Fractional exclusion statistics:
A generalised Pauli principle

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Outline of talk

- Exchange Statistics: The orthodoxy.
- Exclusion Statistics: Pauli - Haldane
- Realisation: Interacting systems in one and two dimensions
- New rules of occupancy: Generalised Pauli principle

Exchange Statistics

Identical particles are indistinguishable– Consider a two particle wave function in quantum mechanics:

$$|\Psi(\vec{x}_1, \vec{x}_2)|^2 = |\Psi(\vec{x}_2, \vec{x}_1)|^2$$

Thus

$$\Psi(\vec{x}_1, \vec{x}_2) = \Psi(\vec{x}_2, \vec{x}_1) \quad \text{Symmetric, Bosons}$$

$$\Psi(\vec{x}_1, \vec{x}_2) = -\Psi(\vec{x}_2, \vec{x}_1) \quad \text{Anti-symmetric, Fermions}$$

Furthermore

$$\Psi(\vec{x}, \vec{x}) = 0 \quad \text{Fermions}$$

Leads to Pauli Exclusion Principle

Thus Exclusion ⇒ State Counting

Anyons

Can we have

$$\Psi(\vec{x}_1, \vec{x}_2) = \exp i\theta(\vec{x}_1, \vec{x}_2) \ \ \Psi(\vec{x}_2, \vec{x}_1) \quad ?$$

Consistency with QM demands that

- $\theta(\vec{x}_1, \vec{x}_2) = \theta$ – Constant
- $\theta = 0$ (Bosons) $\theta = \pi$ (Fermions) in –$d > 2$– space dimensions
- θ may be arbitrary in $d = 1, 2$ $\Rightarrow$Anyons

Thus Exchange Statistics may be generalised, but only in lower space dimensions.

Is it possible generalise Exclusion Statistics ala Pauli ?

Pauli Exclusion Principle

For Fermions

Exchange (anti-)symmetry$\Rightarrow$ Pauli Exclusion Principle

Let n_k – Occupancy of a state labelled by k.

$n_k = 1, 0$	For identical Fermions
$n_k =$ arbitrary	For identical Bosons

Is it possible for n_k to be some thing else ? Say $n_k = 2, 1, 0$

Haldane [*PRL 67, 937(1991)*] proposed one such generalisation.

Haldane Proposal

Consider a system of N identical particles described by

$$\Psi(\vec{x}_1, \vec{x}_2, \vec{x}_3, \cdots, \vec{x}_N)$$

Freeze $N-1$ coordinates–expand Ψ. The single particle space spanned by Ψ is d_N. – Dimension of the single particle space in the presence of $N-1$ others (identical).

How does d_N change as N changes ? Propose

$$\Delta d_N = -g \Delta N$$

g is the Exclusion Statistics Parameter

$$g = 1 \qquad \text{For Fermions}$$

$$g = 0 \qquad \text{For Bosons}$$

Can g be Fractional ?

Illustration

Consider a lattice with d sites:

	Fermions	Bosons
$N = 1$	d	d
$N = 2$	d-1	d
$N = 3$	d-2	d
N	$d_N^F = d - (N-1)$	$d_N^B = d$

Dimension of the N- Particle space:

Fermions	Bosons
$D_N^F = \frac{d!}{N!(d-n)!}$	$D_N^B = \frac{(d+(N-1))!}{N!(d-1)!}$
$D_N^F = \frac{(d_N^F+(N-1))!}{N!(d_N^F-1)!}$	$D_N^B = \frac{(d_N^B+(N-1))!}{N!(d_N^B-1)!}$

g-ons

	Fermions($g = 1$)	g-ons	Bosons($g = 0$)
$N = 1$	d	d	d
$N = 2$	d-1	d-g	d
$N = 3$	d-2	d-2g	d
N	$d_N^F = d - (N-1)$	$d_N^g = d - g(N-1)$	$d_N^B = d$

Dimension of the N- Particle space:

Fermions($g = 1$)	g-ons	Bosons($g = 0$)
$D_N^F = \frac{d!}{N!(d-n)!}$		$D_N^B = \frac{(d+(N-1))!}{N!(d-1)!}$
$D_N^F = \frac{(d_N^F+(N-1))!}{N!(d_N^F-1)!}$	$D_N^g = \frac{(d_N^g+(N-1))!}{N!(d_N^g-1)!}$	$D_N^B = \frac{(d_N^B+(N-1))!}{N!(d_N^B-1)!}$

The partition function

Quantum theories must be based on observables !!
Hilbert space dimension?

