Generalized Mathieu Moonshine
and Siegel Modular Forms

Daniel Persson
Chalmers University of Technology

Mock Modular Forms and Physics
IMSc, Chennai, April 18, 2014

Talk based on:
[arXiv:1312.0622] (w/ R. Volpato)
[arXiv:1302.5425] (w/ M. Gaberdiel, & R. Volpato)
[arXiv:1211.7074] (w/ M. Gaberdiel, H. Ronellenfitsch, R. Volpato)
What is **Moonshine**?
What is **Moonshine**?

The term “moonshine” generally refers to surprising connections between a priori unrelated parts of mathematics and physics, involving:

representation theory
of finite groups
What is **Moonshine**?

The term “**moonshine**” generally refers to surprising connections between a priori unrelated parts of mathematics and physics, involving:

- representation theory of finite groups
- modular forms
What is **Moonshine**?

The term “**moonshine**” generally refers to surprising connections between a priori unrelated parts of mathematics and physics, involving:

representation theory of finite groups

modular forms  conformal field theory
What is Moonshine?

The term “moonshine” generally refers to surprising connections between a priori unrelated parts of mathematics and physics, involving:

- Representation theory of finite groups
- Modular forms
- Infinite-dimensional algebras
- Conformal field theory
What is **Moonshine**?

The term “**moonshine**” generally refers to surprising connections between a priori unrelated parts of mathematics and physics, involving:

- Representation theory of finite groups
- Modular forms
- Infinite-dimensional algebras
- Conformal field theory
What is **Moonshine**?

The term “moonshine” generally refers to surprising connections between a priori unrelated parts of mathematics and physics, involving:

- representation theory of finite groups
- modular forms
- infinite-dimensional algebras
- conformal field theory

The most famous example is **Monstrous Moonshine**.
Monstrous Moonshine

$J(\tau) = q^{-1} + 196884q + \cdots$

modular function

monstrous Lie algebra $\mathcal{M}$

bosonic string theory on $(T^{24}/\Lambda_{\text{Leech}})/\mathbb{Z}_2)$
(holomorphic VOA $V^{\mathcal{M}}$)

(Figure stolen from Jeff’s talk!)
Monstrous Moonshine

\[ J(\tau) = q^{-1} + 196884q + \cdots \]

modular function

\[ M(\mathbb{T}_{24}/\Lambda_{\text{Leech}}/\mathbb{Z}_2) \]

holomorphic VOA \( V^\mathbb{Z} \)

(Figure stolen from Jeff's talk!)
In 2010, Eguchi, Ooguri, Tachikawa conjectured that there is **Moonshine** in the elliptic genus of K3 connected to the finite sporadic group $M_{24} \subset S_{24}$

**EOT observation**: Fourier coefficients of K3-elliptic genus are (sums of) dimensions of irreps of $M_{24}$

A completely new moonshine phenomenon to explore!
<table>
<thead>
<tr>
<th>Monstrous Moonshine</th>
<th>Mathieu Moonshine</th>
</tr>
</thead>
<tbody>
<tr>
<td>monster group $\mathbb{M}$</td>
<td>Mathieu group $\mathbb{M}_{24}$</td>
</tr>
<tr>
<td>bosonic CFT</td>
<td>superconformal field theory</td>
</tr>
<tr>
<td>Virasoro algebra</td>
<td>$\mathcal{N} = (4, 4)$ superconformal algebra</td>
</tr>
<tr>
<td>$J$-function</td>
<td>elliptic genus of K3</td>
</tr>
<tr>
<td>McKay-Thompson series</td>
<td>twining genera</td>
</tr>
<tr>
<td>monster module $V^\mathfrak{g}$</td>
<td>?</td>
</tr>
<tr>
<td>monster Lie algebra $\mathfrak{m}$</td>
<td>?</td>
</tr>
</tbody>
</table>

[Refs]: [Eguchi, Ooguri, Tachikawa][Cheng][Gaberdiel, Hohenegger, Volpato][Eguchi, Hikami][Taormina, Wendland][Gannon]
Despite this amazing progress, we still don’t understand why Mathieu moonshine holds. More precisely, we cannot answer the question:

What does $M_{24}$ act on?
Despite this amazing progress, we still don’t understand why Mathieu moonshine holds. More precisely, we cannot answer the question:

What does $M_{24}$ act on?

