There are five fundamental operations in mathematics:
addition, subtraction, multiplication, division, and modular forms.

— Apocryphal quote ascribed to Martin Eichler

Mock modular forms and physics: an invitation

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Modular forms

What are modular forms?

Holomorphic function $f(\tau)$, $\quad \tau \in \mathcal{H}$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Weight

Ring structure

Periodicity $\tau \to \tau + 1 \quad \Rightarrow \quad$ Fourier series

$$f(\tau) = \sum_n a(n) q^n \,, \qquad q = e^{2\pi i \tau}$$

Interesting numbers

Basic examples: Eisenstein series

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240\, q + 2160\, q^2 + \cdots\ ,$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} = 1 - 504\, q - 16632\, q^2 - \cdots\ ,$$

$$E_{2k}(\tau) = \cdots$$

The space of modular forms of a given weight is finite-dimensional

(see e.g. Zagier, 1-2-3 of modular forms)

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24\, q^2 + 252\, q^3 + \ldots$$

$$= \left(E_4(\tau)^3 - E_6(\tau)^2\right)/1728$$

Ramanujan tau function

$$j(\tau) = (7E_4(\tau)^3 + 5E_6(\tau)^2)/\Delta(\tau)$$

$$= q^{-1} + 24 + 196884\, q + \cdots$$

Partition function of Leech lattice

$$1 + 196883$$

(c.f. Talks of Harvey, Hikami, Taormina, Wendland)

Relations to physics

1. Modular forms are generating functions of solutions to interesting counting problems

e.g.: Heterotic string theory has 16 supersymmetries

Number of 1/2 BPS states $d(N)$ at $m^2 = Q^2 = N - 1$

Fundamental string states with right-movers in ground state

Left-moving energy N distributed in 24 oscillators

(Dabholkar, Harvey '89)

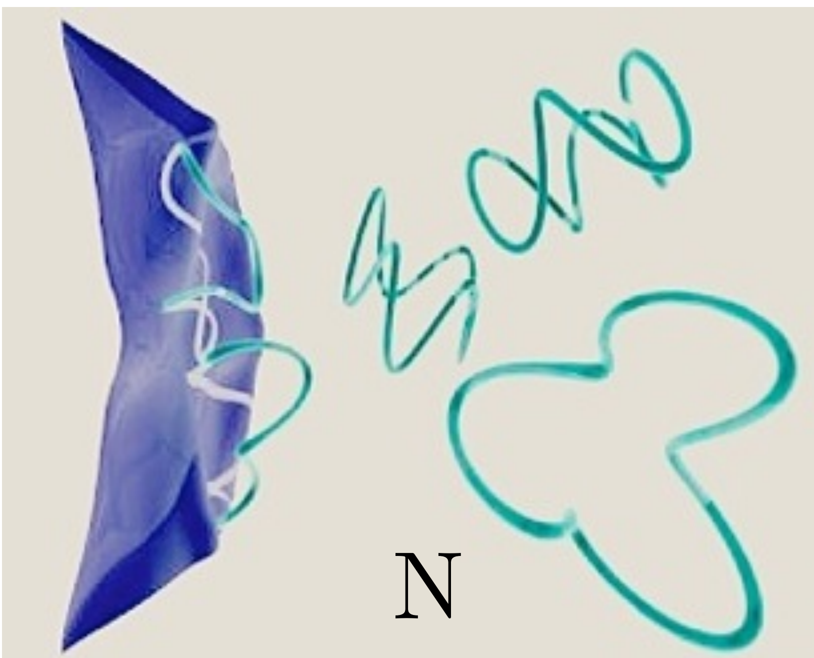
$$\sum_{N=0}^{\infty} d_{\text{micro}}(N)\, q^{N-1} = \frac{q^{-1}}{\prod_{n=1}^{\infty}(1-q^n)} = \frac{1}{\Delta(\tau)} = \frac{1}{\eta(\tau)^{24}}$$

