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Fractional Exclusion Statistics

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Abstract

Haldane introduced the idea of generalised Pauli principle or fractional exclusion statistics. We generalize Haldane's definition of exclusion statistics to systems with infinite dimensional Hilbert spaces and examine its consequences. First we give a detailed a derivation of the connection between the fractional exclusion statistics parameter and the second virial coefficient. We then show that systems of anyons, both free and in a magnetic field, have nontrivial exclusion statistics. While free anyons may be considered as interacting fractional exclusion particles, we show that in one-dimension there exist systems like the particles of Calogero-Sutherland model and quasiparticles of the Luttinger model which may be described as ideal exclusion statistics particles through their thermodynamics. Using these models, we derive a set of constraints on the occupation numbers of exclusion statistics particles which will eliminate the problem of having to deal with negative probabilities encountered in the distribution function.

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In recent years there has been much interest in the physics of particles obeying fractional statistics. A way of characteristing the statistics of particles is through their properties under exchange as in the case of ideal fermions and bosons. Thus the anyons [1] are particles whose many particle wave functions pick up a phase $e^{i\alpha\pi}$ under the exchange of any two particles. Standard bosons and fermions correspond to the case when $\alpha = 0, 1$. However arbitrary values of α are allowed by the configuration topology of the many particle system in one and two spatial dimensions [2]. The parameter α is called the statistics of the particle. Here after we refer to α as the fractional exchange statistics parameter. It is also well known that $\alpha = 1$, implies that the wave functions are antisymmetric under exchange which inturn implies the Pauli exclusion principle. For $\alpha = 0$, the wave function is symmetric and no restriction operates. The existence of such a principle has nontrivial consequences for the counting of states in a many body system. The question may then be raised as to whether this correspondence between statistics and exclusion principle be exploited to define fractional statistics through a generalised exclusion principle? Indeed the answer is yes as is indicated by the developments in recent years.

In a seminal paper [3], Haldane proposed an alternate definition of statistics based on a generalized exclusion principle. His definition is for particles with finite dimensional Hilbert spaces and is motivated by physical examples, such as quasi-particles in the fractional quantum Hall systems and spinons in anti ferromagnetic spin chains. Haldane first defines d_N, the dimension of the available single particle space when there are N particles present, as follows: Consider the N particle wave function $\psi(x_1, x_2, ..., x_N)$. Fix any of the $N-1$ points and look upon the wave function as a function of the remaining single particle variable. The dimension of the space spanned by these functions, d_N , then defines the single particle Hilbert space of dimension d_N in the presence of the other $N-1$ particle. To clarify the definition, consider particles on a lattice of dimension d_1. For fermions, if there are N particles and we fix the position of $N-1$ other particles on the lattice, then the remaining single particle can only occupy any of the $d_1 - (N-1)$ remaining positions due to Pauli blocking. Thus the fermionic single particle dimension as defined by Haldane is $d_N^F = d_1 - (N-1)$. Bosons on the other hand can occupy any of the d_1 sites irrespective of how many other particles are there. Therefore $d_N^B = d_1$. Haldane then considers systems with a generalized Pauli blocking whose single particle space is given by, $d_N = d_1 - g(N-1)$, where d_N, as stated above, is the number of states accessible to a particle in the presence of N-1 other particles. The Fractional Exclusion Statistics(FES) parameter g is then given by,

$$g = -\frac{\Delta d}{\Delta N}, \tag{1}$$

where Δd is the change in the dimension of the single particle space and ΔN is the change in the number of particles when the size of the system and the boundary conditions are unchanged. Thus the parameter g is a measure of the partial Pauli blocking in the system [4]. The value $g = 0(1)$ corresponds to bosons(fermions). Here after we refer to this redefinition of statistics as Fractional Exclusion Statistics(FES). Clearly the definition of statistics is independent of space dimension and is allowed to be fractional in general. In contrast the definition of anyons through the exchanges statistics requires non-trivial configuration space topology.

In important contributions, Isakov [5] and Wu [6] have independently derived the distribution function obeyed by particles with FES. In the specific case of particles occupying the lowest Landau level, the same distribution has also been derived by Ouvry [7]. Specifically, a definition of an ideal gas of particles with nontrivial exclusion statistics was given in references [6,5]. Recently the many body states and operator algebra for exclusion statistics has also been obtained [8–10]. Haldane's idea was also tested numerically for the fractional quantum Hall systems(FQHE) by Johnson and Canright [11]. In an earlier paper [12] we had generalized Haldane's definition to the case where the Hilbert space of the particles is infinite dimensional. We showed that the statistics parameter g is determined by the high temperature limit of the second virial coefficient. This was then applied to the case of anyons in two dimensions and to the case of quasiparticles in the Luttinger model.

While in most of the cases cited above, the existence of fractional exclusion statistics was assumed and consequences worked out, it turns out that in one-dimension there exist a class of models of interacting fermions which can be looked upon as a model of noninteracting anyons obeying FES. These are the well known Calogero-Sutherland model (CSM) where the particles interact via an inverse square interaction in some confining potential [13] and quasiparticles in Luttinger Model [14]. That the particles of CSM obey FES has been shown by several authors recently [5,15–17]. In an earlier paper we had argued that the quasiparticles of the Luttinger model also obey FES [12]. Infact just as the Hamiltonian of the particle-flux composite describes the anyons in two dimensions, the Hamiltonian in these models provides the framework for fractional statistics particles in one dimension.

A crucial property of exclusion statistical interactions is that they should cause shifts in single particle energies at all scales [12]. This property is realized by a large class of one dimensional models of interacting fermions where Fermi liquid theory breaks down [39,19]. In fact it has been shown exactly that quasiparticles with nontrival exclusion statistics exist in a class of models that are solved by the Bethe ansatz [5,15]. In particular the quasiparticles of the Calogero-Sutherland model (CSM) behave like ideal exclusion statistics system [5,15–17]. A feature of the exclusion statistics as gleaned from the analysis of various models is that the exclusion acts across a set of levels unlike in the case of Fermi or Bose statistics where the exclusion principle is stated with a single level in mind. It is this crucial difference that results in the occurance of the so called negative probabilities [20,21]. We will show that the particles obeying fractional exclusion statistics can be characterised by constraints on the sets of occupation numbers. There are no negative probabilities if these constraints are obeyed. If these constraints are relaxed then the negative weights arise in order to compensate for the resulting over counting. Indeed this is the way we encounter negative probabilities in other systems in physics- for example in gauge theories, they arise in the ghost sectors. Ghosts come from the Jacobian associated with nonlinear gauges which essentially ensure the correct counting of states. Another example is that of the Wigner distribution function in quantum mechanics which is not positive definite precisely because some constraints are relaxed. A formulation based on the variable number of single particle states, which depends on the total number of particles in the system, has been discussed by Isakov [22] as a way to avoid the problem of negative weights. Recently, a microscopic interpretation of exclusion statistics systems has been advanced by Chaturvedi and Srinivasan [23] where they show how this problem of negative probabilities may be solved for semions, $g = 1/2$. They have also indicated how their method may be generalised to other values of g.

In these lectures we elaborate on our earlier works [12,16] and clarify some of the outstanding issues in FES. In section 2 we provide a detailed a proof of the connection between the FES parameter g and the second virial coefficient. In section 3, we apply this definition to the case of anyon gas and anyons confined in a one-body oscillator potential in the presence of a constant magnetic field. In the later case we show by explicit calculation the scale dependence of g. In the second virial coefficient there exist precisely two plataue regions- the one at low temperature reproduces the result for g corresponding to the particles confined to LLL while the high temperature result reproduces the anyon gas result. In section 4, we consider the one-D examples. We first show the equivalence of the spectra of quasiparticles in the Luttinger model with that of the spectra of CSM particles. Applying our definition of g through the second virial coefficient we show that the CSM particles indeed obey Haldane's definition of fractional statistics. We also give the details of the derivation of the distribution function which is the same as that given by Wu. The distribution function is given interms of a function $w(E)$ which can be obtained as a solution of a polynomial equation. Next we discuss the partition function and the equation of state of the system and show that the statistics affects only the second virial coefficient. In section 5 we discuss generalised counting rules applicable to FES and discuss the problem of negative probabilities first pointed out by Nayak and Wilczek [20]. The solution of this problem lies in the proper implementation of the counting rules. We then extend these arguements in the thermodynamic limit and describe a connection between the distribution function derived by Wu [6] and solutions of a class of equations given by the mathematician Ramanujam [24].

I. FES AND THE SECOND VIRIAL COEFFICIENT

The original definition of g is given in eq.(1) interms of the effective single particle dimensions embedded in a finite dimensional many particle Hilbert space. While this is a useful definition in systems in which the many particle spectrum admits an interpretation in terms of an effective single particle level spectrum, in general it is not a very useful definition. It is our goal here to first generalise the definition given by Haldane to infinite dimensional spaces. In doing so we show that the statistics parameter g can be measured through the second virial coefficient in the equation of state under conditions.

To be specific, consider a system of N-identical particles. For a system of Bosons occupying d-states the dimension of the N-particle Hilbert space is given by,

$$D_N^B = \frac{(d+(N-1))!}{N!(d-1)!}$$

and for a system of fermions occupying d-states,

$$D_N^F = \frac{d!}{N!(d-N)!}.$$

The FES parameter g may be introduced by requiring that there exist a simple interpolation between D_N^B and D_N^F. A generalization of D_N for arbitrary statistics is therefore,

$$D_N(g) = \frac{(d+(1-g)(N-1))!}{N!(d-1-g(N-1))!}. \tag{2}$$

Note that D_N yields the correct dimension for $g=0$ and $g=1$, it is a matter of definition for arbitrary values of g and is consistent with Haldane formulation of FES. Infact a neat interpretation of the above formula arbitrary g can be given interms interms of the number of hole and particle states as in the case of Fermions. If $d+(1-g)(N-1)$ is the number occupied states in a d dimensional lattice, then the number available states for the Nth particle (holes) is given by $d-1-g(N-1)$. Note, however, that the combinatorial expression introduced above is only a way of realising the original definition of g given by eq.(1) which admits more than one interpretation in terms of the dimension of the many particle states [25].