We have considered a “two-step generalization” of Mathieu moonshine that sheds light on this question.
Despite this amazing progress, we still don’t understand *why Mathieu moonshine holds*. More precisely, we cannot answer the question:

**What does $M_{24}$ act on?**

We have considered a “two-step generalization” of Mathieu moonshine that sheds light on this question.
1. Generalized Mathieu moonshine
Generalized Mathieu Moonshine

[Gaberdiel, D.P., Ronellenfitsch, Volpato]

Introduce a family of functions, the **twisted twining genera**:

\[ \phi_{g,h} : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C} \]

for each commuting pair \( g, h \in M_{24} \)

such that for \( g = e \) we recover the twining genera \( \phi_{e,h} = \phi_h \)
Generalized Mathieu Moonshine

[Gaberdiel, D.P., Ronellenfitsch, Volpato]

Introduce a family of functions, the \textit{twisted twining genera}:

\[ \phi_{g,h} : \mathbb{H} \times \mathbb{C} \to \mathbb{C} \]

for each commuting pair \( g, h \in M_{24} \)

such that for \( g = e \) we recover the twining genera \( \phi_{e,h} = \phi_{h} \)

This is the analogue of Norton’s \textit{generalized monstrous moonshine}

\[ Z_{g,h} : \mathbb{H} \to \mathbb{C} \quad \quad g, h \in M \]

\[ Z_{e,h}(\tau) = T_{h}(\tau) \quad \quad \text{McKay-Thompson series} \]
Generalized Mathieu Moonshine

[Introduce a family of functions, the *twisted twining genera*:

\[ \phi_{g,h} : \mathbb{H} \times \mathbb{C} \to \mathbb{C} \]

for each commuting pair

\[ g, h \in M_{24} \]

such that for \( g = e \) we recover the twining genera \( \phi_{e,h} = \phi_h \)

This is the analogue of Norton’s *generalized monstrous moonshine*

\[ Z_{g,h} : \mathbb{H} \to \mathbb{C} \]

\[ g, h \in M \]

Partially explained by orbifolds of the FLM monster VOA \( V^m \).

Proven in special cases but the full conjecture still open.

[Dixon, Ginsparg, Harvey][Tuite]

[Dong, Li, Mason][Höhn][Tuite][Carnahan]
Generalized Mathieu Moonshine
[Gaberdiel, D.P., Ronellenfitsch, Volpato]

Introduce a family of functions, the \textit{twisted twining genera}:

\[ \phi_{g,h} : \mathbb{H} \times \mathbb{C} \to \mathbb{C} \]

for each commuting pair \( g, h \in M_{24} \)

such that for \( g = e \) we recover the twining genera \( \phi_{e,h} = \phi_h \)

This is the analogue of Norton’s \textit{generalized monstrous moonshine}

\[ Z_{g,h} : \mathbb{H} \to \mathbb{C} \quad g, h \in \mathbb{M} \]

Can we also interpret generalized Mathieu moonshine in terms of orbifolds?
Holomorphic Orbifolds and Group Cohomology

Our main assumption is that the twisted twining genera behave similarly as for characters of a holomorphic orbifold.
Our main **assumption** is that *the twisted twining genera behave similarly as for characters of a holomorphic orbifold*.

**Fact:** Consistent holomorphic orbifolds are classified by $H^3(G, U(1))$.

[Dijkgraaf, Witten][Dijkgraaf, Pasquier, Roche][Bantay][Coste, Gannon, Ruelle]
Holomorphic Orbifolds and Group Cohomology

Our main assumption is that the twisted twining genera behave similarly as for characters of a holomorphic orbifold

Fact: Consistent holomorphic orbifolds are classified by $H^3(G, U(1))$.  
[Dijkgraaf, Witten][Dijkgraaf, Pasquier, Roche][Bantay][Coste, Gannon, Ruelle]

→ multiplier phases of characters $Z_{g,h}(\tau)$ determined by 2-cocycle

$$c_g \in H^2(C_G(g), U(1))$$

obtained from a class $[\alpha] \in H^3(G, U(1))$ via

$$c_h(g_1, g_2) = \frac{\alpha(h, g_1, g_2)\alpha(g_1, g_2, (g_1g_2)^{-1}h(g_1g_2))}{\alpha(g_1, h, h^{-1}g_2h)}$$
In particular, for the S- and T-transformations we have

\[ Z_{g,h}(\tau + 1) = c_g(g, h)Z_{g,gh}(\tau) \]

\[ Z_{g,h}(-1/\tau) = \overline{c_h(g, g^{-1})}Z_{h,g^{-1}}(\tau) \]
In particular, for the S- and T-transformations we have

\[ Z_{g,h}(\tau + 1) = c_g(g,h)Z_{g,gh}(\tau) \]

\[ Z_{g,h}(-1/\tau) = \overline{c_h(g,g^{-1})}Z_{h,g^{-1}}(\tau) \]

Moreover, under conjugation of \( g, h \) one has the general relation

\[ Z_{g,h}(\tau) = \frac{c_g(h,k)}{c_g(k,k^{-1}hk)}Z_{k^{-1}gk,k^{-1}hk}(\tau) \quad \forall k \in G \]
Cohomological Obstructions from $H^3(G)$

$$Z_{g,h}(\tau) = \frac{c_g(h, k)}{c_g(k, k^{-1}hk)} Z_{k^{-1}gk, k^{-1}hk}(\tau)$$