$$= q^{-1} + 24 + 324\, q + \cdots$$

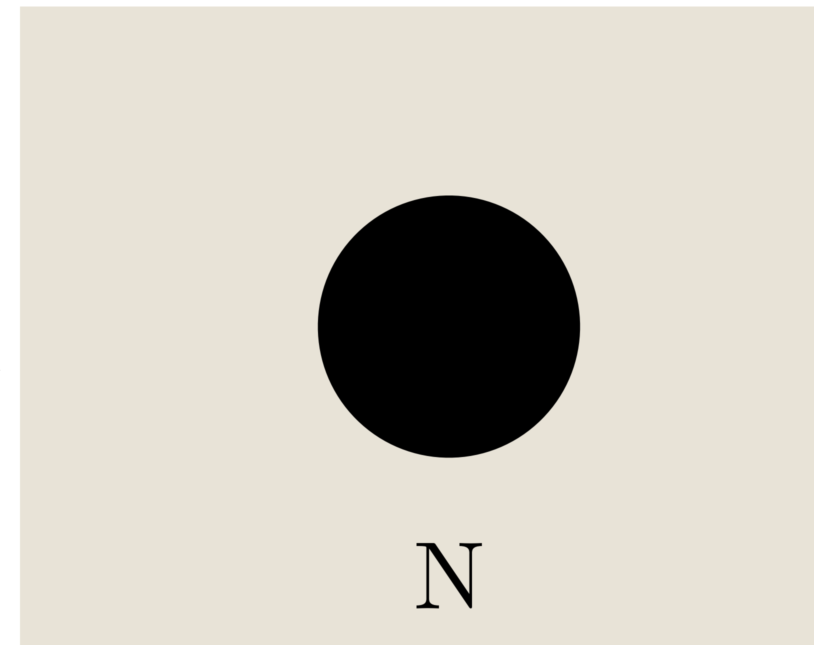
In string theory, ensembles of these microscopic excitations form black holes

Microscopic

Macroscopic



$g_s\mathrm{N} \ll 1$ ← g_s → $g_s\mathrm{N} \gg 1$



Sen '94, Strominger-Vafa '96

$$d_{\text{micro}}(\mathrm{N}) = e^{\pi\sqrt{\mathrm{N}}} + \cdots \quad (\mathrm{N} \to \infty)$$

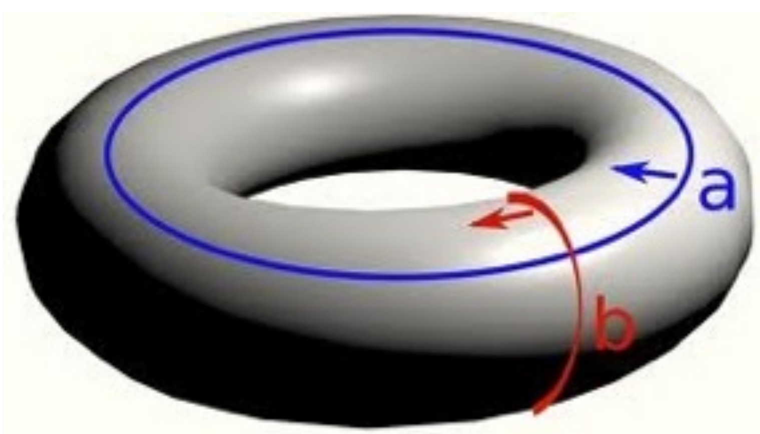
Bekenstein-Hawking '74

$$S_{\mathrm{BH}}^{\text{class}} = \frac{A_H}{4\ell_{\mathrm{Pl}}^2} = \pi\sqrt{N}$$

Asymptotic estimates a very useful guide for Quantum gravity:
Hardy-Ramanujan-Rademacher expansion

2. CFT_2 on a torus naturally produces modular forms

Vibration of a string governed by a two-dimensional CFT.



τ

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

Large coordinate transformations

Symmetry should be reflected in the physics.