We are now in a position to generalize Haldane's definition of FES as given above to the case where the Hilbert space of the particles is infinite dimensional. We are therefore interested in the limit when $d \to \infty$, and N is finite such that $N/d \to 0$. Often this limit may be confused as the zero density limit. To clarify what this limit implies, assume that the particles are spread on a lattice of size L (per dimension) with a spacing a. Then $d = L/a$ and the limit $d \to \infty$ would imply $L \to \infty$ (infinite volume) or the lattice spacing $a \to 0$. Since the density is $\rho = N/L^{d_s}$, where d_s is the space dimension, it is clear that $\rho \to 0$ when $L \to \infty$ or ρ is finite when $a \to 0$(continuum limit). In what follows we are interested in the continuum limit when the density is finite. Expanding the many particle dimension in powers of single particle dimension d, we have,

$$D_N(g) = \frac{1}{N!}[d^N + N(N-1)(\frac{1}{2} - g)d^{N-1} + ...]. \tag{3}$$

In the limit $d \to \infty$, it is easy to see that

$$\frac{1}{2} - g = \lim_{d\to\infty} \frac{d}{N(N-1)} \left[\frac{N!\ D_N(g)}{d^N} - 1\right]. \tag{4}$$

While in the above we have obtained a relation that determines g, it is not yet useful since $D_N(g)$ is not a meaningful quantity for systems in continuum. However, recall that a regulated definition of the dimension fo the Hilbert space is given by the N-particle partition,

$$D_N = \lim_{\beta\to 0} Z_N = \lim_{\beta\to 0} Tr(e^{-\beta H_N}), \tag{5}$$

where β is the inverse temperature and H_N is the N-particle Hamiltonian. Thus for dealing with the infinite dimensional Hilbert space for systems in continuum, we propose a definition of g as a generalization of the Haldane's definition-

$$\frac{1}{2} - g = \lim_{\beta\to 0} \frac{CZ_1}{N(N-1)} \left[N!\frac{Z_N}{Z_1^N} - 1\right], \tag{6}$$

where C is an overall constant of proportionality which we fix next. The definition of g given by eq.(6) forms the basis for the rest of the calculations presented in this paper.

To test the utility of eq.(6), we first consider the case of bosons and fermions in d_s space dimensions. For convenience we confine all the N-particles in an oscillator potential of the form

$$V(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) = \frac{1}{2}\omega^2 \sum_{i=1}^{\infty} r_i^2, \tag{7}$$

where ω is the oscillator frequency and $\vec{r}_i$ are single particle coordinates (we have set $\hbar = m = c = 1$). The choice of the confining is not unique but merely acts as a regulator. The single particle partition function in d_s space dimensions is given by,

$$Z_1(\beta) = \frac{e^{-\beta\omega d_s/2}}{(1-e^{-\beta\omega})^{d_s}}. \tag{8}$$

The N-particle partition function is then given by the [26],

$$Z_N^{B,F} = (\pm)^N \sum_{\mathcal{P}} \prod_l \frac{1}{n!}[\pm\frac{Z_1(l\beta)}{l}]^n, \tag{9}$$

where the $\pm$ signs correspond to bosons or fermions and the sum over $\mathcal{P}$ is given as the number of partitions of N such that $\sum_{l=1}^N nl = N$, n and l being positive integers. Since we are interested in the high temperature limit, it is sufficient to expand Z_N to the next to leading order in powers of Z_1,

$$Z_N^{B,F}(\beta) = \frac{1}{N!}\left[Z_1^N(\beta) \pm Z_1(2\beta)Z_1^{N-2}(\beta)\frac{N(N-1)}{2} + ...\right], \tag{10}$$

Notice that the second term in eq.(10) involves $Z_1(2\beta)$ since the contribution to this term comes when two particles are in the same energy level. Substituting eq.(10) and eq.(8) in eq.(6) we find that,

$$\frac{1}{2} - g = \pm\frac{C}{2}\frac{1}{2^{d_s}}. \tag{11}$$

Since $g = 0, 1$ for bosons and fermions, a consistent definition of g is obtained if we set,

$$C = 2^{d_s}. \tag{12}$$

It is easy to see why the g given by the state counting definition, eq.(4), and through a regulated definition involving partition functions, eq.(6), should differ by a proportionality factor C. Notice that the expansion of D_N involves the single particle space dimension d. The analogous expansion for $Z_N(\beta)$ should therefore be interms of $Z_1(\beta)$ since $d = \lim_{\beta\to 0} Z_1(\beta)$. In actual fact, however, the expansion for $Z_N(\beta)$ involves, in the next-to-leading-order term which is relevent here, the factor $Z_1(2\beta)$. In the high temperature limit, we can write

$$Z_1(2\beta) = \frac{1}{2^{d_s}} \lim_{\beta\to 0} Z_1(\beta)$$

which immediately clarifies the origin of the proportionality factor C. Yet another way of realising the origin of C is to recall that the regulated definition cuts off the energy at 2β when two particles occupy the same state, where as for counting purposes we want to cutoff all states at the same energy irrespective of their occupation. As a result, while the definition given by Haldane is independent of space dimension, the generalization in terms of the partition function for infinite dimensional spaces does depend on the space dimension albeit trivially.

We are now in a position to apply the definition of g to systems of identical interacting particles in the continuum. For simplicity we consider all particles to be confined in an oscillator potential as mentioned before. A glance at the definition of g through eq.(6) for two particles (N=2), immediately suggests that g in this case is determined by the second virial coefficient in the virial expansion of the equation of state of the sytem. We will now elaborate on this connection. To begin with consider a gas of interacting particles in two space dimensions so as to have well defined factors throughout. (The proof can be generalized to arbitrary space dimensions.) The grand partition function of the system is given by,

$$Z(z, V, \beta) = \sum_{N=0}^{\infty} z^N Z_N(V, \beta); \quad z = e^{\beta\mu}, \tag{13}$$

where z is the fugacity parameter and μ is the chemical potential of the system. The equation of state of the system in fugacity expansion may be written as

$$\beta P = \frac{1}{V} ln(Z) = \sum_{l=1}^{\infty} b_l z^l, \tag{14}$$

where b_l's are the expansion coefficients determined completely by $Z_1, Z_2, ...Z_l$. The general expression for b_l can be easily obtained and is given by,

$$b_l = (Z_1)^{l-1} \sum_{\{m_i\}} (-1)^{(\sum_i m_i - 1)} (\sum_i m_i - 1)! \prod_i \frac{(Z_i/Z_1^i)^{m_i}}{m_i!}. \tag{15}$$

The summation over m_i is constrained by $\sum_{i=1}^{l} i m_i = l$. We may also write the equation of state as an expansion in powers of average density of the system, $\rho = N/V$, as

$$\beta P = \rho \left[1 + \sum_{l=1}^{\infty} B_{l+1} (\rho\lambda^2)^l \right], \tag{16}$$

where $\lambda = \sqrt{2\pi\beta/m}$ is the thermal wave length. The virial coefficients B_l are related to the coefficients b in the fugacity expansion. Thus the high temperature limit of the virial expansion of the equation of state exists if all the virial coefficients are finite in this limit. In order to proceed with the computation let us assume that all the B_l's are finite and hence the virial expansion is valid in the high temperature limit. To relate these virial coefficients to the statistics parameter g, first consider the high temperature expansion for the factor Z_N/Z_1^N which appears in eq.(6),

$$\frac{Z_N}{Z_1^N} = \frac{1}{N!} + f_2^{(N)} (\beta\omega)^2 + f_3^{(N)} (\beta\omega)^4 + ..., \tag{17}$$

where $f_k^{(N)}$ are coefficients in the expansion which are yet undetermined. They are in general functions of the interaction strength and may also depend on N. Notice that the high temperature expansion of Z_N/Z_1^N involves expansion in powers of $(\beta\omega)^2$ since we considering N particles confined in an oscillator potential in two space dimensions. In other dimensions this factor is appropriately modified while the proof goes through. Substituting the expression for Z_N/Z_1^N in eq.(17) in eq.(15), we obtain

$$b_l = \frac{1}{(\beta\omega)^{2(l-1)}} [\sum_{n=0}^{l-2} \frac{f_2^{(l-n)}}{l!} (-1)^{l-1} (\beta\omega)^2 + ...]. \tag{18}$$

If we now demand that b_l 's are finite for all l in the high temperature limit, all the coefficients in above expansion up to $(\beta\omega)^{2(l-2)}$ must vanish. In particular therefore,

$$\sum_{n=0}^{l-2} \frac{f_2^{(l-n)}}{l!}(-1)^{l-1} = 0, \tag{19}$$

which is the term of relevence to us. Immediately we obtain a general expression for $f_2^{(N)}$,

$$f_2^{(N)} = \sum_{n=1}^{N-2} \frac{f_2^{(N-n)}}{n!}(-1)^{(n-1)} = N(N-1)\frac{f_2^{(2)}}{N!}, \tag{20}$$

where all $f_2^{(N)}$ are now related to $f_2^{(2)}$ corresponding to the two particle partition function. Substituting eq.(20) and eq.(17) in eq.(6), we find

$$\frac{1}{2} - g = 4f_2^{(2)}, \tag{21}$$

where we have used the value of C which is equal to 4 in two space dimensions. In order to calculate $f_2^{(2)}$ recall the definition of the second virial coefficient B_2 [27,28],

$$B_2 = Z_1 \left[1 - 2\frac{Z_2}{Z_1^2}\right]. \tag{22}$$

Substituting for Z_2/Z_1^2 we find,

$$B_2 = -2f_2^{(2)}. \tag{23}$$

We therefore now have a unique relation between the FES parameter g and the second virial coefficient B_2 in the high temperature limit,

$$\frac{1}{2} - g = -2B_2. \tag{24}$$

A few remarks are in order here. Notice that in order to determine $f_2^{(N)}$ in terms of $f_2^{(2)}$ it is sufficient to demand that the coefficient of the leading divergence in b_l be zero in the high temperature limit. This will uniquely fix g interms of B_2 and is a much weaker condition than demanding all b_l's and hence all the virial coefficients B_l's are finite . Nevertheless, there is no way the weaker condition can be imposed in physical systems except to demand the stronger condition that all the virial coefficients be finite in the high temperature limit. Equation (24) is the main result of the paper: The Haldane exclusion statistics parameter g is completely determined by the high temperature limit of the second virial coefficient of the system of interacting particles in the continuum which admits a virial expansion in this limit.