Whenever $k$ commutes with both $g$ and $h$ one finds

$$Z_{g,h} = \frac{c_g(h, k)}{c_g(k, h)} Z_{g,h}$$
Cohomological Obstructions from $H^3(G)$

$$Z_{g,h}(\tau) = \frac{c_g(h, k)}{c_g(k, k^{-1}hk)} Z_{k^{-1}gk, k^{-1}hk}(\tau)$$

Whenever $k$ commutes with both $g$ and $h$ one finds

$$Z_{g,h} = \frac{c_g(h, k)}{c_g(k, h)} Z_{g,h}$$

So $Z_{g,h} = 0$ unless the 2-cocycle $c_g$ is regular:

$$c_g(h, k) = c_g(k, h)$$

When this is not satisfied we have **obstructions!** [Gannon]
Conjecture (generalized Mathieu moonshine) [GHPV]:

For each \( g \in M_{24} \) there exists a graded unitary representation \( \mathcal{H}_g \) of \( \mathcal{N} = 4 \) with central charge \( c = 6 \) carrying a projective representation

\[
\rho_g : C_{M_{24}}(g) \to GL(\mathcal{H}_g)
\]

commuting with \( \mathcal{N} = 4 \) and determined by a class \([\alpha] \in H^3(M_{24}, U(1))\).
**Conjecture** (generalized Mathieu moonshine) [GHPV]:

For each $g \in M_{24}$ there exists a graded unitary representation $\mathcal{H}_g$ of $\mathcal{N} = 4$ with central charge $c = 6$ carrying a projective representation

$$\rho_g : C_{M_{24}}(g) \to GL(\mathcal{H}_g)$$

commuting with $\mathcal{N} = 4$ and determined by a class $[\alpha] \in H^3(M_{24}, U(1))$.

For each commuting pair $g, h \in M_{24}$ there exists functions $\phi_{g,h}(\tau, z)$ satisfying:

- $\phi_{e,h} = \phi_h$ and $\phi_{e,e} = \chi(K3; \tau, z)$
**Conjecture** (generalized Mathieu moonshine) [GHPV]:

For each $g \in M_{24}$ there exists a graded unitary representation $\mathcal{H}_g$ of $\mathcal{N} = 4$ with central charge $c = 6$ carrying a projective representation

$$\rho_g : C_{M_{24}}(g) \rightarrow GL(\mathcal{H}_g)$$

commuting with $\mathcal{N} = 4$ and determined by a class $[\alpha] \in H^3(M_{24}, U(1))$. For each commuting pair $g, h \in M_{24}$ there exists functions $\phi_{g,h}(\tau, z)$ satisfying:

- $\phi_{e,h} = \phi_h$ and $\phi_{e,e} = \chi(K3; \tau, z)$
- $\phi_{g,h}(\tau, z) = \xi(k)\phi_{k^{-1}gk,k^{-1}hk}(\tau, z), \quad \forall k \in M_{24}$. 
**Conjecture** (generalized Mathieu moonshine) [GHPV]:

For each \( g \in M_{24} \) there exists a graded unitary representation \( \mathcal{H}_g \) of \( \mathcal{N} = 4 \) with central charge \( c = 6 \) carrying a projective representation

\[
\rho_g : C_{M_{24}}(g) \to GL(\mathcal{H}_g)
\]

commuting with \( \mathcal{N} = 4 \) and determined by a class \([\alpha] \in H^3(M_{24}, U(1))\).

For each commuting pair \( g, h \in M_{24} \) there exists functions \( \phi_{g,h}(\tau, z) \) satisfying:

- \( \phi_{e,h} = \phi_h \) and \( \phi_{e,e} = \chi(K3; \tau, z) \)
- \( \phi_{g,h}(\tau, z) = \xi(k)\phi_{k^{-1}g,k^{-1}h}(\tau, z), \quad \forall k \in M_{24} \)
- \( \phi_{g,h}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \chi_{g,h}\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) e^{2\pi i \frac{cz^2}{c\tau + d}} \phi_{g^a h^c, g^b h^d}(\tau, z), \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}) \)
**Conjecture (generalized Mathieu moonshine) [GHPV]:**

For each $g \in M_{24}$ there exists a *graded unitary representation* $\mathcal{H}_g$ of $\mathcal{N} = 4$ with central charge $c = 6$ carrying a *projective representation*

$$\rho_g : C_{M_{24}}(g) \to GL(\mathcal{H}_g)$$

commuting with $\mathcal{N} = 4$ and determined by a class $[\alpha] \in H^3(M_{24}, U(1))$. For each commuting pair $g, h \in M_{24}$ there exists functions $\phi_{g,h}(\tau, z)$ satisfying:

- $\phi_{e,h} = \phi_h$ and $\phi_{e,e} = \chi(K3; \tau, z)$
- $\phi_{g,h}(\tau, z) = \xi(k) \phi_{k^{-1}gk, k^{-1}hk}(\tau, z)$, $\forall k \in M_{24}$
- $\phi_{g,h}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \chi_{g,h}\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) e^{2\pi i \frac{c z^2}{c\tau + d}} \phi_{g^{a\cdot h^c}, g^b h^d}(\tau, z)$, $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z})$
- $\phi_{g,h}(\tau, z) = \sum_{r, \ell} \text{Tr}_{R_{g,r}}(h) \chi_{r+1/4, \ell}(\tau, z)$, $h \in C_{M_{24}}(g)\phantom{R_{g,r}}$ representation of a central extension of $C_{M_{24}}(g)$
**Conjecture** (generalized Mathieu moonshine) [GHPV]:

For each \( g \in M_{24} \) there exists a **graded unitary representation** \( \mathcal{H}_g \) of \( \mathcal{N} = 4 \) with central charge \( c = 6 \) carrying a **projective representation**

\[
\rho_g : C_{M_{24}}(g) \to GL(\mathcal{H}_g)
\]

commuting with \( \mathcal{N} = 4 \) and determined by a class \([\alpha] \in H^3(M_{24}, U(1))\).

For each commuting pair \( g, h \in M_{24} \) there exists functions \( \phi_{g,h}(\tau, z) \) satisfying:

- \( \phi_{e,h} = \phi_{h} \) and \( \phi_{e,e} = \chi(K3; \tau, z) \)

- \( \phi_{g,h}(\tau, z) = \xi(k)\phi_{k^{-1}g,k^{-1}h^{k}k}(\tau, z), \quad \forall k \in M_{24} \)

- \( \phi_{g,h}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \chi_{g,h}\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) e^{2\pi i \frac{cz^2}{c\tau+d}} \phi_{g^{a}h^{c},g^{b}h^{d}}(\tau, z), \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}) \)

- \( \phi_{g,h}(\tau, z) = \sum_{r, \ell} \text{Tr}_{R_g,r}(h) \chi_{r+1/4, \ell}(\tau, z), \quad h \in C_{M_{24}}(g) \)

- **The phases** \( \xi_{g,h}, \chi_{g,h} \) and the **central extension** of \( C_{M_{24}}(g) \) are
determined by the same class \([\alpha] \in H^3(M_{24}, U(1))\)
Example: 8A -twist and 2B -twine:

\[ \phi_{8A,2B}(\tau, z) = \frac{\eta\left(\frac{\tau}{2}\right)^6 \vartheta_1(\tau, z)^2}{\eta(\tau)^6 \vartheta_4(\tau, 0)^2} \]

8A = \(M_{24}\)-conjugacy class of order 8 elements.
**Example:** 8\(A\) -**twist** and 2\(B\) -**twine**:

\[
\phi_{8A,2B}(\tau, z) = \frac{\eta(\frac{\tau}{2})^6 \vartheta_1(\tau, z)^2}{\eta(\tau)^6 \vartheta_4(\tau, 0)^2}
\]

8\(A\) = \(M_{24}\)-**conjugacy class** of order 8 elements.

\(\phi_{8A,2B}(\tau, z)\) is a **Jacobi form** of weight 0 index 1 for the group

\[
\Gamma_{8A,2B} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \mod 4 \right\} = \Gamma^0(4)
\]
Example: $8A$ -twist and $2B$ -twine:

$$\phi_{8A,2B}(\tau, z) = \frac{\eta(\frac{\tau}{2})^6 \vartheta_1(\tau, z)^2}{\eta(\tau)^6 \vartheta_4(\tau, 0)^2}$$

$8A = M_{24}$-conjugacy class of order 8 elements.

$\phi_{8A,2B}(\tau, z)$ is a Jacobi form of weight 0 index 1 for the group

$$\Gamma_{8A,2B} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \mod 4 \right\} = \Gamma^0(4)$$

Multiplier given by:

$$\phi_{8A,2B}(\tau + 4, z) = \frac{\prod_{i=0}^{3} c_g(g, g^i h)}{c_{g^4 h}(g, g^{-1})} \frac{c_{g^{-1}}(g^4 h, k)}{c_{g^{-1}}(g^4 h, g^4 h)} \frac{c_{g^{-1}}(g^4 h, k)}{c_{g^{-1}}(k, h)} \phi_{8A,2B}(\tau) = -\phi_{8A,2B}(\tau)$$
Example: 8A -twist and 2B -twine:

$$\phi_{8A,2B}(\tau, z) = \frac{\eta\left(\frac{\tau}{2}\right)^6 \vartheta_1(\tau, z)^2}{\eta(\tau)^6 \vartheta_4(\tau, 0)^2}$$

8A = \(M_{24}\)-conjugacy class of order 8 elements.

