b at some instant of time. If $a_f : a_b = 1 : k$, what is k? (All displacements, velocities, and accelerations are with respect to the ground.) $\boxed{\ddagger\ \mathbf{1/2}}$

(7) Consider planes passing through the centre of a regular octahedron. With respect to how many of these is the octahedron symmetric? $\boxed{\ddagger\ \mathbf{9}}$
(To say that the octahedron is symmetric with respect to a plane means that the mirror reflection in the plane acts as a bijection on the points of the octahedron.)

(8) Let A be a 4×4 complex matrix with four distinct eigenvalues. Suppose that 0 is one of the eigenvalues. How many distinct square roots does A have? (A square root of A is any 4×4 complex matrix B such that $B^2 = A$.) $\boxed{\ddagger\ \mathbf{8}}$

(9) Let G be a group of order 100. Let n be the order of the normalizer (in G) of a Sylow-2 subgroup of G. What values can n take? $\boxed{\ddagger\ \mathbf{100, 20, 4}}$

(10) Consider the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane oriented in the anti-clockwise direction. Let C be the portion of this oriented curve that lies in the lower half plane. Note that C starts at $z = -1$ and ends at $z = 1$. Evaluate the following line integral: $\boxed{\ddagger\ \mathbf{4/\pi}}$

$$\int_C (1+z)\cos\left(\frac{\pi}{2}z\right)dz$$

(11) Find the volume of the space in $\mathbb{R}^3$ bounded by the six planes $x+y+z=\pm 1$, $x-y+z=\pm 1$, and $x+y-z=\pm 1$. $\boxed{\ddagger\ \mathbf{2}}$

(12) Compute the following limit: $\boxed{\ddagger\ \mathbf{35/36}}$

$$\lim_{n\to\infty} \frac{(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n})\cdot(\sqrt[3]{1}+\sqrt[3]{2}+\cdots+\sqrt[3]{n})}{(\sqrt[6]{1}+\sqrt[6]{2}+\cdots+\sqrt[6]{n})\cdot(\sqrt[3]{1^2}+\sqrt[3]{2^2}+\cdots+\sqrt[3]{n^2})}$$

(13) Find all integers m, $2 \leqslant m \leqslant 999$, such that $m^2 \equiv m \bmod 1000$. $\boxed{\ddagger\ \mathbf{376, 625}}$

(14) Let A be a real $n\times n$ matrix such that $A^{10} = I$, where I denotes the $n\times n$ identity matrix. Suppose that A has no real eigenvalues. Specify all values that n could possibly take. $\boxed{\ddagger\ 4n\textbf{, } n \geqslant 1 \textbf{ integer}}$

(15) Let C_9 denote the cyclic group of order 9. What is the number of subgroups of the Cartesian product $C_9 \times C_9$ that are isomorphic to C_9? $\boxed{\ddagger\ \mathbf{12}}$

(16) Let A be a real matrix of size $3\times n$ and rank r. Given that the equation on the left below has no solution and that on the right below has exactly one solution (as $\mathbf{x}$ varies over all $n\times 1$ real matrices), specify all values that $n+r$ can take. $\boxed{\ddagger\ \mathbf{2, 4}}$

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad A\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(17) Let A, $\mathbf{b}$, and $\mathbf{x}$ be the following matrices:

$$A := \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Let C be the region of $\mathbb{R}^2$ defined as follows: $C := \{\mathbf{x} \in \mathbb{R}^2 \mid A\mathbf{x} \leqslant \mathbf{b}\}$. ($A\mathbf{x} \leqslant \mathbf{b}$ means that the entries of $A\mathbf{x}$ are component-wise less than or equal to those of $\mathbf{b}$.) Find the minimum value of the function $f(\mathbf{x}) = -3x - 4y$ as $\mathbf{x}$ varies over the region C. $\boxed{\ddagger\ \mathbf{-5}}$