II. APPLICATION TO ANYON GAS AND ANYONS IN A MAGNETIC FIELD

We now apply the result of the last section to systems of anyons in two dimensions. We first consider the case of anyon gas. In two space dimensions anyons are defined particles whose many particle wave functions pick up a phase $e^{i\pi\alpha}$ under the exchange of any two particles in the configuration space. Here α is the strength of statistical interaction and arbitrary values are llowed by the configuration space topology of many particle system [2]. In particular $\alpha = 0, 1$ corresponds to bosons and fermions. Traditionally α is the exchange statistics parameter while we continue to denote exclusion statistics by the parameter g.

In order to apply the results of section 2, we need to assume that the virial expansion exists for anyon gas. This is a nontrivial assumption since it is not conclusively proved that all the virial coefficients are finite for anyon gas. However the second virial coefficient is easily calculated and it has also been proved that the third virial coefficient is finite [29]. If indeed all virial coefficients are finite, we can relate the exclusion statistics parameter g to the exchange statistics parameter α through the second virial coefficient. We do this now.

The spectrum of two anyons confined in an oscillator potential is given by,

$$E_{n,l} = (2n + |l - \alpha| + 1)\omega$$

where we have taken out the centre of mass contribution since it does not depend on α. Here n, l are radial and angular momentum quantum numbers. The two particle partition function is easily calculated and therefore the second virial coefficient is given by, [30],

$$B_2 = -\frac{2\alpha^2 - 4\alpha + 1}{4}. \tag{25}$$

This reproduces exactly the second virial coefficient for bosons and fermions, $\alpha = 0, 1$. Using eq.(6) we have,

$$g = \alpha(2 - \alpha) \tag{26}$$

for anyon gas. This again reproduces the correct values of g for bosons and fermions. This is an interesting result since the exchange statistics parameter is not the same as the exclusion statistics parameter unlike in the systems considered by Haldane where it turned out that $g = \alpha$. It is also interesting to note that g is related to the two types of energy shifts produced in the spectrum by the presence of α. For example, for all $l \leq 0$, all the two anyon energy levgels move up by α and for $l > 0$, the levels move down by $-\alpha$. Equivalently if we add the state $l = 0$ for the set of levels with $l > 0$, then the shift $-\alpha$ may be reinterpreted as $2 - \alpha$. Note that all the calculations presented here are done in the bosonic basis where l is even including the $l = 0$ state. Later we will argue that g is indeed related to the energy shift when all levels shift up(or down) no matter where the cutoff lies. In one dimensional systems such a shift immediately gives g interms of the shift. In the case of anyons, however, it is not clear why g is a product of the two types of shifts in the energy spectrum. As has been commented in the literature [20], this is perhaps due to the fact that anyons may be considered as interacting exclusion statistics particles.

Next we consider anyons confined in a uniform magnetic field while keeping the oscillator confinement. The single particle spectrum with is given by,

$$E^{(1)}_{n,l} = (2n + |l| + 1)\tilde{\omega} - l\omega_c, \tag{27}$$

where

$$\tilde{\omega} = \sqrt{\omega_c^2 + \omega^2}$$

and ω_c denotes the cyclotron frequency of the magnetic field and ω as usual denotes the strength of oscillator confinement. The single particle partition function is given by,

$$Z_1 = \frac{1}{2\sinh(\beta\tilde{\omega})}[\frac{e^{-\beta\omega_+}}{2\sinh(\beta\omega_+/2)} + \frac{e^{-\beta\omega_-}}{2\sinh(\beta\omega_-/2)}], \tag{28}$$

where $\omega_\pm = \tilde{\omega} \pm \omega_c$.

In order to compute the second virial coefficient we need to know the two particle spectrum which is given by,

$$E^{(2)}_{n,l} = (2n + |l-\alpha| + 1)\tilde{\omega} - (l-\alpha)\omega_c, \tag{29}$$

where we have taken out the centre of mass energy which does not depend on α. In particular when $\alpha = 0$ the above spectrum is the spectrum of two bosonic oscillator in a uniform magnetic field in relative coordinates. In calculating the partition function we use the bosonic basis and hence the sum is confined to even l only. The result can be easily translated to the fermionic basis by replacing α by $1-\alpha$. The factor of interest to us is the ratio Z_2/Z_1 which is given by,

$$\frac{Z_2}{Z_1} = \frac{1}{2\sinh(\beta\tilde{\omega})}\left[\frac{e^{-\beta\omega_-(1-\alpha)}}{2\sinh(\beta\omega_-)} + \frac{e^{\beta\omega_+(1-\alpha)}}{2\sinh(\beta\omega_+)}\right], \tag{30}$$

and the second virial coefficient is given by,

$$\begin{aligned} B_2 &= \lim_{\beta\to 0}\left[Z_1 - 2\frac{Z_2}{Z_1}\right] \\ &= \lim_{\beta\to 0}\frac{1}{2\sinh(\beta\tilde{\omega})}\left[\frac{e^{-\beta\omega_-/2}}{2\sinh(\beta\omega_-/2)} + \frac{e^{\beta\omega_+/2}}{2\sinh(\beta\omega_+/2)} - \frac{e^{-\beta\omega_-(1-\alpha)}}{\sinh(\beta\omega_-)} - \frac{e^{\beta\omega_+(1-\alpha)}}{\sinh(\beta\omega_+)}\right], \end{aligned} \tag{31}$$

Now consider the high temperature limit, ie $\beta \to 0$. The problem now involves two scales, namely the oscillator frequency and the cyclotron frequency of the magnetic field. To begin with we might set both $\beta\omega_+ << 1$ and $\beta\omega_- << 1$. Immediately we see, expanding to the order β the terms in the square brackets, B_2 reporduces the second virial coefficient of the anyon gas given earlier. In this case both ω and ω_c act merely as regulators to produce the anyon gas result at high temperatures. But the case of interest to us is when the trace over levels is confined to a single (lowest) Landau level. This can be implemented by demanding that $\beta\omega_+ >> 1, \beta\omega_c << 1$ and therefore $\beta\tilde{\omega} >> 1$. Immediately we find that

$$B_2 = -e^{-\beta\tilde{\omega}}(1/2 - \alpha), \tag{32}$$

where we have kept the overall exponential factor as it is. This factor arises from the contribution of the zero point energy to the partition function. Obviously since $\beta\tilde{\omega} >> 1$, we get an absurd result for B_2. It is easy to see why- the existence of the zero point energy ensures that there is an energy gap and naively taking the limit $\beta\tilde{\omega} >> 1$ implies that the cutoff in energy stays inside the gap. To recover the trace over lowest Landau level, one should move the cutoff above the gap, or equivalently substract the zero point energy so the energy cutoff for the purposes of tracing lies above the LLL. This is achieved by simply taking out the exponential factor in B_2 and hence we have the simple result

$$B_2 = -(1/2 - \alpha). \tag{33}$$

By using the definition of g in eq.(6), we immediately obtain

$$\frac{1}{2} - g = \frac{C}{2}[\frac{1}{2} - \alpha], \tag{34}$$

where we have retained the factor C which is given by 2^{d_s}. Although we started with anyons in two space dimensions confined in a magnetic field as well as oscillator field, restricting the trace to LLL is equivalent to treating this system as an one-dimensional system for the purposes of state counting. Therefore we set $d_s = 1 (C = 2)$ and immediately obtain,

$$g = \alpha. \tag{35}$$

We also wish to point out that unlike the case of anyon gas, where we had to assume the existence of the virial expansion, the above result is exact without requiring any assumption about the finiteness of the virial coefficients in the high temperature limit. This is so since the energy of the N-particle bosonic ground state in the presence statistical interaction is exactly known. Basically in the N-particle sector the angular momentum J in the absence of the statistical interaction gets shifted to $J - \nu\frac{N(N-1)}{2}$ in the presence of statistical interactions which also results in energy shifts. The factor ν depends on how the wave functions approach zero as two particle coordinates coincide and is completely determined by α, the strength of the statistical interaction. The form of the LLL spectrum is then very similar to that of the Calogero Model which we will discuss in detail in the next section.

It may of interest to point out that the result for g given above also depends on whether the magnetic field is pointing in the positive z-direction or negative z-direction. In the former case the ground state is degenerate with all the positive angular states populating the state. The energy shift is given by $-\alpha$ in the presence of the statistical interaction for all $l > 0$. This is equivalent to a shift $2 - \alpha$ if we count the states in the partition function by including also the $l = 0$ state. In this case therefore $g = 2 - \alpha$. This is exactly the result proved numerically by Johnson and Canright [11] for FQHE system where they find that for quasiparticles of statistics -1/3, $g = 2 - 1/3$. The second case yields $g = \alpha$ as noted above.

