HIGH ENERGY PHYSICS

Draft notes

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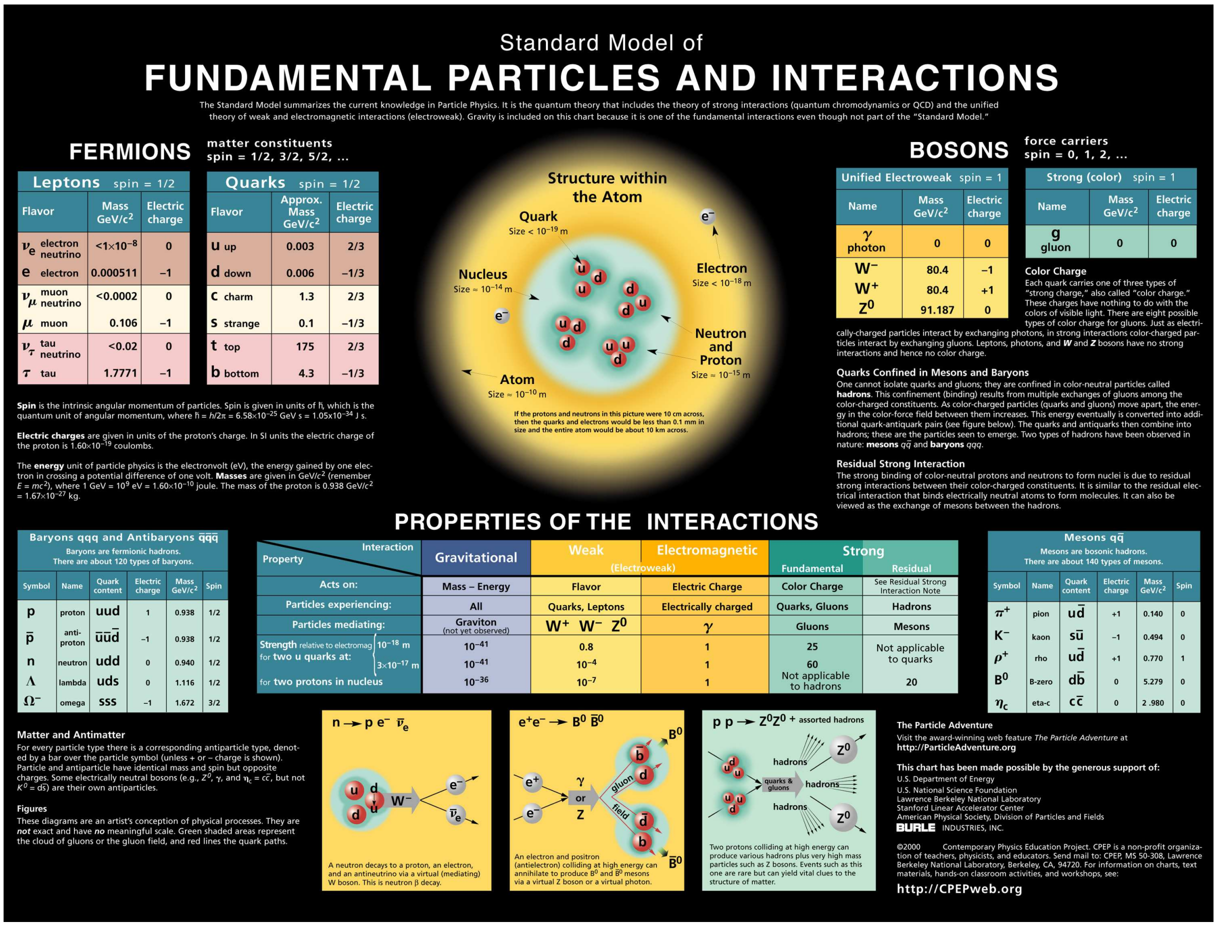

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Chapter 1

Introduction: Particles and Interactions

The study of the underlying structure of elementary particles that make the matter and their interactions is usually referred to as *Elementary Particle Physics*. The study of "elementary particles" as we presently know begins with the discovery of the electron in 1897. By the early 1930's we knew about the existence of the electron, proton and the neutron. Neutrino, more precisely the electron neutrino, was conjectured around this time by Pauli to account for the continuous electron spectrum emitted in the β-decay process to save the collapse of the principle of conservation of energy.

The approximate equality of the masses of two nuclei, He^3(ppn) and H^3(pnn) posed a puzzle since their charges were very different. A new force responsible for binding the protons and neutrons, strong force, had to be conjectured. Further Heisenberg proposed that the masses of these two nuclei are approximately equal due to a symmetry of this force, namely the charge symmetry of the strong force. In today's language we understand this charge symmetry as resulting from the isospin invariance of the strong interaction Hamiltonian.

To account for the short range ($\approx 10^{-13} cms$) of these forces, unlike the electomagnetic force which has infinite range, Yukawa conjectured a new massive particle called the pi-meson or pion. Just as the massless photon mediated the electro-magnetic interactions, the pion was assumed to mediate the strong or nuclear forces. Yukawa assumed that the potential energy between any two protons in a nucleus is of the form

$$V(r) \sim \frac{\exp(-\mu r)}{r}$$

so that the range of the nuclear force is short determined by the parameter μ which is indeed the mass of the pion. Thus apart from electromagnetic interaction, by now we had strong or nuclear force (responsible for binding protons and neutrons) and weak interactions responsible for β decay of the neutrons.

With the advent of powerful accelerators, more of these so called "elementary particles" were discovered and some of them behaved quite "strangely". For example

$$\pi^- + p \to \Lambda + K^0, \tag{1.1}$$

where the interaction between a negatively charged pion and a proton produced two new particles Λ and K^0. The typical time scale for the production of these particles is about

10^{-23} secs*. However the unstable Λ particle decayed rather slowly,

$$\Lambda \to \pi^- + p \tag{1.2}$$

with a time scale of 10^{-8} secs. To understand why a particle produced on a strong scale decays weakly, a new conservation law called "conservation of strangeness" had to be brought in to account for their stability in strong interactions. This new quantum number called strangeness quantum number is conserved in strong interactions where as weak interactions do not respect it. Thus when Λ and K are produced in association the process conserves the new quantum number (the initial state has no strangeness) where as the decay of Λ can not go through without violating the same.

With the discovery of more particles, and their anti-particles, many such discrete quantum numbers were added to the list, the zoo was enlarging into a periodic table analogous to the Mendeleev's periodic table of atoms. Further some of the so called "elementary particles" also began to display signatures of an internal structure. Just as the systematics of periodic table indicated an underlying structure of atoms made up of electrons and nuclei, the table of particle data indicated a possible organisation of this zoo in terms of more fundamental constitutents: the quarks and leptons whose interactions are mediated by gluons (strong force), photons (electromagnetic force) and the W, Z bosons (weak force).

The reactions involving elementary particles obey the following exact conservation laws: energy, linear and angular momentum, charge. In addition one of the clearest results experimentally observed in the reactions of elementary particles is the conservation of Fermion number (provided the antiparticles which are also fermions are assigned a negative fermion number as compared to the particle). On the otherhand there is no such principle with respect to bosons. For example, the photon number or pion number is not conserved. This suggests that the conservation of fermion number is a fundamental feature of all interaction. In particular the conservation of fermion number comes in various shades which puts further constraints:

- From observed transition rates, and the absence of processes which are kinematically allowed conserving the well known charge conservation principle, we can infer the presence of a conservation law. For example we know that the *baryon number is conserved.* A process such as,

  $$p \to e^+ + \pi^0$$

  is not observed even though it is kinematically allowed and charge is conserved. We account for this lack of decay by assigning a baryon number to the proton (+1 and -1 for antiproton) which should be conserved in any interaction involving baryons.

- Similarly non-observation of processes which do not conserve the number of leptons, leads us to the conclusion that the *Lepton number is conserved.* For example

  $$e^- + e^- \to \pi^- + \pi^-$$

  is not allowed even though it conserves charge and is kinematically allowed.

*The time scales of strong and electromagnetic interactions are typically of the order of 10^{-23} and 10^{-16} seconds. The corresponding time-scale for weak interaction however varies widely, the faster of these decays have a time scale of around 10^{-8} seconds. The reaction rates which determine these time scales depend on the corresponding matrix elements for transition which we shall discuss later.

- We have already referred to the conservation of strangeness which is, unlike the previous two, is conserved in strong and electro-magnetic interactions, it is not conserved in weak interactions which makes strange particles to decay to non-strange hadrons.

- Similarly the conservation of Iso-spin is only approximate-it is exact in reactions when only strong forces are operating.

- To this list we have to also add parity(P), charge conjugation(C) and time reversal(T). There is experimental evidence that individually these are violated in weak interactions. A combined operation of these, namely CPT is however an exact symmetry.