\(\phi_{8A,2B}(\tau, z)\) is a Jacobi form of weight 0 index 1 for the group

$$\Gamma_{8A,2B} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \mod 4 \right\} = \Gamma^0(4)$$

Multiplier given by:

$$\phi_{8A,2B}(\tau + 4, z) = \frac{\prod_{i=0}^3 c_g(g, g^i h)}{c_{g^4 h}(g, g^{-1}) c_{g^{-1}}(g^4 h, g^4 h)} \frac{c_{g^{-1}}(g^4 h, k)}{c_{g^{-1}}(k, h)} \phi_{8A,2B}(\tau) = -\phi_{8A,2B}(\tau)$$

using our result for \(c_g(g_2, g_3)\) in terms of \(\alpha \in H^3(M_{24}, U(1))\)
**Theorem [GHPV]:**

- For each commuting pair \( g, h \in M_{24} \) there exists functions \( \phi_{g,h}(\tau, z) \) satisfying all the expected modular properties with respect to subgroups \( \Gamma_{g,h} \subset SL(2, \mathbb{Z}) \).

- There is a unique class \([\alpha] \in H^3(M_{24}, U(1))\) which determines all the modular phases.

- Many of the \( \phi_{g,h} \) vanish due to cohomological obstructions controlled by \( H^3(M_{24}, U(1)) \).

(in deriving these results we use the fact that \( H^3(M_{24}, U(1)) \cong \mathbb{Z}_{12} \) [Ellis, Dutour-Sikiric])
**Theorem [GHPV]:**

- For each commuting pair $g, h \in M_{24}$ there exists functions $\phi_{g,h}(\tau, z)$ satisfying all the expected modular properties with respect to subgroups $\Gamma_{g,h} \subset SL(2, \mathbb{Z})$.

- There is a unique class $[\alpha] \in H^3(M_{24}, U(1))$ which determines all the modular phases.

- Many of the $\phi_{g,h}$ vanish due to cohomological obstructions controlled by $H^3(M_{24}, U(1))$.

(in deriving these results we use the fact that $H^3(M_{24}, U(1)) \cong \mathbb{Z}_{12}$ [Ellis, Dutour-Sikiric])

**“Almost theorem” [GHPV]:**

- For each element $g \in M_{24}$ there exists projective reps $R_{g,r}$ of $C_{M_{24}}(g)$ such that

  $$\phi_{g,h}(\tau, z) = \sum_{r, \ell} \text{Tr}_{R_{g,r}}(h) \chi_{r+1/4, \ell}(\tau, z), \quad h \in C_{M_{24}}(g)$$

This was verified for the first 500 coefficients.
Theorem [GHPV]:

- For each commuting pair $g, h \in M_{24}$ there exists functions $\phi_{g,h}(\tau, z)$ satisfying all the expected modular properties with respect to subgroups $\Gamma_{g,h} \subset SL(2, \mathbb{Z})$.

- There is a unique class $[\alpha] \in H^3(M_{24}, U(1))$ which determines all the modular phases.

- Many of the $\phi_{g,h}$ vanish due to cohomological obstructions controlled by $H^3(M_{24}, U(1))$.

(in deriving these results we use the fact that $H^3(M_{24}, U(1)) \cong \mathbb{Z}_{12}$ [Ellis, Dutour-Sikiric]

“Almost theorem” [GHPV]:

- For each element $g \in M_{24}$ there exists projective reps $R_{g,r}$ of $C_{M_{24}}(g)$ such that

$$\phi_{g,h}(\tau, z) = \sum_{r, \ell} \text{Tr}_{R_{g,r}}(h) \chi_{r+1/4, \ell}(\tau, z), \quad h \in C_{M_{24}}(g)$$

This was verified for the first 500 coefficients.

This is very strong evidence that generalized Mathieu Moonshine holds!

But what is the physical interpretation?
2. Second quantization & black hole counting
Second quantized elliptic genus

Let $X$ be a Calabi-Yau manifold and $\chi(X; \tau, z)$ its elliptic genus.

$\chi(X; \tau, z)$ is a weak Jacobi form of weight zero and index $(\dim_{\mathbb{C}} X)/2$ [Gritsenko]
Second quantized elliptic genus

Let $X$ be a Calabi-Yau manifold and $\chi(X; \tau, z)$ its elliptic genus.