(18) For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, let $\|\mathbf{x}\|_\infty := \max\{|x_1|, |x_2|, |x_3|\}$ and $\|\mathbf{x}\|_1 := |x_1| + |x_2| + |x_3|$. What is the largest positive λ such that $\lambda\|\mathbf{x}\|_1 \leqslant \|\mathbf{x}\|_\infty$ for all $\mathbf{x} \in \mathbb{R}^3$. $\boxed{\ddagger\ \mathbf{1/3}}$

(19) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(y) := (y^2 + y)/2$. Given a real number x, we take it to be the zeroth term x_0 of a sequence $\{x_n\}_{n\geqslant 0}$ where, for every $n \geqslant 1$, the number x_n is defined by induction as $f(x_{n-1})$. We define $g(x)$ to be $\lim_{n\to\infty} x_n$ provided that the limit exists as a real number ($\pm\infty$ are not considered real numbers for the present purpose), and say that $g(x)$ is undefined otherwise. What is the domain of definition of the function $g(x)$? $\boxed{\ddagger\ [\mathbf{-2, 1}]}$

(20) Let C be a $n\times n$ real matrix of rank r and consider the real vector space:

$$V_C := \{A \text{ is a real } n\times n \text{ matrix} \mid CAC = 0\}$$

What is the dimension (as a real vector space) of V_C? $\boxed{\ddagger\ \mathbf{n^2 - r^2}}$
If this dimension is only a function of n and r, then determine that function (in terms of n and r); otherwise write "**varies**" in the answer box.

Part M, Section 2: QUESTIONS 21 TO 25, **True or False** TYPE, MAXIMUM RAW SCORE: 20

This section has 5 questions. Each question has a pair of assertions. For each assertion, you are required to determine its truth value and write either "True" or "False" in the corresponding answer box, as the case may be. Marking scheme: $+2$ for each correct response, but there is negative marking: -2 for each incorrect response; 0 if the answer box is left empty.

(21) Let f be a real valued function on the interval $[0,1]$ such that:

$$f(\lambda x + (1-\lambda)y) \geqslant \lambda f(x) + (1-\lambda)f(y) \qquad \text{for all } x, y, \lambda \text{ in } [0,1]$$

Then:

(a) The set $\{u \in [0,1] \mid f(u) \leqslant \alpha\}$ is closed for all real α. **‡ True**

(b) The set $\{u \in [0,1] \mid f(u) \geqslant \alpha\}$ is closed for all real α. **‡ False**

(22) Let $\{\alpha_n\}$ be a strictly increasing sequence of positive integers. For k a positive integer, set:

$$\beta_k := \begin{cases} 1 & \text{if } k \text{ belongs to the sequence } \{\alpha_n\} \\ 0 & \text{otherwise} \end{cases}$$

In each of the following cases, determine whether the series $\sum_{k\geqslant 1}(-1)^{\beta_k}/k$ converges.

(a) The above series converges when $\alpha_n := n^2$. **‡ False**

(b) The above series converges when $\alpha_n := 2n-1$. **‡ True**

(23) Let $\{a_1, \ldots, a_n\} \subseteq G$ be a conjugacy class in a finite group G, with $n \geqslant 2$. Let $H_1, \ldots, H_n$ be the centralisers of $a_1, \ldots, a_n$ respectively: $H_i := \{g \in G \mid ga_i = a_i g\}$. Then:

(a) The union of $H_1, \ldots, H_n$ can never equal G. **‡ True**

(b) No two H_i can ever be equal. ‡ False

(24) Let $f : [0,\infty) \to \mathbb{R}$ be continuous and such that $\int_0^\infty f(x)\,dx < \infty$. Then:

(a) If f is uniformly continuous, then $\lim_{t\to\infty} f(t)$ exists. **‡ True**

(b) The function f is uniformly continuous. **‡ False**

(25) Let V be a real 3-dimensional vector space and let $T : V \to V$ be a linear transformation of V. Recall that a subspace W of V is called invariant (for T) if $T(w) \in W$ for every $w \in W$.