The conservation laws are intimately connected with some symmetries in nature. The law of conservation of momentum results when the system is translationally invariant. Similarly the conservation of angular momentum of an isolated system is conserved if it is rotationally invariant. We will discuss later in more detail the relation between symmetries and conservation laws.

The present knowledge of particles, both experimental and theoretical, is put together in a model which describes the reactions of all known particles in terms of the fundamental forces between quarks and leptons. The unified model of the interactions of quarks and leptons is called the "Standard Model" of particle physics which has been eminently successful in explaining much of the observed data interms of few parameters, like the masses and coupling constants, whose origins are not well understood yet.

The table**??** summarises the stable and unstable particles which constitute the basic elements of the Standard Model.

Particle	mass (MeV)	charge	spin	stability	interaction
(a) Leptons					
e^-	0.511	-1	1/2	stable	weak, electromagnetic
μ^-	106	-1	1/2	unstable	weak, electromagnetic
τ^-	1777.1	-1	1/2	unstable	weak, electromagnetic
ν_e	??	0	1/2	stable	weak
ν_μ	??	0	1/2	stable	weak
ν_τ	??	0	1/2	stable	weak
(b) Quarks:					
u	3	2/3	1/2	-	weak, electromagnetic, strong
d	6	-1/3	1/2	-	weak, electromagnetic, strong
c	1300	2/3	1/2	-	weak, electromagnetic, strong
s	100	-1/3	1/2	-	weak, electromagnetic, strong
t	175000	2/3	1/2	-	weak, electromagnetic, strong
b	4300	-1/3	1/2	-	weak, electromagnetic, strong
(c) Gauge Bosons:					
g	0	0	1	stable	strong
γ,	0	0	1	stable	electromagnetic
$W^\pm$	80400	$\pm$	1	unstable	weak
Z	91187	0	1	unstable	weak

Table 1.1: In addition there are antiparticles of quarks and leptons. The masses of the quarks shown are approximate. Gravitational interaction is ofcourse common for all the particles.

Some fundamental problems remain to be answered: The standard model is based on a symmetry which leaves the neutrinos massless. However, recent evidence suggests that the neutrinos are indeed massive and have other properties which can not be accommodated in the standard model in its present form. The Higgs scalar, responsible for generating the masses of the gauge bosons, is yet to be discovered.

The questions "beyond standard model" of particle physics will have to be addressed in the on going search for a fundamental theory of all matter. The question of accommodating Gravity in a unified theory of all forces that govern the various phenomena is still an open question.

Problems

Find out which of the following reactions is allowed or forbidden (and why):

1. $n \to e^+ + e^-$
2. $n \to p + e^- + \gamma$
3. $n \to p + e^- + \bar{\nu}_e$
4. $p \to e^+ + \pi^0$

Chapter 2

Scales and Units

2.1 Natural units

Natural units are units of measurement defined such that some physical constants are set to unity. The physical constants are so chosen as to simplify the formulae. The energy of all fundamental particles is measured in units of eV. An eV is defined to be the kinetic energy that a particle carrying the fundamental charge e of the electron gains when it is accelerated across a potential difference of 1 volt. Using the Einstein's relation $E = Mc^2$, all masses are measured in units of eV/c^2. The velocity is expressed as a fraction of the velocity of light c, $\beta = v/c$ and the momentum p is measured in units of eV/c.

It is a common practice therefore in particle physics to set fundamental unit of action $\hbar = 1$ and the velocity of light in vacuum, $c = 1$. The system of units are then completely defined if one specifies either the unit of energy or unit of length, for example. Thus all masses (m), momenta (mc) and energies (mc^2) are given in units of eV (or KeV, MeV, GeV). The length ($\hbar/mc$) and time ($\hbar/mc^2$) are then given in units of eV^{-1}. One may alternate between the units of energy and length using the conversion factor $\hbar c = 1 \approx 200 MeV fm$.

For example the de Broglies wavelength associated with a 1 GeV photon is given by,

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{E} = \frac{2\pi(200)\text{MeV fm}}{1000\text{MeV}} = 0.4\pi\text{fm}.$$

2.2 Scales

Every particle massive or massless is subject to gravitational interaction. Particle with electric charge feel the Coulomb interaction. There are two more forces responsible for happenings in the subatomic domain, namely the strong force responsible for binding nucleons inside a nucleus and the weak force which figures itself in decay processes. There is no classical analogue for these two short ranged forces unlike the electromagnetic and gravity which are long ranged.

Though all the forces act at the same time, they may be distinguished because they have different strengths and ranges. The four distinct interactions are governed by phenomenological coupling constants given by,

$$\begin{aligned}
\text{strong: } \alpha_s &\approx 1 \\
\text{electromagnetic: } \alpha &\approx 1/137 \\
\text{weak: } Gm_p^2 &\approx 10^{-5} \\
\text{gravity: } \mathcal{G}m_p^2 &\approx 6 \times 10^{-39}.
\end{aligned}$$

The order of magnitude estimates of many physical quantities may be given based on simple physical considerations and dimensional analysis.

1. Radius of the Hydrogen atom:

   $$E = \frac{p^2}{2m_e} - \frac{\alpha}{r}$$

   where the first term is the kinetic energy and the second term is the electrostatic energy governed by the fine structure constant. The momentum p scales as $1/r$ and hence

   $$E = \frac{1}{2m_e r^2} - \frac{\alpha}{r}$$

   whose minimum determines r,

   $$r = \frac{1}{m_e \alpha} = 137/0.5\ MeV^{-1} \approx 5 \times 10^{-9}\ cm$$

   The three important scales in quantum electrodynamics differ from each other by powers of α, namely

   - Bohr radius : $\frac{1}{m_e \alpha}$
   - electron compton wave length : $\frac{1}{m_e}$
   - electron classical radius : $\frac{\alpha}{m_e}$

2. Strong interactions:

   The charge radius of the proton as measured by experiments (electron- proton scattering) is about 0.81 fm ($10^{-13} cm$). This is infact larger than the compton wavelength of the proton. Because the strong interaction strength is close to unity, the cross-section for proton-proton scattering is given by

   $$\sigma_{pp} = \pi r_p^2 = 3 \times 10^{-26} cm^2 = 30 mb$$

   using the classical analogy for the cross-section. Indeed the experimental value is close to this, about 45mb close to GeV energy.

3. Electromagnetic scattering: $e^+ + e^- \rightarrow \mu^+ + \mu^-$

   The probability amplitude for this process is propotional to e^2 or α, the fine structure constant. Hence the cross-section must be proportional to α^2. The cross-section may therefore be written as

   $$\sigma_{e^+e^-} = \alpha^2 f(s, m_e, m_\mu)$$

where f is a function of the invariant $s = (p_1 + p_2)^2$, and the masses. Note that the total cross-section is in general a function of Lorentz invariant variables, which in this case is the square of the sum of the two four momenta in the initial state (or final state).

At very high energies we may neglect the masses of the particles, and purely by dimensional considerations the cross section must be given by

$$\sigma_{e^+e^-} \approx \alpha^2/s$$

Indeed the exact cross section is

$$\sigma_{e^+e^-} \approx 4\pi\alpha^2/3s$$

4. Weak interaction: $\nu + N \rightarrow ...$

   The total cross section may be written as

$$\sigma_{\nu N} = G^2 f(s, m_e)$$

   Unlike the electromagnetic and strong interactions the coupling strength $G = [L^2]$ is not dimensionless. Therefore from dimensional arguments the cross section must go as

$$\sigma_{\nu N} = G^2 s = 10^{-38} cm^2$$

   for $s = 1GeV^2$. This is again close to the experimental result.

Problems:

Below are given some problems of a very general nature:

1. Use natural units and express the following important length scales in units of Fermi.
   - Bohr radius
   - electron classical radius
   - compton wave length of electron, pion and the proton.
2. In units of the electron Bohr radius, what would be the Bohr radius for a muonic atom and pionic atom.
3. Consider the decay of a particle of mass M to two particles of masses M_1 and M_2. Show that the energy momentum of the decay products could be entirely fixed in terms of the masses alone.
4. The size of the proton (charge radius) is approximately 1 fm. Typically one needs a probe whose wave length is much less than this size to probe the structure of the proton. Suppose we assume that a photon probe has a wavelength less than 1/10 fm, calculate the energy of the photon required to probe the internal structure of the photon.

5. If a proton is allowed to decay, what possible quantum number/s are violated? What about the electron? Can it ever decay into any of the known particles?

6. The pions are unstable particles. Investigate the decay modes of charged and neutral pions. Assuming an equal number $\pi^{\pm}$ are enter the earths atmosphere (approximately correct), what particles are left in what ratios after all the pions and even their decay products have decayed.

7. Neutral pion, of mass 135 MeV, decays into two photons. If the mean life of the neutral pion is 10^{-16} seconds, calculate the distance that a 1GeV pion will travel prior to decay? What is the approximate opening angle of the two photons in the laboratory frame?