$\chi(X; \tau, z)$ is a weak Jacobi form of weight zero and index $(\dim_{\mathbb{C}} X)/2$ [Gritsenko]

Dijkgraaf, Moore, Verlinde, Verlinde defined the **second quantized elliptic genus** as

$$\Psi_X(\sigma, \tau, z) := \sum_{n=0}^{\infty} p^n \chi(S^n X; \tau, z)$$

$p = e^{2\pi i \sigma}$
Second quantized elliptic genus

Let $X$ be a Calabi-Yau manifold and $\chi(X; \tau, z)$ its elliptic genus.

$\chi(X; \tau, z)$ is a weak Jacobi form of weight zero and index $(\dim_{\mathbb{C}} X)/2$ [Gritsenko]

Dijkgraaf, Moore, Verlinde, Verlinde defined the second quantized elliptic genus as

$$\Psi_X(\sigma, \tau, z) := \sum_{n=0}^{\infty} p^n \chi(S^n X; \tau, z)$$

$p = e^{2\pi i \sigma}$

This is the generating function of elliptic genera of symmetric products of $X$
Second quantized elliptic genus

Let $X$ be a Calabi-Yau manifold and $\chi(X; \tau, z)$ its elliptic genus.

$\chi(X; \tau, z)$ is a weak Jacobi form of weight zero and index $(\dim_{\mathbb{C}} X)/2$ [Gritsenko]

Dijkgraaf, Moore, Verlinde, Verlinde defined the **second quantized elliptic genus** as

$$\Psi_X(\sigma, \tau, z) := \sum_{n=0}^{\infty} p^n \chi(S^n X; \tau, z)$$

$p = e^{2\pi i \sigma}$

DMVV proved the following remarkable formula:

$$\Psi_X(\sigma, \tau, z) = \exp \left[ \sum_{L=1}^{\infty} p^L T_L \chi(X; \tau, z) \right] = \prod_{n>0, m \geq 0, \ell \in \mathbb{Z}} (1 - p^n q^m y^\ell)^{-c_X(mn, \ell)}$$
Second quantized elliptic genus

Let $X$ be a Calabi-Yau manifold and $\chi(X; \tau, z)$ its elliptic genus.

$\chi(X; \tau, z)$ is a weak Jacobi form of weight zero and index $\left(\dim_{\mathbb{C}} X\right)/2$ \[\text{[Gritsenko]}\]

Dijkgraaf, Moore, Verlinde, Verlinde defined the \textbf{second quantized elliptic genus} as

$$\Psi_X(\sigma, \tau, z) := \sum_{n=0}^{\infty} p^n \chi(S^n X; \tau, z) \quad \text{with} \quad p = e^{2\pi i \sigma}$$

DMVV proved the following remarkable formula:

$$\Psi_X(\sigma, \tau, z) = \exp \left[ \sum_{L=1}^{\infty} p^L T_L \chi(X; \tau, z) \right] = \prod_{n>0, m \geq 0, \ell \in \mathbb{Z}} (1 - p^n q^m y^\ell)^{-c_X(mn, \ell)}$$

Hecke operator $T_L : J_0, m \to J_0, mL$

Fourier coefficients of $\chi(X; \tau, z) = \sum_{k \geq 0, \ell \in \mathbb{Z}} c_X(k, \ell) q^k y^\ell$
Second quantized elliptic genus

Gritsenko later showed that

\[ \Phi_X(\sigma, \tau, z) := \frac{A_X(\sigma, \tau, z)}{\Psi_X(\sigma, \tau, z)} \]

is a Siegel modular form of weight \( c_X(0, 0)/2 \)
Second quantized elliptic genus

Gritsenko later showed that

\[ \Phi_X(\sigma, \tau, z) := \frac{A_X(\sigma, \tau, z)}{\Psi_X(\sigma, \tau, z)} \]

is a Siegel modular form of weight \( c_X(0, 0)/2 \)

\( A_X \) is called the “Hodge anomaly”; only depends on the Hodge numbers of \( X \)

This is an example of a (multiplicative) Borcherds lift:

\[ \Phi \] : \( SL(2, \mathbb{Z}) \) \( \rightarrow \) \( SO(3, 2; \mathbb{Z}) \)
For $X$ a K3-manifold we have that

$$\Phi_X = \Phi_{10} = pqy \prod_{m,n,\ell > 0} (1 - p^m q^n y^\ell)^{c(mn,\ell)}$$

Igusa cusp form of weight 10 for $Sp(4; \mathbb{Z})$
Second quantized elliptic genus

For $X$ a K3-manifold we have that

$$\Phi_X = \Phi_{10} = pqy \prod_{m,n,\ell > 0} (1 - p^m q^n y^\ell)^{c(mn, \ell)}$$

This is a multiplicative Borcherds lift of the **K3 elliptic genus**

$$\chi(K3; \tau, z) = 2\phi_{0,1}(\tau, z) = \sum_{n \geq 0, \ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell$$

The inverse is the partition function of 1/4 BPS dyons in $\text{Het}/T^6$ or $\text{IIA}/(K3 \times T^2)$

