Fractional exclusion statistics: A generalised Pauli principle

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work done with R. Shankar
Outline of talk

- Exclusion Statistics: Pauli - Haldane
- Realisation: Interacting systems in one and two dimensions
- New rules of occupancy: Generalised Pauli principle
Identical particles are indistinguishable—Consider a two particle wave function in quantum mechanics:

\[ |\Psi(x_1, x_2)|^2 = |\Psi(x_2, x_1)|^2 \]

Thus

\[ \Psi(x_1, x_2) = \Psi(x_2, x_1) \quad \text{Symmetric, Bosons} \]

\[ \Psi(x_1, x_2) = -\Psi(x_2, x_1) \quad \text{Anti-symmetric, Fermions} \]

Furthermore

\[ \Psi(x, x) = 0 \quad \text{Fermions} \]

Leads to Pauli Exclusion Principle

Thus Exclusion \(\Rightarrow\) State Counting
Anyons

Can we have

$$\Psi(\vec{x}_1, \vec{x}_2) = \exp i\theta(\vec{x}_1, \vec{x}_2) \quad \Psi(\vec{x}_2, \vec{x}_1)$$

Consistency with QM demands that

- $\theta(\vec{x}_1, \vec{x}_2) = \theta$ – Constant
- $\theta = 0$ (Bosons) $\theta = \pi$ (Fermions) in $-d > 2$– space dimensions

- $\theta$ may be arbitrary in $d = 1, 2 \Rightarrow$ Anyons

Thus Exchange Statistics may be generalised, but only in lower space dimensions.

Is it possible generalise Exclusion Statistics ala Pauli?
Pauli Exclusion Principle

For Fermions

Exchange (anti-)symmetry $\Rightarrow$ Pauli Exclusion Principle

Let $n_k$ – Occupancy of a state labelled by $k$.

$n_k = 1, 0$ \hspace{1cm} For identical Fermions

$n_k = \text{arbitrary}$ \hspace{1cm} For identical Bosons

Is it possible for $n_k$ to be some thing else? Say $n_k = 2, 1, 0$

Haldane [ *PRL* 67, 937(1991)] proposed one such generalisation.
Haldane Proposal

Consider a system of $N$ identical particles described by

$$\Psi(\vec{x}_1, \vec{x}_2, \vec{x}_3, \cdots, \vec{x}_N)$$

Freeze $N - 1$ coordinates—expand $\Psi$. The single particle space spanned by $\Psi$ is $d_N$. – Dimension of the single particle space in the presence of $N - 1$ others (identical).

How does $d_N$ change as $N$ changes? Propose

$$\Delta d_N = -g \Delta N$$

$g$ is the Exclusion Statistics Parameter

$g = 1$ For Fermions

$g = 0$ For Bosons

Can $g$ be Fractional?
Consider a lattice with \( d \) sites:

<table>
<thead>
<tr>
<th></th>
<th>Fermions</th>
<th>Bosons</th>
</tr>
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<tbody>
<tr>
<td>( N = 1 )</td>
<td>( d )</td>
<td>( d )</td>
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<tr>
<td>( N = 2 )</td>
<td>( d-1 )</td>
<td>( d )</td>
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<td>( N = 3 )</td>
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<td>( d )</td>
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<tr>
<td>( N )</td>
<td>( d_F^N = d - (N - 1) )</td>
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Dimension of the \( N \)-Particle space:

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### $g$-ons

<table>
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<th>Bosons ($g = 0$)</th>
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<td>$d-1$</td>
<td>$d-g$</td>
<td>$d$</td>
</tr>
<tr>
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<td>$d-2$</td>
<td>$d-2g$</td>
<td>$d$</td>
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Quantum theories must be based on observables!! Hilbert space dimension?

$$D_N \Rightarrow \lim_{\beta \to 0} Z_N(\beta) = \lim_{\beta \to 0} \sum_{\text{states}} \exp(-\beta E_{\text{state}})$$

$$\beta = 1/T$$

The statistical parameter $g$ is then

$$\frac{1}{2} - g = \lim_{\beta \to 0} \frac{CZ_1}{N(N-1)} \left[ N! \frac{Z_N}{(Z_1)^N} - 1 \right]$$

where

$$C = 2^{\eta}; \quad \eta = \text{space-dimension}$$

Murthy, Shankar: PRL, 72,(94)
Equation of state

An immediate consequence is the Equation of State—Relation between Pressure, density and temperature of a Gas in 2-dimensions

Classical Gas \[ \frac{P}{kT} = \rho \]

Bose Gas \[ \frac{P}{kT} = \rho \left[1 - \frac{1}{4}(\rho \lambda^2) + \cdots \right] : \quad g = 0 \]

Fermi Gas \[ \frac{P}{kT} = \rho \left[1 + \frac{1}{4}(\rho \lambda^2) + \cdots \right] : \quad g = 1 \]