8. Suppose the proton could decay with a life time of 10^{30} years, how many cubic meters of water would have to be observed if one wanted to have about 100 events in a year.

9. Low energy neutrinos pass through a piece of solid iron- if the neutrino-nucleon cross section is about $\sigma \approx 10^{-47}\ m^2$, estimate the mean free path of the neutrinos in iron (density of iron is 8 times the density of water).

10. Suppose a neutral pion decays at rest to an electron and positron pair; if this occurs in a magneitc field of magnitude 2 Tesla, what is the radius of the orbits that these charged particles move.

11. In 1987 scientists from Kamioka observatory in Japan observed neutrinos from a supernova-SN1987a-in the large Magellanic Cloud, which is at a distance of 55 kilo Parsecs from Earth. Antineutrinos of energy between 7 and 20 MeV were detected in an interval of about 12 seconds. Assuming all the antineutrinos were emitted almost instantaneously (actually few millisecs) obtain a bound on the neutrino mass.

12. Consider the decay $\omega \rightarrow \pi^+ + \pi^- + \pi^0$. If the mass of the *omega* particle is 780 MeV and that of the pion is 138 MeV on the average, what is the largest possible momentum that a single pion can have?

13. Verify that the spin of the neutral pion can be deduced from the fact that it decays into two photons. Photons have spin-1 and are massless.

14. Free neutron is an unstable particle with a life time of about 13 minutes. Investigate the decay mode of the neutron. Is it possible to have more than one decay mode for the neutron?

15. Neutrons bound in nucleus like He^4 or O^{16} remain stable. Why? Apply the same reasons to understand why neutrons in some heavier nuclei are allowed to decay.

16. Consider a world in which the masses of neutrons and protons are equal. What would be the consequences, how would this world look like?

Chapter 3

Symmetries and invariances principles

Symmetry considerations are a powerful tool to explore and understand the behaviour of elementary particles. They provide the backbone of our theoretical formulations. Even when some of the apparent symmetries are not exact they provide a basis for classification of states assuming exact symmetry and allow us to look at possible sources and pattern of symmetry breaking. Any particle data table invariably lists particles with their quantum numbers arising from symmetry operations.

The known symmetries may be classified as follows:

- Permutation symmetry which results in Bose-Einstein (bosons) and Fermi-Dirac Statistics (fermions).
- Continuous symmetries: Translation in space and time, rotation etc.
- Discrete symmetries: space inversion, time reversal, charge-conjugation, etc.
- Unitary symmetries: U(1) symmetries associated with charge conservation, baryon number, lepton number, SU(2) (isospin) symmetry, SU(3)(flavour) symmetry, SU(3) (colour) symmetry.

We shall discuss the first three briefly here and will not discuss Unitary symmetries here. It is most conveniently discussed in the context of quark models. It suffices to say that there are many conservation laws which arise from invariance of the Hamiltonian under the so-called U(1) (or phase) transformations, like the conservation of lepton number, Baryon number, etc in a manner that is analogous to the charge conservation.

3.1 Permutation Symmetry

All physical identical many particle states must have definite symmetry under permutation symmetry. It is an observed fact that all particles are either bosons are fermions depending on their behaviour with respect to another particle of the same kind. Thus the state of a system of identical bosons is symmetric under permutations where as a system of identical fermions is anti-symmetric under permutations. While one may be able to define systems with mixed symmetry (symmetric under some exchanges and antisymmetric under others) they are not realised in nature. Indeed this symmetry played fundamental a role in the formulation of quarks with colour as the basic entities in the theory of strong interactions. More about this later.

3.2 Continuous symmetries

A symmetry under space translation implies that the interaction energy between two particles is independent of their positions but depends only on their relative distance. Classically the Lagrangian L which is a function of generalised coordinates q_i and generalised velocities $\dot{q}_i$ is unchanged under the displacement $q_i \rightarrow q_i + \delta q_i$. That is

$$\frac{\partial L}{\partial q_i} = 0$$

Then by virtue of the equations of motion, we have

$$\frac{dp_i}{dt} = 0$$

which is a statement of the conservation of momentum. Similarly time translation invariance leads to the energy conservation.

In quantum mechanics if there is a continuous operation like rotation or translation, say G , it may be generated from transformations which differ infinitesimally from the identity transformation

$$G = 1 - i\epsilon g,$$

where g is the Hermitian generator of the symmetry operator in question. For example for rotations about z-axis, it is the z-component of the angular momentum. By definition G is a unitary transformation. Suppose the Hamiltonian is invariant under G, then we have

$$G^{\dagger} H G = H$$

This is equivalent to

$$[g, H] = 0$$

and by virtue of Heisenberg equation of motion, we have

$$\frac{dg}{dt} = -i[g, H] = 0$$

and hence g, or more precisely its quantum expectation value, is a constant of motion. For example if H is invariant under rotations then the angular momentum about the axis of rotation is a constant of motion.

Furthermore, when two operators commute, they can be simultaneously diagonalised. The set of eigenfunctions will be labelled by the eigenvalues, quantum numbers, of both operators. If the Hamiltonian for a transition is invariant under the transformation, then the quantum numbers labelling the initial state will also be conserved. This is a very powerful result which results in selection rules for reactions to occur.

3.3 Discrete symmetries

All symmetry operations in quantum mechanics are not necessarily continuous. The Hamiltonian may also be invariant under discrete transformations, for example space-time inversion. We consider three important symmetries here, namely, Parity, Charge Conjugation and Time Reversal.

3.3.1 Parity

We first consider parity or space inversion. Classically under a parity transformation $\vec{r} \rightarrow -\vec{r}$ and $\vec{p} \rightarrow -\vec{p}$. That is a right-handed coordinate system is changed to a left-handed coordinate system. This can not be achieved by rotation which is a continuous transformation in three-space dimensions. Hence it is a discrete symmetry. Infact it is easy to verify that the determination of the transformation matrix is positive for rotation matrices where as for Parity it is negative.

If $|\alpha\rangle$ is a quantum mechanical state then we require under space inversion,

$$\langle\alpha|P^\dagger\vec{r}P|\alpha\rangle = -\langle\alpha|\vec{r}|\alpha\rangle$$

We accomplish by stating that under parity transformation,

$$P^\dagger\vec{r}P = -\vec{r}$$

or

$$\vec{r}P = -P\vec{r}$$

where we have used the fact that P is unitary. Thus the position and parity anticommute. Further, since two inversions cancel the effect of each other, we have,

$$P^2 = 1$$

or equivalently,

$$P^{-1} = P^\dagger = P$$

The Parity operator is not only unitary but also hermitian with eigenvalues +1 or -1.

By definition the angular momentum is $\vec{L} = \vec{r} \times \vec{p}$. Clearly,

$$[L, P] = 0$$

Since L is the generator of rotations, parity commutes with rotations,

$$[R, P] = 0$$

.

If the Hamiltonian is invariant under parity transformation, then the states are definite eigenstates of the parity. Consider the wavefunction of a rotationally invariant system in three dimensions:

$$\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta, \phi)$$

for example hydrogen atom. Under parity transformationwe have, $r \rightarrow r, \theta \rightarrow \pi-\theta, \phi \rightarrow \pi+\phi$ in spherical coordinates. Thus

$$P\psi_{nlm} = (-1)^l\psi_{nlm}$$

using the property of the spherical harmonics.

"Intrinsic Parity" is a notion that is applied to all the elementary particles. The word intrinsic is used in the same sense in which spin is referred to as intrinsic. To clarify consider for example the orbital angular momentum operator $L_i = (\vec{r} \times \vec{p})_i$. In quantum mechanics the operator L_i is defined as,

$$L_i = -i(x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j})$$

where the indices are taken around cyclically. Further L_i satisfy the angular momentum algebra,

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

The commutation relation is very general and applies to spin-angular momentum also,

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

and to the total angular momentum

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

where $J_i = L_i + S_i$. However there is no spacial representation for S analogous to L. In this sense the spin has no classical analogue and is an intrinsic property of quantum mechanical objects. Consequently the parity of a state described by the eigenfunction of orbital angular momentum is given by,

$$P\psi_{nlm} = \eta_\psi (-1)^l \psi_{nlm}$$

where η_ψ denotes the intrinsic parity of the quantum particle. Further as in the other case,

$$\eta_\psi^2 = 1$$

It is in this sense we refer to parity as an intrinsic property of the state when it is an eigenstate of parity. There is no classical analogue.

Intrinsic parity of the Photon : As an example consider the intrinsic parity of photon. The electromagnetic interaction conserves parity. The current j_μ of a charged particle couples to the electromagnetic field (photon) through

$$j_\mu A^\mu$$

where

$$j_\mu = (\vec{j}, \rho)$$

and

$$A^\mu = (\vec{A}, A_0)$$

in the four-vector notation. Under parity,

$$(\vec{j}, \rho) \to (-\vec{j}, \rho)$$

since $\vec{j} = \rho\vec{v}$, where $\vec{v}$ is the velocity. The electromagnetic interaction is invariant under parity only if

$$(\vec{A}, A_0) \to (-\vec{A}, A_0)$$

Thus the intrinsic parity of the photon has to be negative just like any position vector.