[Dijkgraaf, Verlinde, Verlinde][Shih, Strominger, Yin]
Counting dyons in $\mathcal{N} = 4$ string theory

Large moduli space of such theories:

$$\mathcal{M} = O(6, 22; \mathbb{Z}) \backslash O(6, 22; \mathbb{R}) / (O(6) \times O(22))$$

The discrete duality group preserved the lattice of electric-magnetic charges:

$$(P, Q) \in \Gamma^{6,22} \oplus \Gamma^{6,22}$$

The full non-perturbative duality group is

$$SL(2, \mathbb{Z}) \times O(6, 22; \mathbb{Z})$$

$(P, Q)$ transform as a doublet under $SL(2, \mathbb{Z})$
Hilbert space of states decomposes as

$$\mathcal{H} = \bigotimes_{(P,Q) \in \Gamma^{6,22} \oplus \Gamma^{6,22}} \mathcal{H}_{Q,P}$$

These can be realized as **charged black holes** in the supergravity limit.
Hilbert space of states decomposes as

\[ \mathcal{H} = \bigotimes (P,Q) \in \Gamma^{6,22} \oplus \Gamma^{6,22} \mathcal{H}_{Q,P} \]

These can be realized as charged black holes in the supergravity limit.

We are interested in **BPS-states**:

- **1/2 BPS**: Purely electric \((0,Q)\) or magnetic \((P,0)\)

- **1/4 BPS (generic)**: Dyonic \((Q,P)\)
\textbf{1/2 BPS-states} are counted by \cite{Dabholkar, Harvey}

\[
\frac{1}{\eta(\tau)^{24}} = \sum_{n \in \mathbb{Z}} d(n) q^n
\]

\[\implies d(n) = \text{number of 1/2 BPS-states with charge } Q \text{ such that } n = Q^2/2\]
**1/2 BPS-states** are counted by [Dabholkar, Harvey]

\[
\frac{1}{\eta(\tau)^{24}} = \sum_{n \in \mathbb{Z}} d(n) q^n
\]

⇒ \(d(n) = \) number of 1/2 BPS-states with charge \(Q\) such that \(n = Q^2/2\)

In general, **1/4 BPS states** are counted by the 6th helicity supertrace [Kiritsis]

\[
B_6(P, Q) := \frac{1}{6!} \text{Tr}_{\mathcal{H}_{P,Q}} \left( (-1)^J (2J)^6 \right) \quad J = \text{helicity}
\]
**$1/2$ BPS-states** are counted by \[ \frac{1}{\eta(\tau)^{24}} = \sum_{n \in \mathbb{Z}} d(n) q^n \]

\[ d(n) = \text{number of } 1/2 \text{ BPS-states with charge } Q \text{ such that } n = \frac{Q^2}{2} \]

In general, **$1/4$ BPS states** are counted by the **6th helicity supertrace** \cite{Kiritsis}

\[ B_6(P, Q) := \frac{1}{6!} \text{Tr}_{\mathcal{H}_{P,Q}} \left( (-1)^J (2J)^6 \right) \quad J = \text{helicity} \]

- invariant under $SL(2, \mathbb{Z}) \times SO(6, 22; \mathbb{Z})$
- locally constant on $\mathcal{M}$
Generating function: \[ \Phi_{10}(\sigma, \tau, z) = \frac{1}{\Phi_{10}(\sigma, \tau, z)} = \sum_{m,n,\ell} d(m, n, \ell) p^m q^n y^\ell \]

with the identification

\[ B_6(P, Q) = d \left( \frac{Q^2}{2}, \frac{P^2}{2}, P \cdot Q \right) \]

\[ q := e^{2\pi i \tau}, y := e^{2\pi i z}, p := e^{2\pi i \sigma} \]
Generating function: 

\[ \frac{1}{\Phi_{10}(\sigma, \tau, z)} = \sum_{m,n,\ell} d(m, n, \ell) p^m q^n y^\ell \]

with the identification

\[ B_6(P, Q) = d \left( \frac{Q^2}{2}, \frac{P^2}{2}, P \cdot Q \right) \]

\[ \Phi_{10} \] has a double pole at \( z = 0 \). In the limit, we have a factorization

\[ \lim_{z \to 0} \frac{\Phi_{10}(\sigma, \tau, z)}{(2\pi iz)^2} = \eta(\sigma)^{24} \eta(\tau)^{24} \]

“wall-crossing formula”
3. Second quantization of generalized Mathieu moonshine
Second quantized twisted twining genera

Inspired by the aforementioned results we seek a similar *spacetime interpretation* for the twisted twining genera $\phi_{g,h}(\tau, z)$ of generalized Mathieu moonshine.