It is therefore of interest to study the second virial coefficient and hence g as a function of temperature since the results for LLL and the anyon gas can both be obtained in different limits. We have numerically evaluated the second virial coefficient (eq.(31)) exactly as a function of temperature (β) and the results are shown in Fig.1 for fixed values of $\omega << \omega_c$. It is easily seen that there exist two clear flat regions corresponding to $\beta\omega_c \to 0$ (anyon gas) and $\beta\omega_+ >> 1, \beta\omega_- << 1$ (anyons in a magnetic field). The value of the second virial coefficient at the plataue regions correspond to the two cases discussed above.

We would like to stress here that this does not mean that g as defined by us is a temperature dependent property. Nor is it a high temperature property of the system. The high temperature limit of the partition function is taken in order to count all the states. All states, at low and high energy, contribute equally to the partition function in this limit. The hamiltonian serves the purpose of ordering the states according to energy. To define fine difference of two divergent series it is necessary to order them. Energy gives us a physical basis for doing so. Thus g does not depend on the details of the hamiltonian but only on the ordering. However g can be scale dependent. This can be seen and understood in the $\alpha = 0$ case (free bosons) of the above example (see Fig.1). At high temperatures ($\beta\omega_c << 1$), the magnetic field is irrelevant and the system behaves like a free Bose gas in two dimensions.

In the range $\beta\omega << 1, \beta\omega_c >> 1$, the system effectively behaves like a one dimensional free Bose gas. Thus the second virial coefficient and hence g have the two corresponding plateaus. Exactly the same thing happens for the non-zero values of α also. Therefore g can be scale dependent. In fact this is how we expect non-trivial values of g to appear in physical systems of interacting fermions. Though the full Hilbert space may obey the dimension formula corresponding to fermions ($g = 1$), the low energy sector could deviate from it and correspond to some non-trivial value of g.

Before closing this section we remark that we have a nontrivial and interesting result for anyon gas where g is determined by the exchange parameter α as in other one-d examples. The anyon gas result is surprising since for lattice anyon gas, Haldane argued that the particles are classified as hard core bosons (hence fermions) since the coupling of the particlese to the Chern-Simons gauge field does not affect the Hilbert space dimension. Obviously our result shows that this is not so in the continuum.

III. MODELS FOR IDEAL FES IN ONE DIMENSION

It has lately been recognized that models with the inverse square law interaction, the CSM [13], Haldane-Shastry [31] model and other related models behave very much like ideal gases and that the particles have fractional statistics [32,33]. In this section, we concentrate on the CSM, which is a model defined in the continuum. Just as the Hamiltonian of the particle-flux composite describes the anyons in two dimensions, the CSM Hamiltonian provides the framework for fractional statistics particles in one dimension. However, though the spectrum of the model and its thermodynamics are exactly known [13] the ideal anyon gas nature has not been brought out so far. We demonstrate this now in detail. The results of this section serve as template for other one dimensional systems.

As preamble we first mention some recent results obtained for particles obeying Haldane's fractional exclusion statistics. In a recent paper Wu [6] (the same result was also derived earlier by Isakov [5]) has considered a system with flat dispersion and assuming that Haldane exclusion principle holds (as in the case of anyons in a magnetic field in two dimensions confined to the lowest Landau level) has shown that the statistical distribution function of this system is,

$$\bar{n}(\epsilon) = \frac{1}{w(\epsilon) + g}. \tag{36}$$

Here ϵ is the single particle energy, and $w(\epsilon)$ is given as the solution of the equation,

$$w^g(\epsilon, g)(1 + w(\epsilon, g))^{1-g} = e^{\beta(\epsilon-\mu)}, \tag{37}$$

where β is the inverse temperature and μ is the chemical potential of the system. The distribution function smoothly interpolates between the Bose ($g = 0$) and Fermi($g = 1$) distribution functions. This distribution function has also been derived earlier for the specific case of anyons confined to the lowest Landau level [7]. Wu had also analysed the more general case of many species with different energies ϵ_i and had shown that the above distribution holds when the matrix of statistical parameters g_{ij} is of the form $g\delta_{ij}$. If the species index i could be identified with the momentum k, then indeed it could be the general form for the ideal (exclusion) anyon distribution function. We will demonstrate [16] by a first principles calculation that this is so for model hamiltonian systems in one space dimension. These are the Calogero-Sutherland Model(CSM) [13] and the quasiparticles in the Luttinger model [14]. In particular the inverse square interaction can thus be looked upon as a pure statistical interaction in one dimension.

A. The Calogero-Sutherland model

The hamiltonian of the system of interacting particles in this model is given by ($\hbar = 1, m = 1, c = 1$),

$$H = \sum_{i=1}^{N} \left[-\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\omega^2 x_i^2 \right] + \lambda \sum_{i<j=1}^{N} (x_i - x_j)^{-2}, \tag{38}$$

where the particles are confined in a harmonic well and the thermodynamic limit is obtaining by taking $\omega \to 0$. The particles can also be put on a circle with $|x_i - x_j|$ replaced by the chord length $|sin\frac{\pi}{L}(x_i - x_j)|$ in the interaction term. At special values of the coupling the model can be mapped on to particular matrix models [34]. While all these models have fractional exclusion statistics in the sense defined by Haldane, not all of them may be regarded as ideal in the thermodynamic limit unlike the CSM.

The spectrum of CSM is exactly known. The states can be labelled by a set of occupation numbers $\{n_k\},\ k = 1, ..., \infty$. The energy is given by,

$$E[\{n_k\}] = \sum_{k=1}^{\infty} \epsilon_k n_k + \omega\gamma\frac{N(N-1)}{2}, \tag{39}$$

where $\epsilon_k = k\omega,\ N = \sum_{k=1}^{\infty} n_k$. We have substracted out the zero point energy in the above expression for the spectrum of the CSM Hamiltonian. The factor γ determines how the wave function vanishes in the limit of two particle coordinates approaching each other. In terms of the interaction parameter λ it is given by,

$$\lambda = (\gamma)(\gamma + 1).$$

for antisymmetric wave functions and

$$\lambda = (\gamma)(\gamma - 1)$$

for symmetric wave functions. For bound states γ has to be real and therefore we have the condition that $\lambda \geq -1/4$. Obviously the solution for γ in the above equation has two branches. Though well known, we briefly discuss these branches [35]: Since it is also well known that the virial coefficients are finite for the CSM system in the high temperature limit, we can directly apply the definition of the FES parameter g through the second virial coefficient to the CSM. In the fermionic basis we obtain,

$$g = 1 + \gamma = \frac{1}{2}[1 \pm \sqrt{1 + 4\lambda}].$$

Since $\lambda > -1/4$, the lower branch gives $\gamma < -1/2$ while the upper branch gives $\gamma > 1/2$. Consequently we have $g \leq 1/2$ in the lower branch and $g > 1/2$ in the upper branch. Therefore the semion point is somewhat special. If we now demand that the noninteracting fermionic value of $g = 1$ be contained in the solution, we should choose the upper branch. A similar analysis can be done in the bosonic basis. In this case,

$$g = \gamma.$$

Therefore we have $\gamma > 1/2$ in the upper branch and $0 \leq \gamma < 1/2$ in the lower branch. If we now choose to include the noninteracting bosonic value $g = 0$ then we choose the lower branch. It straightforward to see that switching branches in the same basis is equivalent to changing the basis across the semion point ($g = 1/2$).

Hereafter we choose to work with the Fermionic basis. Independent of which basis we are working, it is easy to see that the Hamiltonian may always be written as,

$$H = \sum_{i=1}^{N} \left[-\frac{1}{2} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \omega^2 x_i^2 \right] + g(g-1) \sum_{i<j=1}^{N} (x_i - x_j)^{-2}, \tag{40}$$

and the spectrum of states as,

$$E[\{n_k\}] = \sum_{k=1}^{\infty} \epsilon_k n_k - (1-g)\omega \frac{N(N-1)}{2}, \tag{41}$$

where n_k are the fermionic occupancies (0,1). Therefore the spectrum as a function of g remains the same once we correct for the difference in the oscillator energies for bosons and fermions. For all practical purposes involving only the spectra, one may assume that the spectra is valid for all $g \geq 0$. In the following discussions we work in the fermionic basis with this proviso.

The energy functional can also be written as,

$$E[\{n_k\}] = \sum_{k=1}^{\infty} \epsilon_k n_k - \omega(1-g) \sum_{k_1<k_2=1}^{\infty} n_{k_1} n_{k_2}. \tag{42}$$

The exact spectrum of the model is thus reproduced by an effective hamiltonian of quasi-particle with constant density of states and constant Landau parameters [37]. As we had discussed in our earlier in the previous section this scale invariant energy shift is the basic reason for the occurrance of nontrivial exclusion statistics. We elaborate on this interpretation later in the conclusions again.

Thus the particles in the CSM are anyons in the sense of Haldane with the exclusion statistics parameter equal to g. We will now show this more directly by deriving the formula for the N-particle Hilbert space dimension. This is infact same as the formula given by Haldane. To this end, we define $N(\epsilon, 0)$ as the number of particles with energy $\epsilon_k < \epsilon$, ie., $N(\epsilon, 0) = \sum_{k=1}^{\infty} \theta(\epsilon - \epsilon_k) n_k$, where $\theta(x) = 0$ for $x \leq 0$ and 1 for $x > 0$. We may now define the shifted single particle energies as

$$\epsilon_A(k, g) = \epsilon_k - \omega(1-g) N(\epsilon_k, 0). \tag{43}$$

We identify this shifted energy with anyon single particle energy since the total energy can now be written as

$$E[\{n_k\}] = \sum_{k=1}^{\infty} \epsilon_A(k,g) n_k \tag{44}$$

In Fig.2 we display the single particle anyon energies as a function of g starting from a fermionic ground state where all the levels up to a given cutoff are occupied. Notice at $g=1$, the occupencies are 0 or 1 corresponding to fermionic occupencies and it can be arbitrary at the bosonic end, $g=0$. While for $g=0$ or $g=1$, we can define these occupencies without any reference to the "cell size", in order to define the possible occupation numbers for arbitrary g, we choose the oscillator frequency ω to provide the size [20]. This does not alter the results at boson and fermion ends. For simplicity we look at occupation numbers for special values of $g = 1/m$ where m is an integer. For the ground state spectrum displayed in Fig.2, the occupation of each cell of size ω is maximal, that is, it is equal to m. However, when one removes particles below the cutoff the occupation can be less than or equal to m. The position of the holes also changes because of the spectral flow in g. The rule that emerges for occupation number of anyons purely by inspection may be stated thus:

Let $m = 1/g$, and let N_i be the number of particles in the occupied states below some ith level. Then an occupation $n_i (n_i \leq m)$ is allowed iff $(N_i \ mod \ m) \leq (m - n_i)$.