Intrinsic parity of the pion : When parity is conserved the intrinsic parity of a particle may be determined relative to others whose intrinsic parity is known: For example consider a reaction

$$A \to B + C$$

Conservation of parity implies

$$\eta_A = \eta_B \eta_C (-1)^L$$

where L is the relative angular momentum of the final state particles.

Thus the intrinsic parity of pion may be determined using the scattering process

$$\pi^- + d \to n + n$$

Using the relation

$$(parity\ \pi)(parity\ d) = (parity\ nn)$$

it is easy to show that the intrinsic parity of the pion should be negative. One needs to assume that the intrinsic parity of proton and neutron to be the same. Infact we **define** the intrinsic parity of the proton to be +1 and define the parity of various other particles relative to that of the proton. While the pion has spin zero, deuteron has spin 1. Since the pion is absorbed almost at rest by the deuteron the relative angular momentum in the initial state is zero. Thus the total angular momentum in the initial state is $J = 1$ entirely due to the spin of the deuteron. Since the neutron is a $S = 1/2$ particle, using angular momentum conservation we have the following options in the final state:

$$\begin{aligned}
|\psi_{nn}^{(1)}\rangle &= |J = 1, S = 1, L = 0, 2\rangle \\
|\psi_{nn}^{(2)}\rangle &= |J = 1, S = 1, L = 1\rangle \\
|\psi_{nn}^{(3)}\rangle &= |J = 1, S = 0, L = 1\rangle
\end{aligned}$$

Antisymmetry excludes all but the second wave function. Hence the parity of the pion is given by,

$$\eta_\pi \eta_d = \eta_n \eta n (-1)^L$$

$$\eta_\pi = -1$$

Hence the parity of the pion with respect to proton or neutron is negative.

A system whose dynamics is given by Schrödinger or Klein-Gordon equation, the wave function in the inverted system describes a particle with opposite momentum. However, with Dirac equation the situation is more complicated since it is first order in space coordinates-hence the form of the equation changes: Suppose ψ satisfies the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(\vec{x}, x_0) = 0.$$

Suppose in the space inverted system

$$(i\gamma_0\partial_0 - i\gamma_i\partial_i - m)\psi(\vec{x}, x_0) \xrightarrow{P} (i\gamma_0\partial_0' - i\gamma_i\partial_i' - m)\psi'(\vec{-x}, x_0)$$

Note that while parity invariance is respected by strong and electromagnetic interactions, it is violated in weak interactions. The famous $\tau - \theta$ puzzle was understood interms of the parity violation in weak interactions proposed by Lee and Yang and later experimentally demonstrated by Wu in the $\beta - decay$ of polarized Co^{60}.

3.3.2 Charge Conjugation

Charge conjugation operator C is in many ways similar to Parity. By definition it inverts all internal charges (electric, baryon number, lepton number etc) of a particle thus relating it to its anti-particle and vice versa. The space-time coordinates are unchanged. For example electric charge

$$Q \xrightarrow{C} -Q$$

that is

$$|\psi(Q, \vec{p}, \vec{s}\rangle \xrightarrow{C} |\psi(-Q, \vec{p}, \vec{s})\rangle$$

Thus the quantum mechanical state of a proton, say, under charge conjugation is transformed into the state of an anti-proton.

$$|p\rangle \xrightarrow{C} |\bar{p}\rangle$$

Therefore as in the case of parity we have,

$$C^2 = 1$$

or equivalently,

$$C^{-1} = C$$

Thus C is not only unitary but also hermitian with eigenvalues +1 or -1.

Since C reverses the charges, it also reverses the electric and magnetic fields. As a result the photon has negative eigenvalue under C. However Maxwell equations are invariant under charge conjugation. From the decay

$$\pi^0 \to \gamma + \gamma$$

we conclude that the pion is even under charge conjugation. While some charge neutral states like photon, neutral pion are eigenstates under C, it is not always so- for example

$$|n\rangle \xrightarrow{C} |\bar{n}\rangle$$

Invariance under P or C would then mean that the transitions would occur to only states with the same eigenvalue in the initial and final states. Note that the eigenvalues of these operators are multiplicative. Strong and electromagnetic interactions respect these symmetries, where as in weak interactions these are violated. However, the combination of P and C is still a symmetry to a good approximation though it is violated in some systems.

3.3.3 Time Reversal

The discussion of time reversal symmetry is some what more complicated. Classically both Newton's equations and Maxwell's equations are invariant under time reversal. We briefly discuss the situation in quantum mechanics where at the outset it appears not to be so since the Schroedinger equation is first order in time.

Suppose $\psi(x,t)$ is a solution of the Schroedinger equation,

$$i\frac{\partial\psi}{\partial t}(x,t) = (-\frac{1}{2m}\nabla^2 + V)\psi(x,t)$$

then it is easy to see that the time reversed state $\psi(x,-t)$ is not a solution because of the first order time derivative. However, it is easy to check that $\psi^*(x,-t)$ is a solution by complex conjugation:

$$i\frac{\partial\psi^*}{\partial t}(x,-t) = (-\frac{1}{2m}\nabla^2 + V)\psi^*(x,-t)$$

Thus we can conjecture that the time reversal has some thing to do with complex conjugation.

Another way of looking at this is to preserve the probability invariant under time reversal. Following Wigner we may then require

$$\langle\psi|\psi\rangle = \langle T\psi|T\psi\rangle$$

There are two ways of achieving this which is obvious if we look at two different quantum states. We may have

$$\langle\phi|\psi\rangle = \langle T\phi|T\psi\rangle$$

as in ordinary transformations or

$$\langle\phi|\psi\rangle^* = \langle T\phi|T\psi\rangle$$

Since the first choice leads to the trouble mentioned above with respect to the dynamical equation, we may choose,

$$T\psi(x,t) = \psi^*(x,-t)$$

Therefore for any Hermitian operator O,

$$\langle\psi|O|\phi\rangle = \langle T\phi|TOT^{-1}|T\psi\rangle$$

Taking the absolute square gets rid of the complex conjugate problem and the probability remains invariant.

How do we choose TOT^{-1}? Here are some examples,

$$TxT^{-1} = x$$

$$TpT^{-1} = -p$$

etc.

Thus for any process $i \to f$

$$|M_{i\to f}| = |M_{f\to i}|$$

where M denotes the matrix element for a given transition. Thus the probability is the same if the initial and final states are reversed as it happens in any time reversal transformation. This is known as the **principle of detailed balance**. The physical cross-section however is not necessarily the same since the flux and finals state phase space are different. Using Fermi's golden rule the transition rates are given by

$$W_{i\to f} = \frac{2\pi}{\hbar}|M_{i\to f}|^2\rho_f,$$

$$W_{f\to i} = \frac{2\pi}{\hbar}|M_{i\to f}|^2\rho_i.$$

While the probabilities are the same the rates may be different since the density of states of the end products $\rho_{i,f}$ are not necessarily the same. These can be quite different depending on the masses and number of particles. This is how one reconciles the time reversal invariance with the law of entropy increase.

3.3.4 CPT theorem

While the discrete symmetries C,P and T appear to violated, the combined operation CPT is an exact symmetry. Any theory that is invariant under Lorentz transformations must have CPT symmetry- CPT theorem. There is no known violation of the CPT symmetry and is consistent with all known experimental observations. The theorem has many consequences:

1. Spin-Statistics theorem: The connection between the spin of the particle and its statistics- for example the spin half particles obey Fermi statistics where as the integer spin particles obey Bose-Einstein statistics.
2. Particles and anti-particles have identical masses and life times.
3. All internal quantum numbers of anti-particles are opposite to those of the particles.

3.4 Problems:

1. Find reasons that could forbid $\gamma \rightarrow \gamma\gamma$. What would happen if the photon had mass?

2. Can an electron and a positron annhilite to a single photon?

3. Consider the decay of the particle $\Delta \rightarrow \pi + N$, where the spin of the Δ particle is 3/2. Determine the parity of the Δ.

4. Consider the process $Co^{60} \rightarrow Ni^{60} + e^- + \bar{\nu}_e$. Show that

$$\langle \cos\theta \rangle = \langle \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} \rangle$$

   is non-zero if parity is violated. Here S is the spin of the nucleus and p is the momentum of the electron.

Chapter 4

Hadrons and the Quark Model

During the 50's and 60's hundreds of hadrons, or strongly interacting particles, were discovered. The concept of "elementary" particle took a beating. The picture was dramatically simplified when it was realised that they could be organised in multiplets, which in turn could be understood in terms of combinations of elementary constituents called quarks. The quark model proposed by Gell-Mann accounts qualitatively for the masses of light hadrons in the region of 1-2 GeV mass range.