Superconformal theories produce holomorphic partition functions

N=(2,2) SCFT $\quad (L_0,\, Q_0^{\pm},\, J_0),\ (-1)^F$

Elliptic genus

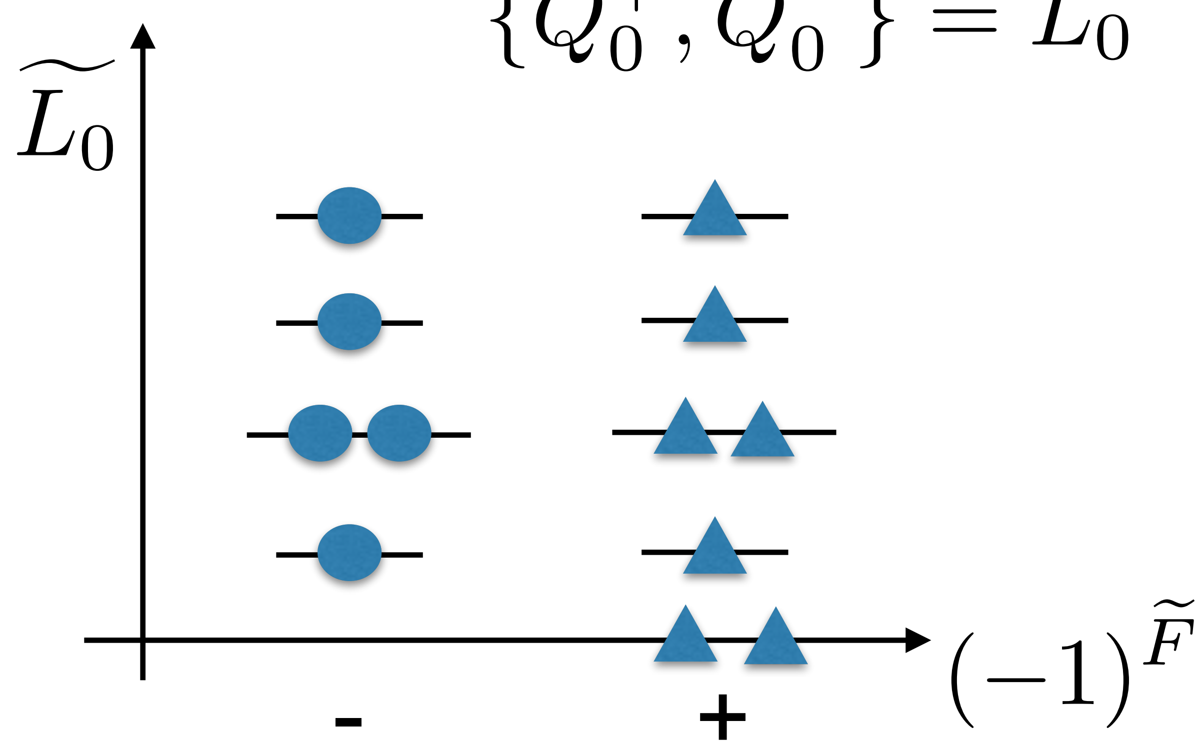
$$Z_{\rm ell}(M;\tau,z) = {\rm Tr}_{\mathcal{H}(M)}\,(-1)^{F+\widetilde{F}}\, q^{L_0}\,\overline{q}^{\widetilde{L_0}}\,\zeta^{J_0}$$

$$q := e^{2\pi i\tau},\ \zeta := e^{2\pi i z}\,.$$

$Z_{\rm ell}$ holomorphic in τ (and z). (Witten)

(Subtlety! Troost,+Ashok, Eguchi-Sugawara, Talk of Troost)

$$\{Q_0^+, Q_0^-\} = L_0$$



Jacobi forms

Jacobi forms: basic definitions

(Eichler-Zagier)

$\varphi(\tau, z)$ holomorphic in $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$

Weight

M: $$\varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \varphi(\tau, z) \qquad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$$

E: $$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)} \varphi(\tau, z) \qquad \forall \quad \lambda, \mu \in \mathcal{Z}$$

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Interesting numbers

Fourier expansion: $$\varphi(\tau, z) = \sum_{n,r} c(n,r)\, q^n \zeta^r$$

Growth condition (weak Jacobi form): $c(n,r) = 0$ unless $n \geq 0$.

Relation between Jacobi forms and modular forms

Elliptic property $\Rightarrow$ *Theta expansion*:

$$\varphi(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\, \vartheta_{m,\ell}(\tau, z)\,,$$

where
$$\vartheta_{m,\ell}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \ell \,(\mathrm{mod}\, 2m)}} q^{r^2/4m}\, \zeta^r\,.$$

$$\Rightarrow h_\ell(\tau) = \int \varphi(\tau, z)\, e^{-2\pi i \ell z}\, dz$$

vector valued modular form

Examples of Jacobi forms

$$A = \varphi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6} = \frac{(\zeta - 1)^2}{\zeta} - 2\,\frac{(\zeta - 1)^4}{\zeta^2}\,q + \cdots$$

$$B = \varphi_{0,1}(\tau, z) = \sum_{i=2}^{4} \frac{\vartheta_i(\tau, z)^2}{\vartheta_i(\tau, 0)^2}$$

