There are five fundamental operations in mathematics: addition, subtraction, multiplication, division, and modular forms.

— Apocryphal quote ascribed to Martin Eichler
Mock modular forms and physics: an invitation

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Modular forms
What are modular forms?

Holomorphic function $f(\tau)$, $\tau \in \mathcal{H}$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z})$$

Periodicity $\tau \rightarrow \tau + 1 \quad \Rightarrow \quad $ Fourier series

$$f(\tau) = \sum_{n} a(n) q^n , \quad q = e^{2\pi i \tau}$$

Interesting numbers

Ring structure

Weight
Basic examples: Eisenstein series

\[ E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240 q + 2160 q^2 + \cdots, \]

\[ E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504 q - 16632 q^2 - \cdots, \]

\[ E_{2k}(\tau) = \cdots \]
The space of modular forms of a given weight is finite-dimensional

\[
\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 + \ldots
\]

\[
= \left( E_4(\tau)^3 - E_6(\tau)^2 \right) / 1728
\]

Ramanujan tau function

\[
j(\tau) = \left( 7E_4(\tau)^3 + 5E_6(\tau)^2 \right) / \Delta(\tau)
\]

\[
= q^{-1} + 24 + 196884q + \ldots
\]

Partition function of Leech lattice

\[
1 + 196883
\]

(c.f. Talks of Harvey, Hikami, Taormina, Wendland)

(see e.g. Zagier, 1-2-3 of modular forms)
Relations to physics
1. Modular forms are generating functions of solutions to interesting counting problems

e.g.: Heterotic string theory has 16 supersymmetries

Number of 1/2 BPS states \( d(N) \) at \( m^2 = Q^2 = N - 1 \)

Fundamental string states with right-movers in ground state

Left-moving energy \( N \) distributed in 24 oscillators

(Dabholkar, Harvey ‘89)

\[
\sum_{N=0}^{\infty} d_{\text{micro}}(N) \, q^{N-1} = \frac{q^{-1}}{\prod_{n=1}^{\infty} (1 - q^n)} = \frac{1}{\Delta(\tau)} = \frac{1}{\eta(\tau)^{24}}
\]

\[
= q^{-1} + 24 + 324 q + \cdots
\]
In string theory, ensembles of these microscopic excitations form black holes.

Microscopic

\[ d_{\text{micro}}(N) = e^{\pi \sqrt{N}} + \cdots \quad (N \to \infty) \]

Sen '94, Strominger-Vafa '96

Macroscopic

\[ S_{\text{BH}} = \frac{A_H}{4\ell^2_{Pl}} = \pi \sqrt{N} \]

Bekenstein-Hawking '74

Asymptotic estimates a very useful guide for Quantum gravity:
Hardy-Ramanujan-Rademacher expansion
2. CFT$_2$ on a torus naturally produces modular forms

Vibration of a string governed by a two-dimensional CFT.

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

Large coordinate transformations

Symmetry should be reflected in the physics.
Superconformal theories produce holomorphic partition functions

\[ N=(2,2) \text{ SCFT} \quad (L_0, Q_0^\pm, J_0), \quad (-1)^F \]

Elliptic genus

\[ Z_{\text{ell}}(M; \tau, z) = \text{Tr}_{\mathcal{H}(M)} (-1)^{F+\tilde{F}} q^{L_0} \overline{q}^{\tilde{L}_0} \zeta^J_0 \]

\[ q := e^{2\pi i \tau}, \quad \zeta := e^{2\pi iz}. \]

\[ Z_{\text{ell}} \text{ holomorphic in } \tau \text{ (and } z). \quad \text{(Witten)} \]

(Subtlety! Troost,+Ashok, Eguchi-Sugawara, Talk of Troost)
Jacobi forms
Jacobi forms: basic definitions

\( \varphi(\tau, z) \) holomorphic in \( \tau \in \mathcal{H} \) and \( z \in \mathbb{C} \)

**M:** \( \varphi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k e^{\frac{2\pi imcz^2}{c\tau + d}} \varphi(\tau, z) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z}) \)

**E:** \( \varphi(\tau, z + \lambda \tau + \mu) = e^{-2\pi im(\lambda^2 \tau + 2\lambda z)} \varphi(\tau, z) \quad \forall \lambda, \mu \in \mathbb{Z} \)

**Fourier expansion:**
\[
\varphi(\tau, z) = \sum_{n,r} c(n, r) q^n \zeta^r
\]

**Growth condition (weak Jacobi form):** \( c(n, r) = 0 \) unless \( n \geq 0 \).
Relation between Jacobi forms and modular forms

Elliptic property $\Rightarrow$ **Theta expansion:**

$$
\varphi(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau) \vartheta_{m, \ell}(\tau, z),
$$

where

$$
\vartheta_{m, \ell}(\tau, z) := \sum_{r \in \mathbb{Z}} q^{r^2/4m} \zeta^r.
$$

$$
\Rightarrow h_\ell(\tau) = \int \varphi(\tau, z) e^{-2\pi i \ell z} \, dz
$$

vector valued modular form
Examples of Jacobi forms

\[ A = \varphi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6} = \frac{(\zeta - 1)^2}{\zeta} - 2 \frac{(\zeta - 1)^4}{\zeta^2} q + \cdots \]

\[ B = \varphi_{0,1}(\tau, z) = \sum_{i=2}^{4} \frac{\vartheta_i(\tau, z)^2}{\vartheta_i(\tau, 0)^2} \]
\[ = \frac{\zeta^2 + 10\zeta + 1}{\zeta} + 2 \frac{(\zeta - 1)^2 (5\zeta^2 - 22\zeta + 5)}{\zeta^2} q + \cdots \]

\[ C = \varphi_{-1,2}(\tau, z) = \frac{\vartheta_1(\tau, 2z)}{\eta(\tau)^3} = \frac{\zeta^2 - 1}{\zeta} - \frac{(\zeta^2 - 1)^3}{\zeta^3} q + \cdots \]

Ring of weak Jacobi forms generated by A, B, C.
What is new?
Wall-crossing and BH phase transitions

Phase I

\[(Q, P)\]

Phase II

\[(Q, P)\] + \[Q \rightarrow P\]

Serious problem: throwing out multi-centered BHs (Denef-Moore 2007) destroys the modular symmetry.
A concrete realization: $N=4$ string theory

Partition function of 1/4 BPS dyons

\[
Z^{(N=4)}_{(dyon)}(\tau, z, \sigma) = \frac{1}{\Phi_{10}(\tau, z, \sigma)} = \sum_{m=-1}^{\infty} \psi_m(\tau, z) e^{2\pi i m \sigma}.
\]

Meromorphic Jacobi forms of weight -10, index m. (poles in $z$)

\[
\psi_m(\tau, z) = \sum_{n,r} d_{\text{micro}}(n, r) q^n \zeta^r.
\]

(Dijkgraaf, Verlinde, Verlinde; Gaiotto, Strominger, Yin; David, Sen)

Has zeros (divisors) in the Siegel upper half plane.

Igusa cusp form

(c.f. talks of Hohenneger, Govindarajan, Persson, Volpato)
Questions

- What is the correct expansion of the meromorphic Jacobi forms?
- Can we extract the degeneracies of the single-centered black hole?
- What are the modular properties of the corresponding Fourier coefficients?

Mock modular forms.
Solution of BH wall-crossing problem

Canonical decomposition of the partition function:

\[ \psi_m = \psi_m^{BH} + \psi_m^{\text{multi}} \]

Partition function of the isolated BH is a *mock modular form*.

Multi-centers and wall-crossing info in Appell-Lerch sum.