$$D_N \Rightarrow \lim_{\beta \to 0} Z_N(\beta) = \lim_{\beta \to 0} \sum_{states} \exp(-\beta E_{state})$$

$$\beta = 1/T$$

The statistical parameter g is then

$$\frac{1}{2} - g = \lim_{\beta \to 0} \frac{C Z_1}{N(N-1)} \left[N! \frac{Z_N}{(Z_1)^N} - 1 \right]$$

where

$$C = 2^\eta; \quad \eta = \text{space-dimension}$$

Murthy, Shankar: PRL, 72,(94)

Equation of state

An immediate consequence is the Equation of State– Relation between Pressure, density and temperature of a Gas in 2-dimensions

Classical Gas $\frac{P}{kT} = \rho$

Bose Gas $\frac{P}{kT} = \rho[1 - \frac{1}{4}(\rho\lambda^2) + \cdots] : \quad g = 0$

Fermi Gas $\frac{P}{kT} = \rho[1 + \frac{1}{4}(\rho\lambda^2) + \cdots] : \quad g = 1$

Haldane Gas $\frac{P}{kT} = \rho[1 - \frac{1}{2}(\frac{1}{2} - g)(\rho\lambda^2) + \cdots]$

λ— Thermal wavelength; k— Boltzmann constant

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Distribution function

The distribution function probability of occupation of a state is obtained by maximising entropy$s = k \log D_N^g$

Bose Gas $n_k = \dfrac{1}{\exp[(\epsilon_k - \mu)/kT] - 1} : \quad g = 0$

Fermi Gas $n_k = \dfrac{1}{\exp[(\epsilon_k - \mu)/kT] - 1} : \quad g = 1$

Haldane Gas $n_k = \dfrac{1}{w(\exp(\epsilon_k - \mu)/kT) + g}$

where w is a solution of the equation

$w^g (1 + w)^{1-g} = \exp[(\epsilon_i - \mu)/kT]$: Ramanujan's Eq. in LNs

Wu, PRL, 73,(94), Isakov PRL 73, (94)

We have studied Exclusion Statistics as a possibility

- Unlike anyons (exchange - specific to two-space dimesions), exclusion statistics is defined in any space dimension.
- At zero temperature the maximal occupancy of a state is $1/g$. Finite temp. dist. is given by $n_k(g)$– nicely interpolates between Bose and Fermi statistics.
- Statistical mechanics of g-ons are well studied and understood.

Q: How does it arise in physical systems ?

We illustrate using a one-dimensional Model: CSM

Calogero Sutherland Model

Realisation in 1-d: An exactly solvable many body Hamiltonian with non-trivial correlations

The Hamiltonian: $\hbar = 1, m = 1, c = 1$

$$H = \frac{1}{2}\left[\sum_i p_i^2 + \omega^2 \sum_i x_i^2 + \lambda \sum_{i,j} \frac{1}{(x_i - x_j)^2}\right]$$

The Spectrum of system of N interacting Fermions:

$$E(g) = E(g=0) - (1-g)\omega\frac{N(N-1)}{2}$$

where $\lambda = g(g-1)$

Single particle picture

Non-interacting: $g = 0$

$$E(g=0) = \sum_m n_m \epsilon_m \; : \quad \epsilon_m = m\omega$$

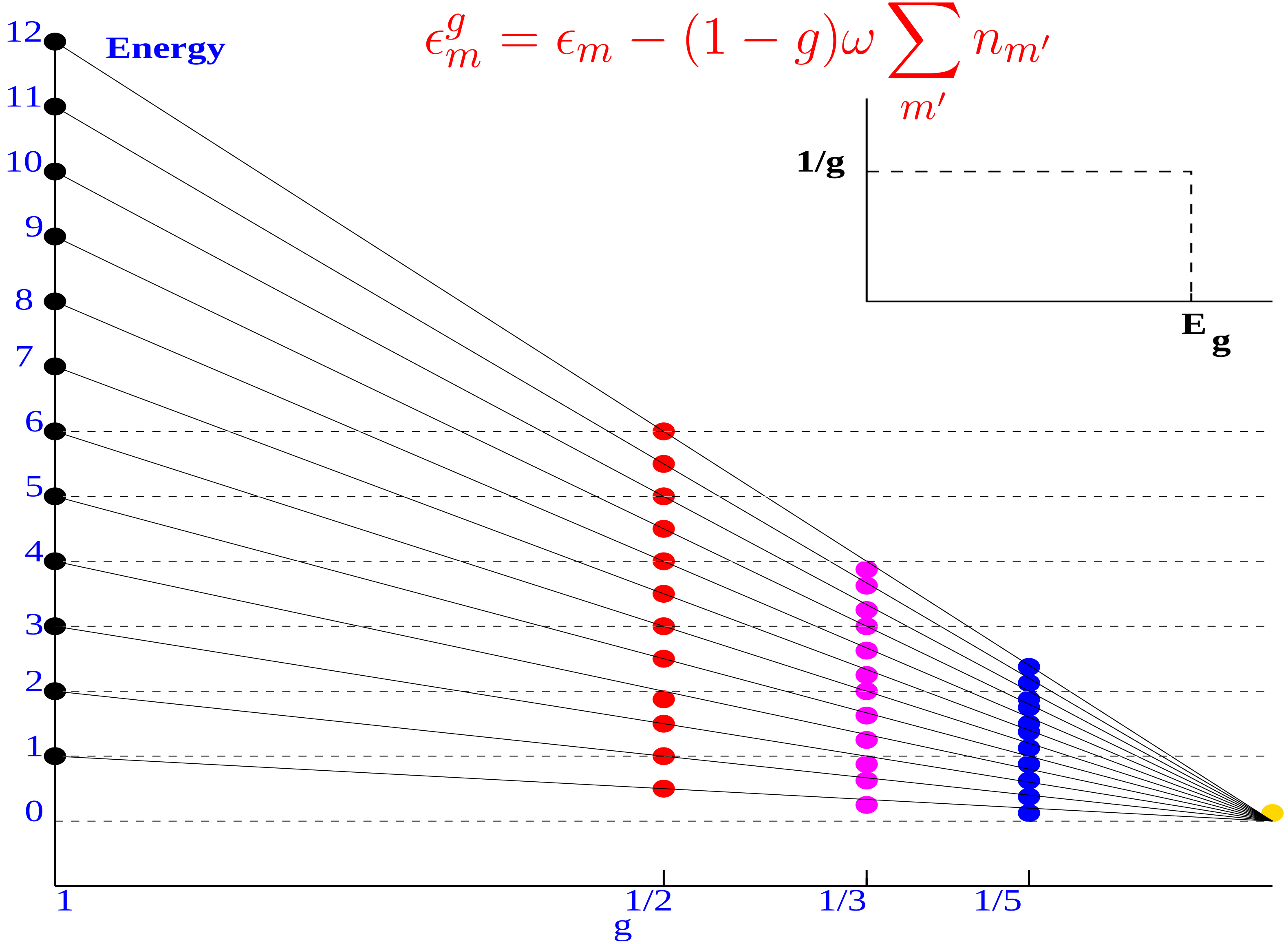
Interacting: $g \neq 0$

$$\begin{aligned}
E(g) &= E(g=0) - (1-g)\omega \tfrac{N(N-1)}{2} \\
E(g) &= \textstyle\sum_m n_m \epsilon_m - (1-g)\omega \sum_{m<m'} n_m n_{m'} \\
E(g) &= \textstyle\sum_m n_m \epsilon_m^g \; : \quad \epsilon_m^g = \epsilon_m - (1-g)\omega \sum_{m'} n_{m'}
\end{aligned}$$

Murthy, Shankar, PRL 73(94), 75(95)

Ground state of g-ons

$$\epsilon_m^g = \epsilon_m - (1-g)\omega \sum_{m'} n_{m'}$$



Generalised Pauli Principle

We can now remove the scaffolding of the Model and *define exclusion statistics system* imposing the constraints:

Let $g = 1/m$ where m is integer. Then

> Let $m = 1/g$, and let N_i be the number of particles in the occupied states below some ith level, $N_i = \sum_{j<i} n_i$. Then an occupation $n_i (n_i \leq m)$ is allowed iff $(N_i \; mod \; m) \leq (m - n_i)$.

For $g = 1 (fermions)$ –identical to Pauli principle, and imposes no constraints when $g = 0 (bosons)$

Murthy, Shankar PRB, 60(99)

Results

- CSM provides a realisation of ideal exclusion gas in 1-d: Interacting Fermions⇒Non-int. q-particles with Ex. statistics
- So does the low energy quasi-particle spectrum of Luttinger liquid
- Distribution function n_k of the model - same as the one derived using Haldane Dimension Formula by Y.S. Wu
- An approximate 2-d realisation is provided by Fermions interacting with a very short range potential

Bhaduri, Murthy, Srivastava, PRL 76(1996)

Finally

“The fundamental character of exclusion statistics caused by interactions is that they cause scale-invariant energy shifts. As a result of interactions among Fermions the effective single particle levels move up and down causing changes in the occupancy, in a given energy bin–det. by inherent scale in the problem"

MS, PRL, 72(94)

“In an important paper, Murthy and Shankar showed ... the linchpin of their argument is that in a theory with a high energy cutoff the transmutation of statistics by attaching flux tubes will generally push some states beyond the cutoff, thereby reducing the Hilbert space dimension. This generates a FES which persists even when the cutoff is taken to infinity"

Nayak- Wilczek, PRL, 72 (94)