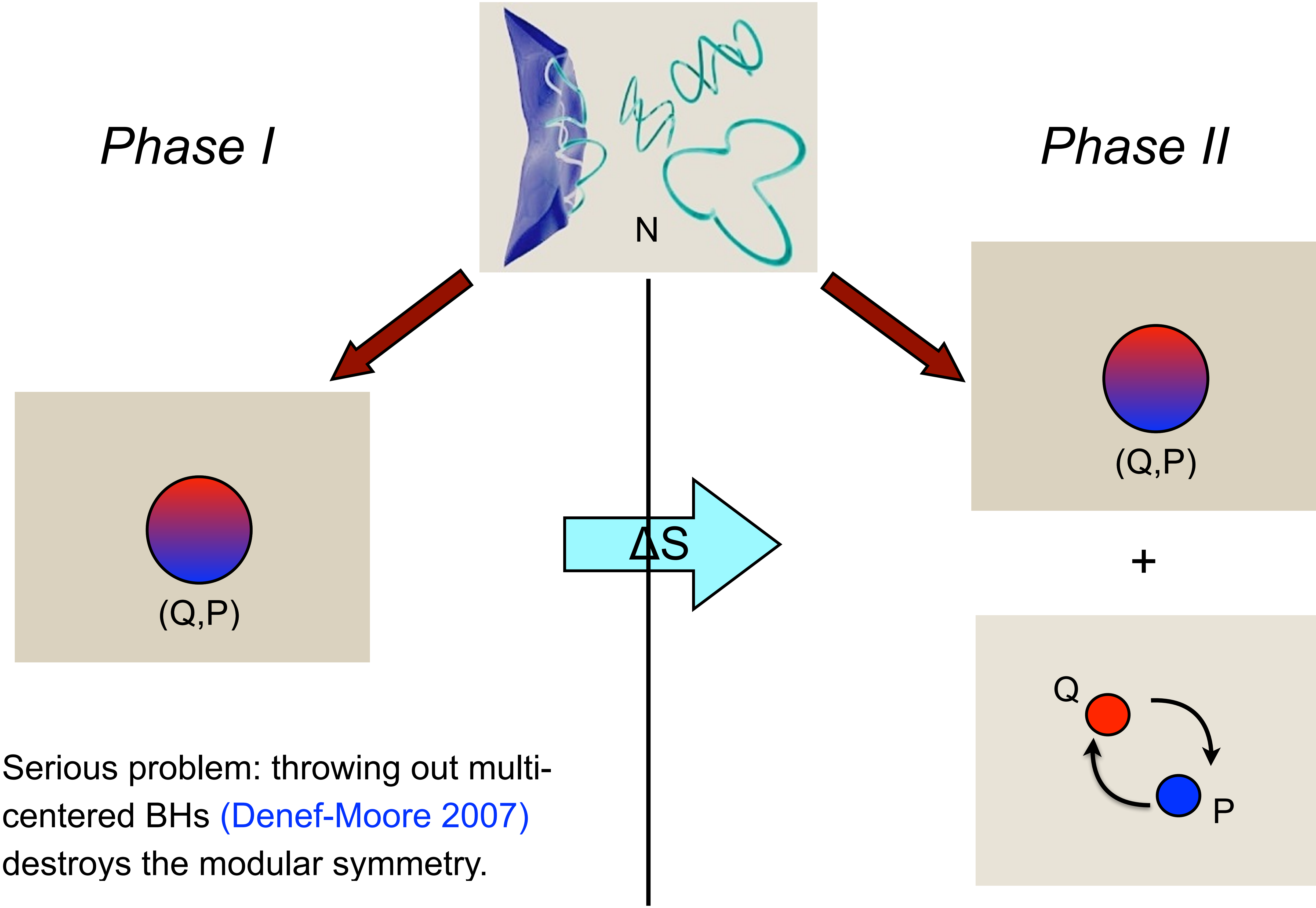
$$= \frac{\zeta^2 + 10\zeta + 1}{\zeta} + 2\,\frac{(\zeta - 1)^2\,(5\zeta^2 - 22\zeta + 5)}{\zeta^2}\,q + \cdots \quad .$$

$$C = \varphi_{-1,2}(\tau, z) = \frac{\vartheta_1(\tau, 2z)}{\eta(\tau)^3} = \frac{\zeta^2 - 1}{\zeta} - \frac{(\zeta^2 - 1)^3}{\zeta^3}\,q + \cdots$$

Ring of weak Jacobi forms generated by A, B, C.

What is new?

Wall-crossing and BH phase transitions



Serious problem: throwing out multi-centered BHs (Denef-Moore 2007) destroys the modular symmetry.

A concrete realization: N=4 string theory

Partition function of 1/4 BPS dyons

(Dijkgraaf, Verlinde, Verlinde; Gaiotto, Strominger, Yin; David, Sen)

$$Z^{(N=4)}_{(\mathrm{dyon})}(\tau, z, \sigma) = \frac{1}{\Phi_{10}(\tau, z, \sigma)}$$

$$= \sum_{m=-1}^{\infty} \psi_m(\tau, z)\, e^{2\pi i m \sigma} \, .$$

Igusa cusp form

Has zeros (divisors) in the Siegel upper half plane.

Meromorphic Jacobi forms of weight -10, index m.
(poles in z)

$$\psi_m(\tau, z) \stackrel{?}{=} \sum_{n,r} d_{\mathrm{micro}}(n, r)\, q^n\, \zeta^r \, .$$

(c.f. talks of Hohenneger, Govindarajan, Persson, Volpato)

Questions

- What is the correct expansion of the meromorphic Jacobi forms?
- Can we extract the degeneracies of the single-centered black hole?
- What are the modular properties of the corresponding Fourier coefficients?

→ **Mock modular forms.**

Solution of BH wall-crossing problem

Canonical decomposition of the partition function:

$$\psi_m = \psi_m^{BH} + \psi_m^{\mathrm{multi}}$$

Partition function of the isolated BH is a *mock modular form*.

Multi-centers and wall-crossing info in Appell-Lerch sum.