(a) Every T as above admits an invariant one-dimensional subspace. **‡ True**

(b) Every T as above admits an invariant two-dimensional subspace. **‡ True**

Part D, Section 1: QUESTIONS 26 TO 35, **Short Answer Type**, MAXIMUM RAW SCORE: 40
YOUR RAW SCORE ON THIS SECTION COUNTS 1.5 TIMES TOWARDS YOUR D-SCORE.

Marking Scheme: $+4$ for every complete and correct response; no negative marking: 0 if either the question is left unanswered or the response is incorrect or the response is incomplete.

(26) Let A be a real matrix of size 7×6 and rank 4. Let n be the number of real (strictly) positive eigenvalues of A^tA, counted with multiplicity, where A^t denotes the transpose of A. Specify all possibilities for n. $\boxed{\ddagger\ \mathbf{4}}$

(27) There are precisely seven open sets in a topological space on a finite set X. What is the minimum possible cardinality of X? $\boxed{\ddagger\ \mathbf{4}}$

(28) Let $C := \{z \in \mathbb{C} \,|\, |z| = 2023\}$ be the circle in the complex plane with centre at the origin and radius 2023, oriented in the anti-clockwise direction. Let $f(z) = e^{\frac{\pi}{2}z} - e^{-\frac{\pi}{2}z}$. Evaluate the contour integral: $\boxed{\ddagger\ \mathbf{2023}}$

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz$$

(29) How many different (non-isomorphic) structures of a ring with identity (not necessarily commutative) can a set with four elements carry? $\boxed{\ddagger\ \mathbf{4}}$

(30) Find the set of real values of y_0 such that there exists a bounded function $y(t) : [0, \infty) \to \mathbb{R}$ that solves the following initial value problem: $\boxed{\ddagger\ [\mathbf{-1}, \infty)}$

$$y'(t) = (1 - y^2(t)) \cosh y(t) \text{ for } t > 0; \qquad y(0) = y_0.$$

(31) Consider the subgroup V of $\mathbb{Q}^2$ (under addition) generated by $\mathbf{v}_1 = (\frac{1}{2}, \frac{1}{3})$ and $\mathbf{v}_2 = (\frac{1}{4}, \frac{1}{5})$: in other words, V consists of integer linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$V := \{a\mathbf{v}_1 + b\mathbf{v}_2 \mid a, b \in \mathbb{Z}\}$$

Let U be the subgroup $\mathbb{Z}^2$ of $\mathbb{Q}^2$. What is the index of U in V, or, in other words, what is the cardinality of the quotient group V/U? $\boxed{\ddagger\ \mathbf{60}}$

(32) Consider the complex equation $e^z = 3z + 1$. How many roots does this equation admit in the region $|z| < 1$? $\boxed{\ddagger\ \mathbf{1}}$

(33) Let G be the group of invertible 2×2 matrices with entries in the field $\mathbb{Z}/5\mathbb{Z}$ of 5 elements, with respect to the usual matrix multiplication. Let S be the subset of those elements of G that can be written in the form LU where L is a lower triangular matrix in G and U an upper triangular matrix in G. How many elements does S have? (A 2×2 matrix is <u>lower triangular</u> if its entry in position $(1, 2)$ is zero; it is <u>upper triangular</u> if its entry in position $(2, 1)$ is zero.) $\boxed{\ddagger\ \mathbf{400}}$

(34) Let A, B, and C be matrices with real entries of sizes 40×20, 20×30, and 30×40 respectively. Given that the ranks of the matrices AB, BC, and B are respectively 11, 12, and 15, what is the minimum possible rank of the matrix ABC? $\boxed{\ddagger\ \mathbf{8}}$

(35) Find $u(\pi/2, 7\pi/4)$, where u is the solution of: $\boxed{\ddagger\ \mathbf{1/32}}$

$$u_{tt}(x,t) = u_{xx}(x,t) \text{ for } x \in (0,\pi),\ t > 0; \qquad u(x,0) = \sin^{10} x, \text{ for } x \in (0,\pi),$$
$$u_t(x,0) = 0 \text{ for } x \in (0,\pi), \qquad u(0,t) = u(\pi,t) = 0, \text{ for } t > 0.$$

Part D, Section 2: QUESTIONS 36 TO 40, **True or False** TYPE, MAXIMUM RAW SCORE: 20
YOUR RAW SCORE ON THIS SECTION COUNTS 1.5 TIMES TOWARDS YOUR D-SCORE.