The hadrons are divided into two broad categories called mesons (integer spin) and baryons (half-odd integral spin with an additional quantum number called the baryon number). The following table summarises the low lying hadrons classified according to their spin and parity. We have already alluded to the isospin and strangeness before. As far as strong interactions are concerned both isospin and strangeness are conserved exactly. The table **??** also shows the assignment of isospin and strangeness quantum numbers. By inspection it is easy to see that there exists a relation between the charges of the particles and other quantum numbers:

$$Q = I_z + \frac{Y}{2} = I_3 + \frac{B+S}{2}, \tag{4.1}$$

where I_3 is the isospin projection, Y is the hypercharge which is the sum of baryon number and strangeness quantum number. This is the well known Gell-Mann-Nishijima relation. Infact the original assignments of quantum numbers were made using this relation as well*.

The deliberate arrangement of mesons into groups of (8+1) and baryons into groups of (8) and (10) is suggestive of a classification scheme about which we will say more.

In quantum mechanics the degeneracy of eigenvalues is an indication of an underlying symmetry. From the table the following facts emerge:

- Isospin multiplets of the same J^P are almost exactly degenerate- for example (p,n), $\pi^{\pm,0}$. Thus isospin symmetry is exact in strong interactions. The generators of isospin transformations commute with the Hamiltonian. Small mass differences among the multiplets may then be attributed to isospin breaking effects due to other interactions.

- The hadrons within each J^P group are approximately "degenerate" to varying degrees. Baryons are degenerate to within 30 percent, where as with mesons it would be questionable. In the following analysis we concentrate more on Baryons and discuss mesons only in passing.

*With the discovery of new flavours or quantum numbers Gell-Mann Nishijima's original relation has been generalised to include an expanded list of particles

Particle	Mass(MeV)	J^P	Isospin	Strangeness
pseudoscalar Mesons: 8 + 1				
$\pi^{\pm,0}$	140	0^-	1	0
K^+, K^0	495	0^-	1/2	1
$\bar{K}^0, K^-$	495	0^-	1/2	-1
η^0	550	0^-	0	0
η'^0	960	0^-	0	0
vector Mesons: 8 + 1				
$\rho^{\pm,0}$	770	1^-	1	0
K^{*+}, K^{*0}	890	1^-	1/2	1
$\bar{K}^{*0}, K^{*-}$	890	1^-	1/2	-1
ω^0	780	1^-	0	0
ϕ^0	1020	1^-	0	0
spin 1/2 Baryons: 8				
p,n	940	$1/2^+$	1/2	0
Λ^0	1115	$1/2^+$	0	-1
$\Sigma^{\pm,0}$	1190	$1/2^+$	1	-1
$\Xi^{0,-}$	1315	$1/2^+$	1/2	-2
spin 3/2 Baryons: 10				
$\Delta^{++,+,0,-}$	1232	$3/2^+$	3/2	0
$\Sigma^{*\pm,0}$	1385	$3/2^+$	1	-1
$\Xi^{*0,-}$	1523	$3/2^+$	1/2	-2
Ω^-	1672	$3/2^+$	0	-3

Table 4.1: Hadrons and their properties

One can construct from the list given above sets of $I_3 - Y$ plots which will be identified with the weight diagrams of the SU(3) group later.

The SU(3) scheme outlined by Gell-Mann had dramatic prediction that Ω^- particle, which was then not yet discovered, should be there to complete the decuplet $J^P = 3/2^+$. Indeed it was found.

We note a couple of important points without details here:

(1) From the known experimental data on Baryon excited states only states with $I = 1/2, 3/2$ have been seen. This fact, as we shall see later, is crucial for the quark model where only the minimal three quarks are require to construct baryons. If $I = 5/2$ state is observed it would require minimum five quarks.

(2) The excitation spectra of N, Δ, Λ are approximately similar. Even though their constituents may be different combinations of various quarks, the approximate similarity indicates a certain universality of the confining potential- namely flavour independence.

4.1 The quark model

With the list of "fundamental particles" increasing following the discovery of more excited states of particles in table **??**, and new particles of even higher masses discovered, the question that whether all of them could be regarded as elementary or fundamental was looming large. The anomalous magnetic moments of nucleons also pointed to the existence of a substructure.

One feature we have noticed of the hadrons when arranged according to their J^P is that they come neatly arranged in various multiplets.

1. Baryons: $8(1/2^+) \oplus 10(3/2^+)$
2. Mesons: $9(0^-) \oplus 9(1^-)$

Gell-Mann and Zweig(1964) proposed that such a multiplet structure naturally arises when hadrons are thought of as composites of more fundamental objects- quarks which are again fermions with spin 1/2.

The minimal non-trivial configuration for generating Baryons, which are also fermions, is to bind three quarks (qqq). Each quark is assigned a baryon number 1/3 which ensures the fundamental baryons have unit baryon number. Note that the baryon number is additive like charge.

Mesons are composites of quark(B=1/3) and anti-quark(B=-1/3) pairs so that the baryon number of mesons is zero. Since the quarks have spin 1/2. mesons will necessarily have integral spin.

The next hypothesis introduced by Gell-Mann is that these quarks span the fundamental representation of the group SU(3) which has dimension 3 and the anti-quarks span the conjugate representation of dimension $\bar{3}$. These assumptions are sufficient to see the emergence of the hadron multiplet structure:

1. Mesons $(q\bar{q})$: $3 \otimes \bar{3} = 1 \oplus 8$
2. Baryons (qqq) : $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus \bar{8} \oplus 10$

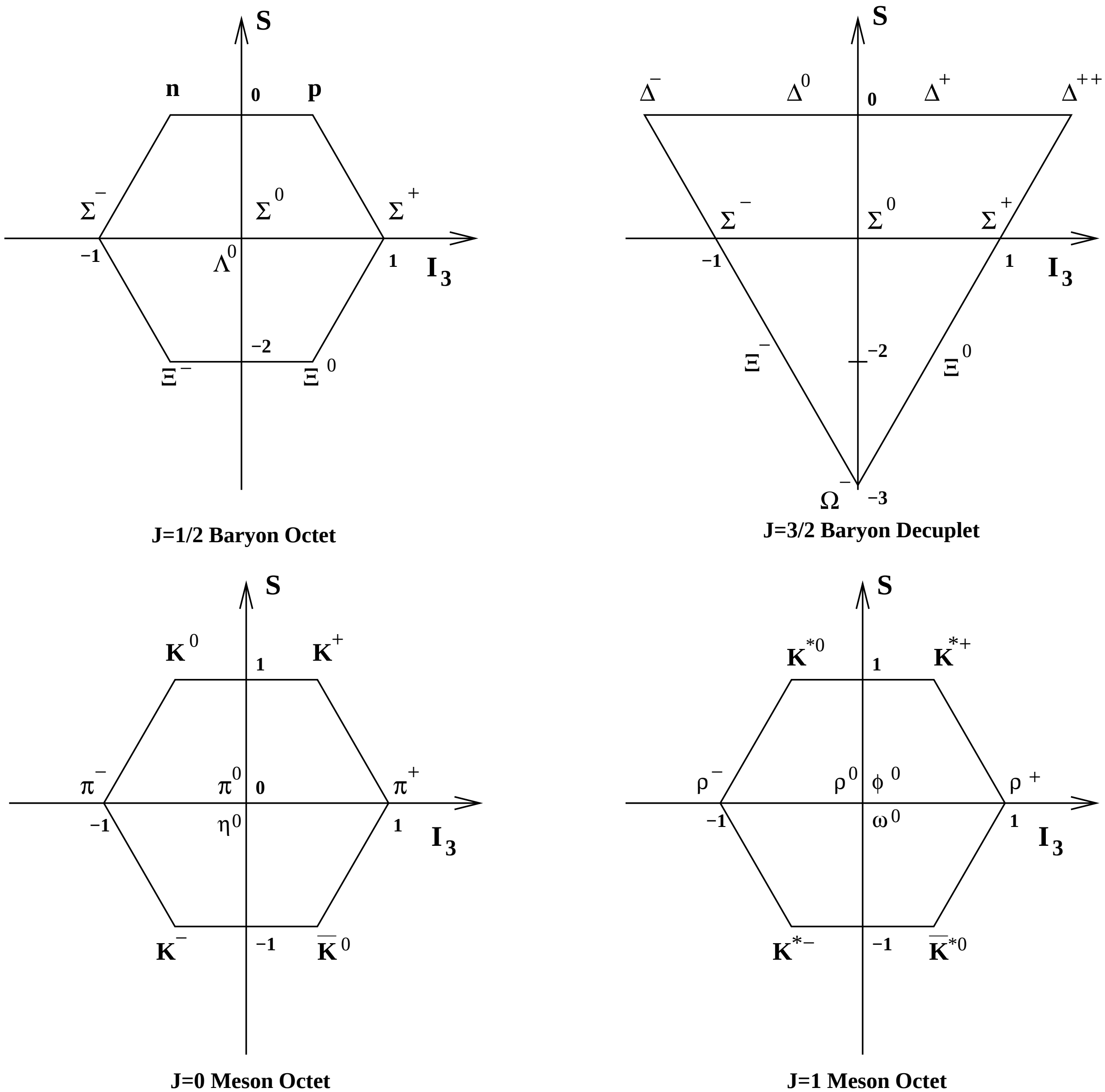

Figure 4.1: SU(3) Weight diagrams of hadrons

where the right hand side shows the dimensionality of higher dimensional representations obtained as a direct sum of the irreducible representations (by taking the Kronecker product of the fundamental representation). The notation will be clarified later but the resemblence to the observed multiplet structure is clear.