(If we allow arbitrary occupation numbers up to a maximum of m without this restriction, then one is lead to statistics different from that of Haldane.) Now assume that there are some d states up to the cutoff and a total of N particles. Using above rule, one can count for each value of m (or g), the total number of states for the N particle system. Indeed it turns out to be the dimension of the N particle Hilbert space given by Haldane, namely

$$D_N(g) = \frac{(d + (1-g)(N-1))!}{N!(d-1-g(N-1))!}.$$

We will come back to this question again in the section on the constraints and counting rules for the occupancies.

B. The distribution function of CSM particles

The distribution function for exclusion statistics particles as given by eq.(36) was derived by Wu. In this the statistics of the particles was assumed as given by Haldane. We now show that the interacting particles in CSM can be thought of ideal fractional exclusion particles. In this section we first define anyon number densities and show that their thermal average is exactly given by the distribution function in eq.(36).

The number of particles with energy less than ϵ is

$$N(\epsilon, g) = \sum_{k=1}^{\infty} \theta(\epsilon - \epsilon_A(k,g)) n_k.$$

The anyon number density in the thermodynamic limit is then given by

$$n_A(\epsilon, g) = \lim_{\Delta\epsilon \to 0} \lim_{\omega \to 0} \frac{N(\epsilon + \Delta\epsilon, g) - N(\epsilon, g)}{\Delta\epsilon}, \tag{45}$$

where the limit $\omega \to 0$ is to be taken first.

We will now derive a differential equation for $\bar{N}(\epsilon, g)$, the thermal average of $N(\epsilon, g)$ and solve it to obtain the anyon distribution function. Consider a state with N-particles labelled by $\{k_i\}, i = 1, ..., N$ ordered such that $k_{i+1} > k_i$. Thus for this state $N(\epsilon_{k_i}, 0) = i-1$. The i^{th} shifted energy is $\epsilon_A(k_i, g) = \epsilon_{k_i} - \omega(1-g)(i-1)$. Note that $\epsilon_A(k_{i+1}, g) - \epsilon_A(k_i, g) \geq \omega g > 0$. Thus the anyon energies $\epsilon_A(k_i, g)$ also increase monotonically with i. We therefore have,

$$N(\epsilon_{k_i}, g) = N(\epsilon_{k_i}, 0) = i - 1 \tag{46}$$

To get some feel for the anyon number densities, let us consider the ground state of the N particle system. The set $\{k_i\}$ is (1,2,3,...,N). The corresponding set of shifted energies is $(\omega, (1+g)\omega, (1+2g)\omega, ..., (1+(N-1)g)\omega)$. The number of particles in the interval ϵ to $\epsilon + \omega$ is then,

$$\begin{aligned} n_A(\epsilon, g) &= \frac{1}{g} \quad , \qquad \epsilon < \epsilon_F = g\bar{\rho} \\ &= 0 \quad , \qquad \epsilon \geq \epsilon_F \ , \end{aligned} \tag{47}$$

where $\bar{\rho} \equiv \omega N$. This is the same as the zero temperature limit of the distribution function in eq.(36)(see also Fig.2).

Next we compute how the anyon density changes with g. From eqs.(45) and (43) we see that as g increases the particles move to the right in ϵ space with a 'velocity' given by $\omega N(\epsilon, g)$. Thus the number of particles crossing a point ϵ when g increases by Δg, is given by the velocity at the point of crossing multiplied by the density at that point. We thus obtain the differential equation,

$$\frac{\partial \rho(\epsilon, g)}{\partial g} = \rho(\epsilon, g) \frac{\partial \rho(\epsilon, g)}{\partial \epsilon}, \tag{48}$$

where $\rho(\epsilon, g) = \omega N(\epsilon, g)$. Denoting the thermal average of $\rho(\epsilon, g)$ as $\bar{\rho}(\epsilon, g)$ and using the fact that in the thermodynamic limit, $\bar{\rho^2}(\epsilon, g) = (\bar{\rho}(\epsilon, g))^2$ we obtain the differential equation,

$$\frac{\partial \bar{\rho}(\epsilon, g)}{\partial g} = \bar{\rho}(\epsilon, g) \frac{\partial \bar{\rho}(\epsilon, g)}{\partial \epsilon}. \tag{49}$$

The distribution function $\bar{n}_A(\epsilon, g)$ can be obtained from the solution of eq.(49) by using $\bar{n}_A(\epsilon, g) = \frac{\partial \bar{\rho}(\epsilon, g)}{\partial \epsilon}$. Thus we need the solution to eq.(49), with the boundary condition,

$$\left.\frac{\partial \bar{\rho}(\epsilon, g)}{\partial \epsilon}\right|_{g=1} = \frac{1}{e^{\beta\epsilon} + 1} \tag{50}$$

so as to match the result at the fermionic end.

We will now show that eq.(49) along with the boundary condition (50) is satisfied by ,

$$\bar{\rho}(\epsilon, g) = \int_0^\epsilon d\epsilon' \frac{1}{w(\epsilon', g) + g} \quad , \tag{51}$$

where $w(\epsilon, g)$ is determined through eq.(37). By changing variables from ϵ to w in eq.(51) the integral can be done to get,

$$\bar{\rho}(\epsilon, g) = \frac{1}{\beta}(ln(\frac{w(\epsilon)}{1+w(\epsilon)}) - ln(\frac{w(0)}{1+w(0)})) \tag{52}$$

The condition $\bar{\rho}(\infty, g) = \bar{\rho}$ and the fact that $\lim_{\epsilon\to\infty} w(\epsilon, g) \to \infty$ implies that $w(0)$ is independent of g and $e^{-\beta\mu} = w(0)^g(1+w(0))^{1-g}$. It can be verified that,

$$\frac{\partial}{\partial g} w(\epsilon) = \frac{w(w+1)}{(g+w)} \bar{\rho}(\epsilon, g), \tag{53}$$

$$\frac{\partial}{\partial \epsilon} w(\epsilon) = \beta \frac{w(w+1)}{(g+w)}. \tag{54}$$

Using the above results it can be verified that the form of $\bar{\rho}(\epsilon, g)$ in eq.(51) does satisfy eq.(49). Thus the anyon distribution function in the CSM is given by,

$$\bar{n}_A(\epsilon, g) = \frac{1}{w(\epsilon, g) + g} \tag{55}$$

which is exactly the distribution function derived by Wu [6]. Notice, however, the main difference lies in the fact that we started with interacting fermion system and derived these results whereas in Wu's method fractional statistics is assumed from the beginning.

C. Partition function and equation of state

The additivity property of the anyon energies suggests that the grand partition function should be expressible in a factorized form. If we assume that the usual definition of the distribution function for the average number of particles holds for anyons, namely,

$$\bar{n}_A(x, g) = \frac{\partial}{\partial x} ln z(x) \tag{56}$$

where $lnZ^A = \int_0^\infty dx\ lnz(x)$, x is the dimensionless energy defined as $x = \frac{\epsilon}{\omega}$. Then it follows that we must have,

$$lnZ^A = \int_0^\infty dx \ \ ln(1 + w^{-1}(x)) \tag{57}$$

If this were true, it would show that the particles in CSM have the basic properties of an ideal gas. We now show that this is indeed the case.

The N-particle partition function for the spectrum in eq.(41) is given by

$$Z_N^A = e^{\tilde{\beta}(1-g)\frac{N(N-1)}{2}} Z_N^F, \tag{58}$$

where $\tilde{\beta} = \beta\omega$ and Z_N^F is the N particle fermion partition function. Setting $g = 0$, the bosonic partition function is obtained,

$$Z_N^B = e^{\tilde{\beta}\frac{N(N-1)}{2}} Z_N^F, \tag{59}$$

Combining eqs.(58) and (59) we may write the anyon partition function as,

$$Z_N^A = (Z_N^F)^g (Z_N^B)^{1-g}. \tag{60}$$

The grandpartition function may also be written in the form,

$$Z^A = \sum_{N=0}^{\infty} e^{\beta\mu_A N} Z_N^A = \sum_{N=0}^{\infty} (e^{\beta\mu_F N}(Z_N^F))^g (e^{\beta\mu_B N}(Z_N^B))^{1-g}, \tag{61}$$

where we have set the anyon chemical potential $\mu_A = g\mu_F + (1-g)\mu_B$. In the thermodynamic limit, the sum is saturated at the value of $N = \bar{N}$ where $\bar{N} = \frac{1}{\beta}\frac{\partial ln Z^A}{\partial \mu_A}$. The grand partition function can then be written as,

$$Z^A(\beta,\mu) = (Z^F(\beta,\mu_F))^g (Z^B(\beta,\mu_B))^{1-g}, \tag{62}$$

where μ_F and μ_B are determined by the conditions, $\bar{N} = \frac{1}{\beta}\frac{\partial ln\ Z^F}{\partial \mu_F} = \frac{1}{\beta}\frac{\partial ln\ Z^B}{\partial \mu_B}$. The solution to these conditions is $e^{-\beta\mu_F} = w(0), e^{-\beta\mu_B} = 1 + w(0)$. It can be verified that this ensures that $\frac{1}{\beta}\frac{\partial ln Z^A}{\partial \mu_A} = \bar{N}$. We therefore have

$$ln\ Z^A(\beta,\mu) = g\int_0^{\infty} dx\ ln(1 + e^{-(\tilde{\beta}x - \beta\mu_F)}) - (1-g)\int_0^{\infty} dx\ ln(1 - e^{-(\tilde{\beta}x - \beta\mu_B)}) \tag{63}$$

We now make the change of variables $(1 + e^{-(\tilde{\beta}x - \beta\mu_F)}) = 1 + w^{-1}$ in the first term and $(1 - e^{-(\tilde{\beta}x - \beta\mu_B)}) = 1 - (1+w)^{-1}$ in the second term. We then have,

$$ln\ Z^A(\beta,\mu) = \int_{w(0)}^{\infty} dw \frac{w+g}{w(1+w)} ln(1 + w^{-1}) \tag{64}$$

We now again make a change of variable from w to ϵ using eq.(37). The partition function then gets written as,

$$lnZ^A = \int_0^{\infty} dx\ ln(1 + w^{-1}(x)) \tag{65}$$

This is exactly the factorized form in eq.(57).