This section has 5 questions. Each question has a pair of assertions. For each assertion, you are required to determine its truth value and respond by writing either "True" or "False" in the corresponding answer box, as the case may be. Marking scheme: $+2$ for each correct response, but there is negative marking: -2 for each incorrect response; 0 if the answer box is left empty.

(36) Let v_1, v_2, v_3, v_4, v_5, and v_6 be six distinct vectors in a vector space. There are $\binom{6}{4} = 15$ distinct subsets consisting precisely of four of these vectors. Suppose that, out of these 15 subsets, only $\{v_1, v_2, v_3, v_4\}$ and $\{v_1, v_2, v_3, v_5\}$ are linearly independent, and the remaining 13 are linearly dependent. Then:

(a) v_6 is the zero vector. **‡ True**

(b) v_4 and v_5 are non-zero scalar multiples of each other. **‡ True**

(37) Let $f(z)$ be an arbitrary non-constant entire function on the complex plane.

(a) It is possible for $f(z)$ not to take any real value; in other words, the image of $f(z)$ could possibly be contained in $\mathbb{C}\backslash\mathbb{R}$. **‡ False**

(b) A complex number z_0 is called a local minimum of $f(z)$ if $|f(z_0)| \leqslant |f(z)|$ for all z in some open set U around z_0. If z_0 is a local minimum of $f(z)$ then $f(z_0) = 0$. **‡ True**

(38) Let k be the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo a prime p. Any polynomial $f(x)$ with coefficients in k can be considered a k-valued function on k: by $\alpha \mapsto f(\alpha)$ for all $\alpha \in k$. Such a polynomial of degree d is called monic if its leading coefficient (the coefficient of x^d) is 1.

(a) For any function $\varphi : k \to k$, and any integer d such that $d \geqslant p$, there exists a monic polynomial $f(x)$ of degree d such that $\varphi(\alpha) = f(\alpha)$ for all α in k. **‡ True**

(b) For any integer $d \geqslant 1$, and any monic polynomial $f(x)$ of degree d, the number of polynomials $g(x)$ (not necessarily monic) of degree exactly d such that $f(\alpha) = g(\alpha)$ for all $\alpha \in k$ is independent of $f(x)$ (that is, the same for all monic $f(x)$ of degree d). **‡ True**

(39) Let $p(x) = x^3 + ax^2 + bx + c$ be a cubic polynomial with real coefficients a, b, c, and define:

$$D := \{(a, b, c) \in \mathbb{R}^3 \mid \text{ the polynomial } p(x) \text{ factors into linear factors over } \mathbb{R}\}$$

Then:

(a) D is connected (as a subset of the topological space $\mathbb{R}^3$). **‡ True**

(b) For any $(a, b, c) \in D$, we have $a^2 \geqslant 3b$. **‡ True**

(40) Let $\mathbb{Q}[x]$ be the ring of polynomials in the variable x with rational coefficients. Let S be the sub-ring of $\mathbb{Q}[x]$ consisting of all those polynomials $f(x)$ with the property that $f(\alpha)$ is an integer whenever α is an integer. Then:

(a) Given a positive integer n, distinct integers $a_1, \ldots, a_n$ and any integers $b_1, \ldots, b_n$, there exists a polynomial $f(x)$ in S such that $f(a_i) = b_i$ for all $1 \leqslant i \leqslant n$. **‡ True**

(b) The ring S is finitely generated as a ring over the ring $\mathbb{Z}$ of integers. **‡ False**