While the above classification scheme is shown to work, the fundamental representation is never realised in nature leading to the notion of **quark confinement**. At this stage therefore the quarks merely serve as mnemonics for the classification of hadrons in which they have been permanantly bound. The recent evidence of the decay of the **top** quark in the D0 experiment in Fermilab has however provided the first solid evidence for the reality of quarks.

In the next few sections we consider simple examples using the spin analogy to clarify many group theoretical notions that are used here.

4.2 SU(2) - Spin and Isospin

To simplify the analysis some what we start with the non-strange hadrons. The only symmetry we have to use here is the isospin which is conserved. Hence the states have definite isospin labels. The non-strange baryons are arranged as follows:

- I=1/2 : p, n
- I=3/2 : $\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}$

Suppose $\psi_i^\alpha, \psi_j^\beta$ are basis vectors corresponding to two unitary irreducible representations of a compact Lie group, where α, β label the representation and i, j label vectors in each representations, the basis vectors (tensors) of the Kronecker product represenation are given by the product $\psi_i^\alpha \psi_j \beta$.

In general these need not form the basis of an irreducible representation. However, the basis of any irreducible representation contained in the product can be expanded interms of the product tensors. The coefficients of such an expansion are called Clebsch-Gordon coefficients generalising from the example of the rotation group where they were formulated first. For example,

$$\psi_k^\gamma = \sum_{i,j} C(\alpha, \beta, \gamma; i, j, k)\psi_i^\alpha \psi_j^\beta$$

where ψ_k^γ form the basis of an irreducible representation contained in the product.

Consider for example the D^j representation of the rotation group $R(3)$. The product is written as,

$$D^{j_1} \otimes D^{j_2} = D^{|j_1+j_2|} \oplus ... \oplus D^{|j_1-j_2|}$$

where each irreducible representation is characterised by well defined permutation symmetry. For example, the group of transformations on a spin 1/2 system is given by the representation $D^{1/2}$. For a system of two spin-half objects, we have

$$D^{1/2} \otimes D^{1/2} = D^1 \oplus D^0$$

which is simply a statement of the fact that the two spin half particles may be combined into a spin-1 or spin-0 system. In terms of dimensionalities this may also be written as,

$$2 \otimes 2 = 3 \oplus 1$$

We note that the representation $D^{1/2}$ defines the unitary irreducible representation of lowest dimension of the group SU(2). The above group theoretical statements may be illustrated easily by the following example. Consder explicitly the states of a spin half particle.

Let,

$$\begin{aligned} \uparrow &= |S = 1/2, S_z = 1/2 > \\ \downarrow &= |S = 1/2, S_z = -1/2 > \end{aligned}$$

be the basis vectors of the fundamental representation of SU(2) which is a group of Unitaary-Unimodular 2×2 matrices. The product states are four in number,

$$\uparrow\uparrow, \quad \uparrow\downarrow, \quad \downarrow\uparrow, \quad \downarrow\downarrow$$

. Except the first and the last others do not have definite symmetry under permutation. One may project these into states with definite permutation symmetry:

$$\begin{aligned} |1,1> &= \uparrow\uparrow \\ |1,0> &= \frac{(\uparrow\downarrow + \downarrow\uparrow)}{\sqrt{2}} \\ |1,-1> &= \downarrow\downarrow \end{aligned}$$

which is equivalent to the statement

$$|1,m> \sum_{m_1,m_2} C(1/2,1/2,1;m_1,m_2,m)|1/2,m_1>|1/2,m_2>$$

which span the Spin-1 representation of a combination of two spin-1/2 particles. Note that the representation is completely symmetric under the exchange of the two spins.

The other combination is antisymmetric and leads to the spin-0 representation of the two particle system.

$$|0,0>= \frac{(\uparrow\downarrow - \downarrow\uparrow)}{\sqrt{2}}$$

which is equivalent to the statement

$$|0,0> \sum_{m_1,m_2} C(1/2,1/2,0;m_1,m_2,0)|1/2,m_1>|1/2,m_2>$$

Note that

$$\begin{aligned} J^2|j,m> &= j(j+1)|j,m> \\ J_z|j,m> &= m|j,m> \end{aligned}$$

While combining two spin 1/2 objects it is sufficient to look at the symmetry properties of CG coefficinets to get the symmetry property of the state

$$C(j_1, j_2, j; m_1, m_2, m) = (-1)^{j_1+j_2-j} C(j_2, j_1, j; m_1, m_2, m)$$

Example of a physical system for two spin-1/2 objects is the deuteron.

4.2.1 A system of three spin-1/2 objects

Applying the CG theorem,

$$D^{1/2} \otimes D^{1/2} \otimes D^{1/2} = [D^1 \oplus D^0] \otimes D^{1/2} = D^{3/2} \oplus D^{1/2} \oplus D^{1/2}$$

or interms of multiplicities we have

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus \bar{2}$$

Thus there are two spin 1/2 representations (distinguished by their permutation symmetry and one spin 3/2 representation.

The states that span these representations may be constructed explicitly:

- $$\begin{aligned} |3/2, m> &= \sum_{m_1, m_2} C(1, 1/2, 3/2; m_1, m_2, m)|1, m_1> |1/2, m_2> \\ |3/2, 3/2> &= \uparrow\uparrow\uparrow \\ |3/2, 1/2> &= \frac{\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow}{\sqrt{3}} \\ |3/2, -1/2> &= \frac{\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow}{\sqrt{3}} \\ |3/2, -3/2> &= \downarrow\downarrow\downarrow \end{aligned}$$

  Collectively we refer to these states as χ_s and are explicitly symmetric.

- $$\begin{aligned} |1/2m> &= \sum_{m_1, m_2} C(1, 1/2, 1/2; m_1, m_2, m)|1, m_1> |1/2, m_2> \\ |1/2, 1/2> &= \frac{2\uparrow\uparrow\downarrow - (\uparrow\downarrow + \downarrow\uparrow)\uparrow}{\sqrt{6}} \\ |1/2, -1/2> &= \frac{2\downarrow\downarrow\uparrow - (\downarrow\uparrow + \uparrow\downarrow)\downarrow}{\sqrt{6}} \end{aligned}$$

  Collectively we call these states χ_λ. Note that these states are not symmetric or antisymmetric under exchange of spins. These are called Mixed-symmetry states-symmetric in 1-2 with no particular symmetry with respect the third spin.

I, I_3	State	Charge Q
I=1 Triplet:		
1,1	uu	4/3
1,0	$\frac{ud+du}{\sqrt{2}}$	1/3
1,-1	dd	-2/3
I=0 Singlet:		
0,0	$\frac{ud-du}{\sqrt{2}}$	1/3

- $$\begin{aligned} |1/2, m> &= \sum_{m_1,m_2} C(0, 1/2, 1/2; 0, m, m)|0, 0> |1/2, m> \\ |1/2, 1/2> &= \frac{(\uparrow\downarrow - \downarrow\uparrow)\uparrow}{\sqrt{2}} \\ |1/2, -1/2> &= \frac{(\downarrow\uparrow - \uparrow\downarrow)\downarrow}{\sqrt{2}} \end{aligned}$$

  Collectively we call these states χ_ρ. These are again called Mixed-symmetry states-antisymmetric in 1-2 with no particular symmetry with respect the third spin.

The precise number of states in each representation correspond to the multiplicities obtained from the CG theorem.

4.2.2 Combining Isospin states

We may carry out the same excercise in the isospin space. The rotations in isospin space are analogous to the rotations in the spin space. The fundamental group is again SU(2) and is spanned by two vectors **u** and **d** referring to the up and down quark states. Analogy with spin is clear once we identify $\uparrow \to u$ and $\downarrow \to d$. The construction of states in the isospin space then proceeds the same way as in the spin space.

By analogy with spin the u-quark has $I = 1/2, I_3 = 1/2$ and the d-quark has $I = 1/2, I_3 = -1/2$. All the quarks carry spin-1/2 and are fermions under permutation symmetry.

Using the Gell-Mann - Nishijima formula the charges of the quarks may be obtained as follows:

$$Q_u = I_3 + (B + S)/2 = 2/3$$

$$Q_d = I_3 + (B + S)/2 = -1/3$$

since strangeness S=0 and Baryon number B=1/3 for u and d quarks by definition. Thus the quarks carry fractional charges.