We now consider the equation of state of the ideal gas of anyons in CSM. This has already been done by Sutherland [38] who however chose to work with the fugacity expansion. The coefficients of terms in this expansion in general depend on g and appear quite complicated. However, the coefficients of the virial expansion in average density $\bar{\rho}$ are extremely simple. From the product form of the partition function in eq. (62), the virial coefficients can all be computed exactly since the virial expansion for the free Fermi and Bose gas is exactly computable when the density of states is constant. We have,

$$lnZ^A = \beta\ P = \bar{\rho}\ \sum_{l=1}^{\infty} b_l\ (\beta\bar{\rho})^{l-1} \tag{66}$$

where b_l are the virial coefficients for the ideal Fermi or Bose gas for $l \neq 2$ ($b_l^F = b_l^B\ \forall\ l \neq 2$ for constant density of states) and $b_2 = -\frac{1}{2}(\frac{1}{2} - g)$. It is interesting to note that the interaction affects only the second virial coefficient. As we had shown [12], it is the second virial coefficient that determines the exclusion statistics. Thus the $\frac{1}{r^2}$ interaction in this system modifies the equation of state in a minimal way. It can therefore be thought of as a purely statistical interaction in this system. One should be careful in defining the thermodynamic limit in which the above result is valid. This thermodynamic limit is specified by $N \to \infty$ while the density N/L is held fixed. Since the oscillator parameter ω plays the role of $1/L$ it is also equivalent to fixing ωN. Thus the thermodynamic limit is completely specified with by taking high temperature limit first and then $\omega \to 0, N \to \infty$ such that the density is finite.

There has been some confusion about the result derived above [40,41]: The confusion mainly arises from the fact that the results for the virial coefficients at bosonic and fermionic ends are identical to the virial coefficients one gets for free bosons and fermions in 2-dimensions whereas the system we are considering is a 1-dimensional system. Naively one would expect that a system in a box with box length $L(L \to \infty)$ and the system in a harmonic well with frequence $\omega(\omega \to 0)$ to be formally equivalent. In actual practice, however, one never works with infinite size systems. We do use the thermodynamic limit to model systems where the thermal energy scale is much larger than the level spacing. As can be easily seen the density of states in the infinite size limit is $g(E) = E^{-1/2}$. However if ω is finite the density of states is constant corrresponding to linear dispersion. Since the thermodynamics of a non-interacting system is determined by its density of states and statistics the two cases have different thermodynamic limits. In this section we are interested in this case where the system has constant density of states.

D. Quasiparticles in the Luttinger Model

We will now consider quasiparticles in the Luttingermodel. We note that this spectrum (eq.(41))is identical to the spectrum of quasi-particle states in a gaussian theory of compact bosons [12,36] with radius $R = 1/\sqrt{(g)}$ with the following identification- while k in the above equation is a state label in CSM, in the gaussian theory of compact bosons k refers to the box quantized momenta. The discussion in the subsequent sections thus applies to this case also and we will elaborate on these cases in the next section. As can be seen from eq.(41), the effect of the interaction is that each particle shifts the energy of every other particle by a constant $\omega(g-1)$. The low energy physics of this model maps on to the massless Thirring model. As is well known [14], the model is exactly solvable. The theory can be written completely in terms of the left(L) and right(R) currents satisfying the algebra

$$[J_n^r, J_m^{s\dagger}] = n\delta_{mn}\delta_{rs},$$

where $n, m = 1, 2, ...$ and $r, s = L, R$. We also have the zero modes

$$[\phi_0^r, J_0^s] = i\delta_{rs}.$$

The hamiltonian is

$$H = \frac{2\pi v_F}{L} \sum_r \left[\frac{1}{2} J_{0r}^2 + \sum_{n=1}^{\infty} J_n^r J_n^{r\dagger} \right]. \tag{67}$$

Quasiparticles are created by the action of the vertex operators $V_r^\dagger(x) =: e^{-iq^r\phi^r(x)}$: on the ground state. The $\phi^r(x)$ are the bosonic phase fields given by,

$$\phi^r(x) = \phi_0^r(x) + \frac{2\pi}{L}\sigma_r \left[xJ_{0r} + \sum_{n=1}^{\infty}\left(\frac{e^{-i\sigma_r\frac{2\pi}{L}nx}}{in}J_n^r + hc\right)\right], \tag{68}$$

where $\sigma_{L,R} = \pm 1$. The periodicity properties of the compact zero modes $\phi_0^{L,R}$ constrain the allowed values of q_r to be $q_r = (NR + \sigma_r\frac{M}{2R})$ and R is the radius of the field ϕ. It is a function of the interaction strength. Our convention correspond to $R = 1$ for the case of noninteracting fermions. The constraints on q_r imply that quasiparticles in the left sector must be created along with quasiparticles in the right sector (except for special values of R). However because the hamiltonians of the two sectors are completely decoupled, it is consistent to analyse the spectrum of the two sectors independently. We will therefore focus on the left sector alone.

We first consider the space of one quasiparticle states which we define to be the space of the states $|x >= V_L^\dagger(x)|0>$. These are not a linearly independent set. From the form of $V_L^\dagger(x)$, it follows that $|x + L >= e^{i\alpha\pi}|x>$, where $\alpha = q_L^2$. Therefore, we have the expansion $|x >= \sum_{n=1}^{\infty} e^{ik_nx}|n>$, where $k_n = \frac{2\pi}{L}(n + \frac{\alpha}{2})$. From the form of $V_L^\dagger(x)|0>$ it follows that $n > 0$. It can be easily shown that $|n>$ form an orthogonal set of eigenstates of the Hamiltonian with eigenvalues $E_n^{(1)} = v_F k_n$. The single particle partition function is then given by,

$$Z_1 = \frac{e^{-\tilde{\beta}\alpha/2}}{1 - e^{-\tilde{\beta}}} = e^{-\tilde{\beta}\alpha/2}Z_1^B; \qquad \tilde{\beta} = \frac{2\pi}{L}\beta v_F. \tag{69}$$

Next we come to the N quasiparticle states which we define to be the span of $|\{x_n\} >= \prod_{n=1}^N V_L^\dagger(x_n)|0>$. From the fact that $V_L^\dagger(x)V_L^\dagger(y) = e^{i\alpha\pi}V_L^\dagger(y)V_L^\dagger(x)$, it follows that these many quasiparticle states pick up a phase $e^{i\alpha\pi}$ under the exchange of two of the coordinates. We normal order the vertex operators to obtain, $|\{x_n\} >= \prod_{n>m}\left[sin\left(\frac{\pi(x_n - x_m)}{L}\right)\right]^\alpha : \prod_{n=1}^N V_L^\dagger(x_n) : |0>$. Again from the form of the vertex operators it follows that we have the expansion,

$$|\{x_n\} >= \prod_{n>m}^{N}\left[sin\left(\frac{\pi(x_n - x_m)}{L}\right)\right]^\alpha \sum_{\{k_n\}} \phi_N^B(\{k_n\}|\{x_n\})|\{k_n\}>, \tag{70}$$

where $\phi_N^B(\{k_n\}|\{x_n\})$, are symmetrized N particle plane waves with momenta $\{k_n\}$, where

$$k_n = \frac{2\pi}{L}(n + \frac{N\alpha}{2}); \quad n > 0.$$

The states $|\{k_n\}>$ can be shown to be eigenstates of the hamiltonian with energy $E^N(\{k_n\}) = v_F\sum_{n=1}^N k_n$. This exactly reporduces the spectrum of states one gets in the CSM(see eq.(43)). We can show that $|\{k_n\}>$ are a linearly independent set of states. The states with the same energy are however not orthogonal.

The N-quasiparticle partition function is then

$$Z_N = e^{-\tilde{\beta}N^2\alpha/2}Z_N^B,$$

where Z_N^B is the N-particle bosonic partition function. We can now exactly compute g using the high temperature expansion for Z_N^B and obtain $g = \alpha$. Thus the exchange and exclusion statistics parameters are identical for these models. Note that the exchange statistics in one dimensions is somewhat arbitary. We could have changed it by multiplying the vertex operators with suitable cocycle factors. The exclusion statistics parameter is however unambigous and unique. When $R = \frac{1}{\sqrt{2}}$, the theory is equivalent to the low energy physics of the SU(2) symmetric quantum antiferromagnetic chain. We then have the spinon excitations with $\alpha = 1/2$. Thus we recover Haldane's result that $g = 1/2$ for this case.