Haldane Gas \[ \frac{P}{kT} = \rho \left[1 - \frac{1}{2}\left(\frac{1}{2} - g\right)(\rho \lambda^2) + \cdots \right] \]

\( \lambda \)— Thermal wavelength; \( k \)— Boltzmann constant

Murthy, Shankar: PRL, 72,(94)
The distribution function probability of occupation of a state is obtained by maximising entropy $s = k \log D_N^g$

**Bose Gas**

$$n_k = \frac{1}{\exp[(\epsilon_k - \mu)/kT] - 1} : \quad g = 0$$

**Fermi Gas**

$$n_k = \frac{1}{\exp[(\epsilon_k - \mu)/kT] - 1} : \quad g = 1$$

**Haldane Gas**

$$n_k = \frac{1}{w(\exp(\epsilon_k - \mu)/kT) + g}$$

where $w$ is a solution of the equation

$$w^g(1 + w)^{1-g} = \exp[(\epsilon_i - \mu)/kT] : \quad \text{Ramanujan’s Eq. in LNs}$$

Wu, PRL, 73,(94), Isakov PRL 73, (94)
We have studied **Exclusion Statistics** as a possibility

- Unlike anyons (exchange - specific to two-space dimensions), exclusion statistics is defined in any space dimension.

- At zero temperature the maximal occupancy of a state is \( \frac{1}{g} \). Finite temp. dist. is given by \( n_k(g) \) – nicely interpolates between Bose and Fermi statistics.

- Statistical mechanics of \( g \)-ons are well studied and understood.

Q: How does it arise in physical systems?

We illustrate using a one-dimensional Model: CSM
Calogero Sutherland Model

Realisation in 1-d: An exactly solvable many body Hamiltonian with non-trivial correlations

The Hamiltonian: \( \hbar = 1, m = 1, c = 1 \)

\[
H = \frac{1}{2} \left[ \sum_i p_i^2 + \omega^2 \sum_i x_i^2 + \lambda \sum_{i,j} \frac{1}{(x_i - x_j)^2} \right]
\]

The Spectrum of system of \( N \) interacting Fermions:

\[
E(g) = E(g = 0) - (1 - g)\omega \frac{N(N - 1)}{2}
\]

where \( \lambda = g(g - 1) \)
Single particle picture

Non-interacting: \( g = 0 \)

\[
E(g = 0) = \sum_m n_m \epsilon_m : \epsilon_m = m\omega
\]

Interacting: \( g \neq 0 \)

\[
E(g) = E(g = 0) - (1 - g)\omega \frac{N(N-1)}{2}
\]

\[
E(g) = \sum_m n_m \epsilon_m - (1 - g)\omega \sum_{m < m'} n_m n_{m'}
\]

\[
E(g) = \sum_m n_m \epsilon_m^g : \epsilon_m^g = \epsilon_m - (1 - g)\omega \sum_{m'} n_{m'}
\]

Murthy, Shankar, PRL 73(94), 75(95)
Ground state of g-ons

\[ \epsilon^g_m = \epsilon_m - (1 - g) \omega \sum_{m'} n_{m'} \]
Generalised Pauli Principle

We can now remove the scaffolding of the Model and define exclusion statistics system imposing the constraints:

Let \( g = 1/m \) where \( m \) is integer. Then

Let \( m = 1/g \), and let \( N_i \) be the number of particles in the occupied states below some \( i \)th level, \( N_i = \sum_{j<i} n_i \). Then an occupation \( n_i (n_i \leq m) \) is allowed iff \( (N_i \mod m) \leq (m - n_i) \).

For \( g = 1(\text{fermions}) \) – identical to Pauli principle, and imposes no constraints when \( g = 0(\text{bosons}) \)

Murthy, Shankar PRB, 60(99)
Results

- CSM provides a realisation of ideal exclusion gas in 1-d: Interacting Fermions $\Rightarrow$ Non-int. q-particles with Ex. statistics
- So does the low energy quasi-particle spectrum of Luttinger liquid
- Distribution function $n_k$ of the model - same as the one derived using Haldane Dimension Formula by Y.S. Wu
- An approximate 2-d realisation is provided by Fermions interacting with a very short range potential

*Bhaduri, Murthy, Srivastava, PRL 76(1996)*
Finally

“The fundamental character of exclusion statistics caused by interactions is that they cause scale-invariant energy shifts. As a result of interactions among Fermions the effective single particle levels move up and down causing changes in the occupancy, in a given energy bin—det. by inherent scale in the problem"

MS, PRL, 72(94)

“In an important paper, Murthy and Shankar showed ... the linchpin of their argument is that in a theory with a high energy cutoff the transmutation of statistics by attaching flux tubes will generally push some states beyond the cutoff, thereby reducing the Hilbert space dimension. This generates a FES which persists even when the cutoff is taken to infinity"

Nayak- Wilczek, PRL, 72 (94)