Following table summarises the states of two isospin 1/2 particles: Obviously no such di-quark systems with non-integral charges appearr in nature. However we need the above construction to construct systems of three I=1/2 particles. Using the spin analogy the following table summarises the system of three quarks (qqq) which will be identified with the baryon states.

I, I_3	State	Charge Q	Baryon
I=3/2 Δ ϕ_s			
3/2,3/2	uuu	2	Δ^{++}
3/2,1/2	$\frac{uud+duu+udu}{\sqrt{3}}$	1	Δ^{+}
3/2,-1/2	$\frac{udd+dud+ddu}{\sqrt{3}}$	0	Δ^{0}
3/2,-3/2	ddd	-1	Δ^{-}
I=1/2 Nucleon doublet:ϕ_λ			
1/2,1/2	$\frac{2uud-(ud+du)u}{\sqrt{6}}$	1	p
1/2,-1/2	$\frac{2ddu-(ud+du)d}{\sqrt{6}}$	1	n
I=1/2 Nucleon doublet:ϕ_ρ			
1/2,1/2	$\frac{(ud-du)u}{\sqrt{2}}$	1	p
1/2,-1/2	$\frac{(ud-du)d}{\sqrt{2}}$	1	n

4.2.3 Spin-Isospin States of definite symmetry

The spin-isospin state of the Δ particle with S=I=3/2 is given by

$$|\Delta>= \chi_s\phi_s$$

However the nucleon states with S=I=1/2 have many possible combinations which have the same quantum numbers as the proton and the neutron, infact too many for comfort since there are exactly two members of the doublet that we should extract.

$$\chi_\rho\phi_\rho, \chi_\rho\phi_\lambda, \chi_\lambda\phi_\lambda, \chi_\lambda\phi_\rho$$

So we have four instead two states. But none of these states has a well defined symmetry or antisymmetry under permutations, while the Δ is completely symmetric under spin as well as isospin indices.

If we demand a completely symmetry under exchange as in the case of Delta states then one gets the following combination:

$$|N>= \frac{\chi_\rho\phi_\rho + \chi_\lambda\phi_\lambda}{\sqrt{2}}$$

On the otherhand a completely antisymmetric state would have the combination

$$|N>= \frac{\chi_\rho\phi_\lambda - \chi_\lambda\phi_\rho}{\sqrt{2}}$$

where $N = p, n$ depending upon the isospin projection of ϕ state.

In nuclear three body problem the nuclei $He^3(ppn)$ and $H^3(pnn)$ play the roles analogous to that of proton(uud) and neutron(ddu). The choice of the particular combination of the spin-isospin state is dictated by the fact that the state of a system of fermions must be antisymmetric in all indices. Since in the ground state wave function of these two nuclei (L=0) is completely symmetric, one choses the antisymmetric wave function given above. The ground state static properties are well reproduced by such a combination. Thus it might seem that there is an unambigious choice for the Nucleon from the above two choices. However, the delta states given above are completely symmetric under spin-isospin indices. *The question therefore hangs on the fate of the Spin-Statistics Theorem.* We will address this issue next.

4.2.4 Spin-Statistics Problem: Origin of colour

Consider the state of Δ particle. As remarked before the spin-isospin state of this particle is completely symmetric under permutations. Its $J^P = 3/2^+$ and hence it is even under parity. It is also the ground state of the $I = 3/2$ state. Quantum mechanics tells us that the ground state of any system with even parity must be spacially symmetric under permutations. For example the ground states of the hydrogen molecule, Helium and Oxygen nuclei, etc. Thus we find ourselves in the piquant situation where the Δ state is a completely symmetric in $space \otimes spin \otimes isospin$ coordinates.

The spin-statistics theorem tells us that a state of a system of ferrmions has to be completely antisymmetric. Thus we encounter a paradoxical situation that spin-statistics theorem may not hold for the Δ states in particular†.

A way out of this dilemma is to introduce a new quantum number called **Colour**. Thus each quark (u or d) comes in three colours and the wave function of the baryons is completely antisymmetric in the colour space. Thus all baryons have

$$B_{colour} = \epsilon_{ijk} q_i q_j q_k$$

where $i, j, k = red, green, blue$, the three colours (you may take 1,2,3 for the indices). The full wave function of the Delta state is then given by,

$$|\Delta >= \epsilon_{ijk} q_i q_j q_k [\psi_{space} \chi_s \phi_s]$$

which is on the whole an antisymmetric state. One may wonder if the above decomposition smells of non-relativistic quantum mechanics which may not be wholly valid for quarks since their masses are not very large. Indeed the situation with nucleons will clarify this issue further.

We may now extend the arguement given above for the nucleon states also. As we have seen there are two combinations available for nucleons:

$$|N >= \frac{\chi_\rho \phi_\rho + \chi_\lambda \phi_\lambda}{\sqrt{2}}$$

$$|N >= \frac{\chi_\rho \phi_\lambda - \chi_\lambda \phi_\rho}{\sqrt{2}}$$

combined from the mixed symmetry states of spin and isospin. Once again we assume the spacial part is symmetric since both nucleon form the ground state of the $J^P = 1/2^+$ spectrum of baryons. Since the second combination is completely antisymmetric, it may seem as though we do not have the spin-statistics problem. However, since the quarks have to be coloured in order to preserve the antisymmetry of the Δ states, it is natural to choose the symmetric states and impose antisymmetry condition by invoking colour. Thus we choose the nucleon states to be,

$$|N >= \epsilon_{ijk} q_i q_j q_k \frac{\chi_\rho \phi_\rho + \chi_\lambda \phi_\lambda}{\sqrt{2}}$$

which is now completely antisymmetric.

An even stronger evidence of the choice of the combinations given above for nucleons, hence for colour, actually comes from the experimental measurement of the static magnetic moment of the nucleons. We discuss this below.

†Historically many solutions wer proposed- Parastatistics by Greenberg and coloured quarks with integral charge called the Han-Nambu model. But the experimental evidence is firmly against these proposals

The experimental data on the neutron and proton magnetic moments gives,

$$\frac{\mu_n = -1.91}{\mu_p = 2.79} = -0.685$$

The corresponding magnetic moment operator in terms of the basic quark operators is given by,

$$M_z = \sum_{i=1}^{3} \mu \sigma_{iz} e_i$$

where μ is the unit of quark magnetic moment which we keep arbitrary since we do not know this. e_i is the charge of the i-th quark and σ_{iz} is the z-component of the Pauli spin vector $\vec{\sigma}$. We are therefore interested in evaluating

$$\mu_{n,p} =< N = n, p|M_z|N = n, p >$$

Note that the operator involves only the spin and isospin operators. We concentrate only this part of the wave-function. Because these states of the nucleon are either fully symmetric or antisymmetric we have the identity,

$$\mu_{n,p} = 3\mu < N = n, p|e_3\sigma_{3z}|N = n, p >$$

The matrix elements in the spin space are given by,

$$\begin{aligned} < \chi_\rho|\sigma_{3z}|\chi_\rho > &= 1 \\ < \chi_\rho|\sigma_{3z}|\chi_\lambda > &= 0 \\ < \chi_\lambda|\sigma_{3z}|\chi_\lambda > &= -1/3 \end{aligned}$$

Similarly in the isospin space we have for protons

$$\begin{aligned} < \phi^p_\rho|e_3|\phi^p_\rho > &= 2/3 \\ < \phi^p_\rho|e_3|\phi^p_\lambda > &= 0 \\ < \phi^p_\lambda|e_3|\phi^p_\lambda > &= 0 \end{aligned}$$

and for neutrons

$$\begin{aligned} < \phi^n_\rho|e_3|\phi^n_\rho > &= -1/3 \\ < \phi^n_\rho|e_3|\phi^n_\lambda > &= 0 \\ < \phi^n_\lambda|e_3|\phi^n_\lambda > &= 1/3 \end{aligned}$$

Substituting these in the spin-isospin wave functions of the neutron and proton we have,

$$\begin{aligned} \mu_n &= -2\mu/3 \\ \mu_p &= \mu \end{aligned}$$

and therefore the ratio is given by,

$$\frac{\mu_n}{\mu_p} = -2/3$$

whereas the experimental value is given by -0.685 which is in excellant agreement considering the crude assumptions made.

On the other hand if we had chosen the antisymmetric combination in the spin-isospin space disregarding the colour hypothesis, we would have obtained,

$$\frac{\mu_n}{\mu_p} = -2$$

in contradiction with experiment. Thus we have now evidence for colour from two independent approaches- the spin-statistics theorem and the experimental data on the static magnetic moments of the neutron and proton. Note that we did not need to fix μ the basic unit of magnetic moment of the quarks- it just cancelled out in the ratios.