The above example gives a clear insight into the mechanism of the phenomenon. What is happening is that the addition of a quasiparticle causes a phase shift of every other quasiparticle, resulting in an energy shift of $\frac{\alpha\pi v_F}{L}$ per particle . When we count the dimension of the single particle space with a fixed (smooth) cutoff, there are α states missing. The important thing here is that the all single particle levels shift up by the same amount however high the energy. This is why we get g to be well defined and nontrivial in the cutoff going to infinity limit. We will come back to this intuitive but physical arguement again in the last section.

IV. COUNTING RULES, NEGATIVE PROBABILITIES AND ALL THAT

The problem of negative probabilities was first pointed out by Nayak and Wilczek [20] and elaborated by Polychronokos in a recent paper [21]. In order to clarify the origin of negative probabilities or weights, we first discuss an equation and its solution due to Ramanujan [24]. Ramanujan considered the following equation:

$$aqX^p - X^q + 1 = 0, \tag{71}$$

where a may be complex and p, q are positive.The general solution for X^d is,

$$X^d \;=\; \sum_{N=0}^{\infty} C_N(p,q,d)a^N, \tag{72}$$

where $C_0(p,q,d) = 1$ and $C_1(p,q,d) = d$ and

$$C_N(p,q,d) = \frac{d}{N!}\prod_{j=1}^{N-1}(d + Np - jq), \quad N \geq 2.$$

To make connection with the result obtained by Polychronokos [21], which is a particular case of the general solution given by Ramanujan, we now put $p = (1-g)$ and $q = 1$, then

$$C_N(1-g,1,d) \;=\; d\frac{(d + (1-g)N - 1)!}{N!(d - gN)!} \tag{73}$$

which is clearly different from the dimension formula of Haldane. However, it correctly reproduces the bosonic and fermionic dimension formula for $g = 0$ and $g = 1$ respectively. This dimension formula was derived independently by Polychronokos [21] with the restriction that any two particles are atleast g sites apart when placed on a periodic lattice. One can also derive the Haldane dimension formula with the restriction that any two particles are g sites apart but without the restriction of periodicity.

Further if we put $X = (1+w^{-1})$ and $a = e^{-\beta\epsilon}$ in eq.(71), we immediately obtain equation (37) derived earlier by Wu. The important point to notice here is that the dimension formula that precisely leads to the distribution function derived earlier [6,5,7] is given by C_N and not the Haldane dimension formula. In the limit $d >> 1$, however, it is easy to see that

$$C_N(1-g,1,d) = D_N(g,d) + O(\frac{1}{d}).$$

Therefore in the continuum limit, the C_N and D_N are approximately the same.

The grand canonical partition function of the system may be written as,

$$Z \;=\; (1+w^{-1})^d \;=\; \sum_{N=0}^{\infty} C_N(1-g,1,d)e^{-\beta N\epsilon}, \tag{74}$$

where w satisfies equation (37). We have also assumed that all the energy levels are degenerate with energy given by ϵ. Note that this is an exact expression and no assumption is required on the single particle dimension d. The negative weights arise [20,21], when one insists on expanding $1+w^{-1}$ in powers of $e^{-\beta\epsilon}$. From equation (72) and the definitions following the equation, it follows that,

$$1+w^{-1} \;=\; \sum_{n=0}^{\infty} C_n(1-g,1,1)e^{-\beta n\epsilon} \tag{75}$$

The weights

$$C_n(1-g,1,1) = \; p_n \;=\; \prod_{m=2}^{n}(1-\frac{gn}{m}) \tag{76}$$

are always negative for $gn > m$ for some m [21]. This is indeed the problem of negative weights associated with exclusion statistics and is claimed to be inherent in the exclusion statistics. There are however a few points to note: The negative probabilities arise because of our insistence on the factorization [23] implied in eq.(74). For example, combining equations (74) and (75) we have,

$$Z \;=\; \sum_{\{n_j\}}(\prod_j C_{n_j}(1-g,1,1))e^{-\beta\epsilon\sum_j n_j}, \tag{77}$$

where the sum is an unconstrained one over all sets of occupation numbers. The over counting resulting from this unconstrained sum is compensated by the occurence of negative weights. We next derive the precise counting rules which impose constraints on this sum and avoids this problem.

A. Realization in an interacting system and counting rules

Any realization of fractional exclusion statistics must have its origins in systems of interacting particles. The expectation is that under certain conditions systems of interacting particles which obey Fermi or Bose statistics may be described in terms of quasiparticles (or quasiholes) which obey fractional statistics. The quasiparticles of the CSM behave like ideal exclusion statistics particles. The main feature of CSM is that the total energy of the many-body system can be written in terms of single quasi-particle energies which involve shifted momenta and these shifts contain the information about the exclusion statistics of the quasiparticles. In this section we analyse these shifted momenta and make explicit connection with the formula in eq. (2). We then use them to obtain constraints on the allowed set of occupation numbers. These are what we refer to as the counting rules that reproduce the formula in eq.(2). The statistical mechanics of the system obeying these constraints is then the same as that defined by Wu [6] and all statistical weights are positive.

We begin with the trignometric Sutherland model [42] of an N-particle system on a ring of unit radius. The Hamiltonian is given by,

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{j<i} \frac{2g(g-1)}{\sin^2[(x_i - x_j)/2]}, \tag{78}$$

where g is the interaction parameter. We will soon identify this with the statistical parameter of the exclusion statistics. While the model can be applied to both interacting bosons and fermions, we choose to work in the fermionic basis here after. The energy of an N-fermion state may be written in terms of shifted momenta as

$$E = \sum_{i=1}^{\infty} k_i^2 n_i, \tag{79}$$

where $n_i = 0, 1$ and the shifted momenta k_i (also called pseudo momenta in Ref. [17]) are given by

$$k_i \;=\; m_i - (1-g)\frac{(N_i^- - N_i^+)}{2}, \tag{80}$$

where m_i are distinct integers, $N_i^{-(+)}$ are the number of particles with shifted momenta less (greater) than k_i. Note that we could have also started with the Calogero-Sutherland model with harmonic confinement considered in detail in the previous section. The results below follow analogously with the proviso that we have shifted energies instead of shifted momenta. We choose to work with shifted momenta here since generalizations to higher dimensions may be easier if one were to derive the constraints in momentum variables instead of assuming specific form of confinement.

First we establish the relationship between the shifted momenta given above and the Haldane's dimension formula (2). Consider the above system with an upper and lower cutoff on the momenta, k_{max} and k_{min} respectively. We divide this range of momenta into cells of unit length (the first and last cells could be smaller) and define the occupancy of the j^{th} cell, n_j to be the number of particles with momenta k_i, such that $j+1 > k_i \geq j$. We identify single particle space dimension d with the number of cells in the range, i.e $d = k_{max} - k_{min}$, where d may be fractional. If we now denote the range of the m_is by d_F, we have

$$d_F = m_{max} - m_{min} = d + (1-g)(N-1). \tag{81}$$

Since there exists an m_i for every k_i, the total number of states in the range $k_{max} - k_{min}$ is the same as that between $m_{max} - m_{min}$. The total number of states is then the number of ways N distinct integers can be picked from d_F distinct integers, i.e ${}^{d_F}C_N$, as in fermionic description. Substituting for d_F from the above expression we immediately reproduce the Haldane dimension formula in eq.(2) [12].

In order to obtain the counting rules we will first derive three properties of the set of momenta $\{k_i\}$. If k_i are ordered such that they increase with increasing i, then we have, $k_{i+1} - k_i = m_{i+1} - m_i - (1-g)$. If $g < 1$, then it follows that $m_{i+1} > m_i$. Further, if $m_{i+1} - m_i = 1$ then $k_{i+1} - k_i = g$ and if $m_{i+1} - m_i > 1$ then $k_{i+1} - k_i > (1+g) > 1$ because m_i's are integers.

We can then draw the following three conclusions from the properties of the shifted momenta k_i:

1. The ordering in k_is is the same as the ordering in m_is.

2. "Close packed" m_is with unit spacing correspond to "close packed" k_is with spacing g.

3. The gaps between any two non-close packed k_is is greater than 1. Therefore all the k_is in any cell are close packed.

We now come to the question as to what are the constraints on the sets of occupation numbers $\{k_i\}$. For example, if $g = 0$, there are no constraints as in the bosonic case. If $g = 1$ the constraints are $n_j \leq 1$ as in the case of fermions. For any other g, one obvious constraint come from the second property derived above, namely the occupancy of the j-th cell $n_j \leq \frac{1}{g}$ which specifies the maximum occupancy of a given cell assumed to be of unit spacing. This is the same constraint one derives from the distribution function of Wu (36). An important departure from the usual bosonic and fermionic case is that the cell size is important and cannot be arbitrarily taken to zero as in the case of bosons and fermions [20].

There are further constraints on the occupancy. To formulate them we use the third property. Let k_L be the lowest momentum in the j^{th} cell. Then from the second and third property, it follows that

$$k_L + g(n_j - 1) < j + 1. \tag{82}$$

We can write k_L as $k_L = j + f(k_L)$, where $f(k_L)$ denotes the fractional part of k_L, that is, $0 \leq f(k_L) < 1$. We then have,

$$f(k_L) + g(n_j - 1) \ < \ 1. \tag{83}$$

From equation (80), we can express $f(k_L)$ as a function of the occupation numbers,

$$f(k_L) \ = \ f\left[-(1-g)\frac{[N_{cj}^{-} - N_{cj}^{+} - (n_j - 1)]}{2}\right], \tag{84}$$

where $N_{cj}^{-} = \sum_{l<j} n_l$ and $N_{cj}^{+} = \sum_{l>j} n_l$. Equations (83) and (84) then constitute a set of constraints on the occupation numbers.