4.2.5 Constituent Quarks

The ratio of the magnetic moments as calculated before does not fix the unit of the quark magnetic moment. As in the case of the electron if we assume that the Dirac magnetic moment of the quarks to be given by the expressions:

$$\mu_u = \frac{e_u}{2m_u} = \frac{2\mu}{3} \quad \mu_d = \frac{e_d}{2m_d} = \frac{-\mu}{3}$$

Assuming $m = m_u = m_d$ we have for the proton magnetic moment

$$\mu_p = 2.79\frac{e}{2M_P} = \frac{e}{2m}$$

where m is the quark mass, we immediately get,

$$m = \frac{M_p}{2.79} = 336MeV$$

This mass is often referred to as the constituent quark mass. Unlike the mass of the electron which enters the QED Lagrangian as a fundamental quantity, the constituent quark mass has no firm theoretical basis except to define a scale for discussing the low energy and static properties of the nucleon.

4.2.6 Other evidences for colour

We conclude this discussion with few more remarks on the colour quantum number: Some of the strongest evidence for colour comes from experiments. Consider the following ratio which is now experimentally measured:

$$\frac{\sigma(e^+e^- \to hadrons(q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

which is the ratio of the total cross-sections for electron-positron annihilation to either quarks or muons. Typically such a total cross-section is obtained by summing over all the final states. Thus in the numerator one sums over all the spin-isospin (around 1 GeV. At higher energies one has to sum over other quarks as well) states and in the denominator we sum over the spin states of the muons. If quarks come in three colours, one needs to sum over these as well. As it turns out merely summing over spin and flavours underestimates the ratio by a factor close to three suggesting the existence of an extra degree of freedom. Imposing the requirement that the quarks come in three colours solves this puzzle as well.

The strongest evidence to date comes from the following decay:

$$\pi^0 \to \gamma\gamma$$

It is some what complicated to discuss this case without a background in quantum field theory. It suffices to say that the π decay to two photons proceeds through the mediation of quarks. Once the amplitude is obtained by summing over all quark states. Without imposing the colour degree of freedom, the decay amplitude is underestimated by a factor of 3, and hence the rate by a factor of 9. Including colour the calculated decay rate agrees with experiments within errors.

4.3 SU(3) Flavour States

We have constructed states of non-strange baryons using the SU(2) isospin doublet of quarks (u,d). Extending these arguements to construct hadrons using the triplet of quarks (u,d,s) is straight-forward if more cumbersome. We shall mention briefly how the hadron octets and decuplets mentioned in the beginning of this section are obtained using three basic quark **flavours**

Regarding the triplet (u,d,s) as the basis spanning the fundamental representation of SU(3), we can combine any two of them first. There are nine such combinations which may be arranged as

$$3 \otimes 3 = 6 \oplus 3$$

using the expansion of Kronecker product. Explicitly these di-quark states can be written as

$$uu, dd, ss, \frac{ud+du}{\sqrt{2}}, \frac{us+su}{\sqrt{2}}, \frac{sd+ds}{\sqrt{2}}$$

which are 6 completely symmetric states and

$$\frac{ud-du}{\sqrt{2}}, \frac{us-su}{\sqrt{2}}, \frac{sd-ds}{\sqrt{2}}$$

which are 3 completely antisymmetric states.

Similarly combining three quarks we obtain,

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

where the representation with dimension 10 is completely symmetric given by,

$$uuu, ddd, sss, (uud)_{sym}, (uus)_{sym}, (udd)_{sym}, (sdd)_{sym}, (ssu)_{sym}, (ssd)_{sym}, (uds)_{sym}$$

where $(uud)_{sym}$ means a completely symmetric arrangement of (uud) etc. These quark states correspond to the spin 3/2 decuplet representation of the baryons.

The singlet under SU(3) with dimensionality 1 is the completely antisymmetric combination of (uds) quarks. The two octets are mixed symmetry representations.

Thus we could generate the weight diagrams of SU(3) analogous to the Gell-Mann's scheme for hadrons interms of their quark contents.

Combining these states with states of definite spin proceeds as in the case of combining isospin and spin states.

Appendix: Introduction to SU(2) and SU(3)

In general SU(N) is a group of $N \times N$ unitary unimodular matrices.

$$UU^\dagger = 1, \quad det(U) = 1$$

In general we may therefore write,

$$U = \exp{(i\theta_a T_a)}, \quad a = 1, ..., N^2 - 1$$

where θ_a are the parameters of the group and T_a are the hermitian(because the elements are unitary) generators of the group.

The generators obey the following properties:

$$\begin{aligned} Trace(T_a) &= 0 \\ Trace(T_a T_b) &= \delta_{ab} \end{aligned}$$

and

$$[T_a, T_b] = i f_{abc} T_c$$

which defines the algebra of the generators completely.

SU(2) is the group of 2×2 unitary unimodular matrices. It is also the lowest dimensional nontrivial representation of the rotation group. The generators may be chosen to be

$$T_a = \frac{1}{2}\sigma_a; \quad a = 1, 2, 3$$

where σ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.2}$$

The basis for this representation is conventionally chosen to be the eigenvectors of σ_3 that is the column vectors,

$$|1/2, 1/2 >= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1/2, -1/2 >= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.3}$$

which describe a spin-1/2 particle with the projection $m = 1/2, -1/2$ respectively. As we have seen this fundamental representation of SU(2) may be combined to build higher dimensional representation corresponding to the spins $J = 1, 3/2, 2, ...$ etc. Note that there is only one diagonal generator. In general for SU(N) there can atmost be $N - 1$ diagonal generators which is known as the rank of the group. The rank of the group is also equal to the number of Casimir operators- the states that span the representation are eigenstates of this operator. For example the Casimir operator of the SU(2) is J^2. The states are simultaneous eigenstates of J^2 and J_z.

The group SU(3) is the group of 3×3 unitary unimodular matrices. The generators may be chosen to be

$$T_a = \frac{1}{2}\lambda_a; \quad a = 1, ...8$$

where λ are given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.4}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \tag{4.5}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} /\sqrt{3} \tag{4.6}$$

We note a few points here:

- The generators T_1, T_2, T_2 generate an SU(2) subgroup of SU(3) and the algebra of these generators closes among themselves.

- The diagonal generators commute among themselves.

$$[H_i, H_j] = 0$$

hence the algebra is closed. The diagonal generators define a subalgebra called the Cartan subalgebra. The elements of this subalgebra are $m = N - 1$ in number where m is the rank of the group. All states in a representation D are labelled by the eigenvalues of H_i such that

$$\{H_i\}| >= \{\mu_i\}| >$$

and $\vec{\mu}_i = \{\mu_i\}$ is called the weight vector.

For the group SU(3) we have chosen $H_1 = \lambda_3/2, H_2 = \lambda_8/\sqrt{3}$. The eigenvectors may be chosen to be,

$$|1/2, 1/3 >= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |-1/2, 1/3 >= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |0, -2/3 >= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.7}$$

We may easily identify the quantum numbers of these states with isospin and hypercharge of u(1/2,1/3), d(-1/2, 1/3) and s(0,-2/3) quarks. Thus the three quarks u,d and s form the basis of the fundamental representation of SU(3).

4.3.1 Conjugate representation

Suppose T_a are generators of some representation D of the group, then

$$[T_a, T_b] = i f_{abc} T_c$$

and $-T_a^*$ also satisfy the same algebra

$$[T_a^*, T_b^*] = i f_{abc} T_c^*$$

Therefore $-T_a^*$ also generate a representation $\bar{D}$ of the same dimension. The states are again eigenstates of the diagonal generators of the group. Thus we have, for example,

$$D \to \bar{D}$$

$$H_1 \to -H_1, \quad H_2 \to -H_2$$

Under this change,

$$u = |1/2, 1/3 > \to \bar{u} = |-1/2, -1/3 >$$

$$d = |-1/2, 1/3 > \to \bar{d} = |1/2, -1/3 >$$

$$s = |0, -2/3 > \to \bar{s} = |0, 2/3 >$$

interms of flavour states of SU(3). Note that in the conjugate representation all the charges (hyper) are reversed.

Thus if we choose the vectors that span the fundamental representation of SU(3) as quarks, the vectors that span the conjugate representation are anti-quarks. Indeed while there were many choices for the fundamental group for three quarks like O(3), SO(3), SU(3) became a natural choice since its representations are not real unlike SO(3).

4.4 Problems:

1. Explicitly construct the wavefunction of the Δ^{++} state which is completely antisymmetric.

2. Using isospin symmetry show that the transition rates for $\Delta \to \pi + N$ are in the following ratio:

$$\Delta^{++} \to p\pi^+ : \Delta^+ \to p\pi^0 : \Delta^{++} \to p\pi^- = 3 : 2 : 1$$

3. Using isospin analysis show that $\rho^0 \to \pi^0\pi^0$ is forbidden.

4. Use isospin invariance to show that the reaction cross-section for $pp \to \pi^+ d$ is twice that of

$$np \to \pi^0 d$$