We will now show that these form a complete set of constraints. Namely, given any set of occupation numbers, $\{n_j\}$, that satisfies the constraints, there exists a set of momenta, $\{k_i\}$, that realizes it. To do this, consider a set $\{n_j\}$, where $j_{min} \leq j \leq j_{max}$. The lowest value of the momentum in the j^{th} cell is uniquely determined by the occupation numbers through equation(84). Because of the third property, all the other momenta are also uniquely determined. Hence we have shown that there are no more constraints. Equations (83) and (84) form a complete set of constraints. Note also that the above logic implies that there is a one to one correspondence between the sets of occupation numbers, $\{n_j\}$, that satisfy the constraints (83) and (84)and the sets of momenta, $\{k_i\}$, that satisfy equation (80).

We can now remove the scaffolding of the Sutherland Model that we started with and *define* exclusion statistics system by the above constraints. The connection to the dimension formula in eq.(2) established earlier implies that

$$\sum_{\{n_i\}} F(\{n_i\}) \; = \; D_N(g,d), \tag{85}$$

where $N = \sum_j n_j$ and $F(\{n_i\}) = 1$ if $\{n_i\}$ satisfy the constraints and zero otherwise. Note that the weights now are positive definite. There are no negative weights once the constraints are imposed.

Next, we construct some simple examples from the above counting rules. For simplicity we look at occupation numbers for special values of $g = 1/m$ where m is an integer. The rules formulated above for the occupation number of exclusion particles may be combined and restated thus:

> *Let $m = 1/g$, and let N_i be the number of particles in the occupied states below some ith level, $N_i = \sum_{j<i} n_i$. Then an occupation $n_i(n_i \leq m)$ is allowed iff $(N_i \ mod \ m) \leq (m - n_i)$.*

This rule now includes all the three constraints stated above.

To see how this rule is implemented, consider a system of N-particles spread over d states. In order that D_N is an integer, we choose $N = mp + 1$, where p is an integer. Since $N \leq md$, we have $p < d$. We shall divide these d states into cells. An allowed configuration may be represented as a string of numbers $(n_1 n_2 n_3, ...)$, where each $n_i \leq m$ denotes the occupancy of levels ordered from left to right. Instead of dealing with a configuration where all N particles are spread over d states (some which may be empty), we can simplify the discussion by considering one cell at a time. Each cell may now have a partition of m. This allows us to fill the subsequent cells without reference to the previous cell according to the counting rules since $N_i \ mod \ m = 0$. We now fill each cell with a partition of m which is allowed by the rules given above. This then generates all possible allowed configurations whose sum is given by D_N.

If, in particular, we are interested in expectation values of symmetric functions of n_i, we can work with symmetrised weights. Consider a symmetric operator $O(\{n_i\})$. The expectation value of this operator may be written as,

$$< O(\{n_i\}) >= \frac{\sum_{\{n_i\}} F_s(\{n_i\}) O(\{n_i\})}{\sum_{\{n_i\}} F_s(\{n_i\})}, \tag{86}$$

where

$$F_s(\{n_i\}) \;=\; \frac{1}{d!}\sum_p F(p_{n_i}). \tag{87}$$

Here p stands for all permutations of the allowed configurations. Every allowed configuration in $\{n_i\}$ may be characterised by the multiplicities q_n, namely a given allowed configuration may be written as a string, $m^{q_m}(m-1)^{q_{m-1}}...1^{q_1}$, where $q_1 + 2q_2 + ... + mq_m = N$. We may now also allow any permutation of these occupancies (with zeros added to make up d-states). The dimension of the N-particle space may then be written as

$$D_N(g,d) = \sum_{\{q_n\}} f_m^N(q_1, q_2, ...) \quad {}^dC_q, \tag{88}$$

where $q = \sum_{n=1}^m q_n$. The new weights f are defined as,

$$f_m^N(q_1, q_2, ...q_m) = \frac{M_a(q_1, q_2, ...q_m)}{M_t(q_1, q_2, ..., q_m)}, \tag{89}$$

where M_a allowed configurations after symmetrising and M_t is the total number of configurations for a given set of q's which define a configuration. We shall clarify this now with specific examples.

B. The case of semion ($g = 1/2, m = 2$)

The maximal occupancy of a state in this case is 2. Hence allowed occupancy of a state is 2 or 1. Zeros may occur any where without changing the rules. Let us implement this in the specific case of $d = 4, N = 5$, say. In this case the allowed configurations are given by the strings (2210),(2111),(1121). In the first configuration, zero can be anywhere and therefore there are four configurations. Notice that a string of the form (1211) or (1112) violates the counting rules. Therefore counting all the allowed configurations we obtain $D_5(1/2, 4) = 6$. This is exactly what one gets from the Haldane formula.

Further if we symmetrise each of these allowed configurations, then the new weights may be computed using eq.(89). In the specific case of $m = 2$, we have

$$M_t(q_1, q_2) ={}^{q_1+q_2} C_{q_2}, \qquad M_a(q_1, q_2) ={}^p C_{q_2}, \tag{90}$$

where p is defined through the equation $N = 2p+1$. The corresponding f is therefore given by,

$$f_2^N = \frac{{}^pC_{q_2}}{{}^{q_1+q_2}C_{q_2}} \tag{91}$$

Note that these weights, wheather in the symmetrised form or unsymmetrised form, are positive definite. Further, this is exactly the formula derived by Chaturvedi and Srinivasan [23] in their microscopic analysis of Haldane statistics for semions.

It is important to stress the differences in these two approaches- in their analysis Chaturvedi and Srinivasan start from a formulation of the statistical mechanics of a system by removing factorizability of the weights as a criterion. They derive the expression for the weights in eq.(91) by imposing the conditions positivity and the requirement of symmetry (all configuratons which are permutations of each other carry the same weight). Our starting point is the Sutherland model. We derive our rules from the properties of shifted momenta. After removing this scaffolding, we obtain not only positive definite weights for each configuration but when symmetrised they reproduce the results of Chaturvedi and Srinivasan.

C. The case with $g = 1/3$ or $m = 3$

The maximal occupancy of a state in this case is 3. The allowed configurations for each cell are (3),(21),(12),(111). That is we can form a string of allowed configuration with any of these cells in any order to make up N particles. Any number of zeros may be added in between to make up a total of d-states.

As in the semion case we may consider expectation values of symmetric functions of n_i. Following the same procedure we can derive the symmetrized weights f_3^N defined in eq.(89). Since, $m = 3$, we have

$$M_t(q_1, q_2, q_3) = ^{q_1+q_2+q_3} C_{q_3} \ ^{q_1+q_2}C_{q_2}, \qquad M_a(q_1, q_2, q_3) = ^{p} C_{q_3} \ ^{p-q_3}C_{q_2} \ (2)^{q_2}, \tag{92}$$

where p, as before, is defined through the equation $N = 3p + 1$. The corresponding weight f is therefore given by,

$$f_3^N = \frac{^{p}C_{q_3} \ ^{p-q_3}C_{q_2} \ (2)^{q_2}}{^{q_1+q_2+q_3}C_{q_3} \ ^{q_1+q_2}C_{q_2}} \tag{93}$$

These weights are again positive definite. Chaturvedi and Srinivasan [23] also suggest how their method may be extended beyond the semion case which they considered in detail. However, this extension requires additional conditions which are not imposed in the semion case. In contrast, our rules as derived from the point of view of an exactly solvable model are completely specified independent of the actual value of g (or m). There is an algorithm to derive f_m^N for arbitrary m though this gets complicated for larger m.

V. DISCUSSIONS AND CONCLUSIONS

We have thus presented a detailed discussion of the generalized Pauli principle as enunciated by Haldane in his seminal paper [3]. Much of this is based on our own work in this area. We have also tried to include discussions of the relevant details from other important contributions as needed. Since this is not a review of the field we have also omitted many contributions. One of the importants conclusions that emerges is that we may look upon certain types interactions as giving rise to new types of statistics, where the word statistics has been used purely in the sense of restrictions on occupation numbers. Given an energy cell, the number of single particle states in the cell depend on the nature of the interaction between particles and hence the occupencies. This fact alone is not sufficient to generalise the occupation probabilities. If however, the occupancies are independent of where the cell is located in the full energy space, one expects a generalised Pauli principle to hold. Haldane's idea realises such a situation. There have also been other complementary ideas which we have not discussed here.

We would like to emphasise this crucial property of exclusion statistical interactions it should cause shifts in single particle energies at all scales [12]. This property is realized by a large class of one dimensional models of interacting fermions where Fermi liquid theory breaks down [39,19]. In fact it has been shown exactly that quasiparticles with nontrival exclusion statistics exist in a class of models that are solved by the Bethe ansatz [5,15]. In particular the quasiparticles of the Calogero-Sutherland model (CSM) behave like ideal exclusion statistics system [5,15–17].

One immediate casuality of the exclusion statistics is the fact the now we can not associate an occupation probability for a single state as the exclusion acts across levels. If we did this in an unconstrained manner, then some of these probabilities may turn out to be negative. However, as we have shown the particles obeying fractional exclusion statistics may be characterised by constraints on the sets of occupation numbers. There are no negative probabilities if these constraints are obeyed. If these constraints are relaxed then the negative weights arise in order to compensate for the resulting over counting. The negative probabilities discussed in literature can be understood as arising when the system constrained by the counting rules is replaced by an unconstrained one. The negative weights then compensate for the introduction of unphysical configurations. This is therefore exactly analogous to other situations in physics where negative probablities arise, for example, the ghosts and negative norm states in gauge theories or as in the case of Wigner distribution in quantum mechanics.

Finally, we have not discussed the realisation of exclusion statistics in higher dimensions than one. There have been several attempts in this direction. For example, in two dimensions a many body system with very short-range two body interaction may be thought of as a candidate for realising exclusion statistics. Unlike CSM this system does not describe an ideal FES system, but to the leading order it does have the features of FES. This was shown first using the Thomas-Fermi approximation and later by doing the exact HF mean-field calculations [43].

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