This generalizes earlier results by Cheng and Govindarajan.
Second quantized twisted twining genera

Inspired by the aforementioned results we seek a similar **spacetime interpretation** for the twisted twining genera $\phi_{g,h}(\tau, z)$ of generalized Mathieu moonshine.

This generalizes earlier results by Cheng and Govindarajan.

We define the **second quantized twisted twining genus** as:

$$
\Psi_{g,h}(\sigma, \tau, z) := \exp \left[ \sum_{L=1}^{\infty} p^L T^\alpha_L \phi_{g,h}(\tau, z) \right]
$$
Second quantized twisted twining genera

Inspired by the aforementioned results we seek a similar spacetime interpretation for the twisted twining genera $\phi_{g,h}(\tau, z)$ of generalized Mathieu moonshine.

This generalizes earlier results by Cheng and Govindarajan.

We define the second quantized twisted twining genus as:

$$\Psi_{g,h}(\sigma, \tau, z) := \exp \left[ \sum_{L=1}^{\infty} p^L T^\alpha_L \phi_{g,h}(\tau, z) \right]$$

where $T^\alpha_L$ are twisted equivariant Hecke operators, generalizing those used in generalized monstrous moonshine by Ganter & Carnahan.
Second quantized twisted twining genera

Inspired by the aforementioned results we seek a similar spacetime interpretation for the twisted twining genera $\phi_{g,h}(\tau, z)$ of generalized Mathieu moonshine.

This generalizes earlier results by Cheng and Govindarajan.

We define the **second quantized twisted twining genus** as:

$$\Psi_{g,h}(\sigma, \tau, z) := \exp \left[ \sum_{L=1}^{\infty} \sum_{L} p^L \mathcal{T}_L^\alpha \phi_{g,h}(\tau, z) \right]$$

where $\mathcal{T}_L^\alpha$ are twisted equivariant Hecke operators, generalizing those used in generalized monstrous monstrous moonshine by Ganter & Carnahan.

Note that this depends on the choice of 3-cocycle $\alpha \in H^3(M_{24}, U(1))$ but different representatives in each class $[\alpha]$ simply amounts to a shift of $\sigma$
Twisted equivariant Hecke operators

Geometric interpretation following Ganter. Let

\[ \mathcal{M}_{M_{24}} = \mathcal{P} \times (\mathbb{H}_+ \times \mathbb{C}) / M_{24} \times (SL(2, \mathbb{Z}) \times \mathbb{Z}^2) \]

\[ = \]

moduli space of principal $M_{24}$-bundles
on the elliptic curve $E_\tau$
Twisted equivariant Hecke operators

Geometric interpretation following Ganter. Let

$$\mathcal{M}_{M_{24}} = \mathcal{P} \times (\mathbb{H}_+ \times \mathbb{C}) / M_{24} \times (SL(2, \mathbb{Z}) \times \mathbb{Z}^2)$$

moduli space of principal $M_{24}$-bundles on the elliptic curve $E_\tau$

The twisted twining genera $\phi_{g,h}$ are sections of a line bundle $\mathcal{L}_{g,h}^\alpha \to \mathcal{M}_{M_{24}}$
Twisted equivariant Hecke operators

Geometric interpretation following Ganter. Let

\[ \mathcal{M}_{M_{24}} = \mathcal{P} \times (\mathbb{H}_+ \times \mathbb{C}) / M_{24} \times (SL(2, \mathbb{Z}) \times \mathbb{Z}^2) \]

\[ = \text{moduli space of principal } M_{24} \text{-bundles on the elliptic curve } E_\tau \]

The twisted twining genera \( \phi_{g,h} \) are sections of a line bundle

\[ L_{g,h}^\alpha \rightarrow \mathcal{M}_{M_{24}} \]

The twisted equivariant Hecke operators provide a map

\[ T^\alpha_L : L_{g,h}^\alpha \rightarrow (L_{g,h}^\alpha) \otimes L \]
Twisted equivariant Hecke operators

Geometric interpretation following Ganter. Let

\[ \mathcal{M}_{M_{24}} = \mathcal{P} \times (\mathbb{H}_+ \times \mathbb{C}) / M_{24} \times (SL(2, \mathbb{Z}) \times \mathbb{Z}^2) \]

\[ = \text{moduli space of principal } M_{24} \text{-bundles on the elliptic curve } E_\tau \]

The twisted twining genera \( \phi_{g,h} \) are sections of a line bundle

\[ \mathcal{L}_{g,h}^\alpha \rightarrow \mathcal{M}_{M_{24}} \]

The twisted equivariant Hecke operators provide a map

\[ \mathcal{T}_L^\alpha : \mathcal{L}_{g,h}^\alpha \rightarrow (\mathcal{L}_{g,h}^\alpha)^\otimes L \]

sections have multiplier phase \( \chi_{g,h} \)  sections have multiplier phase \( (\chi_{g,h})^L \)
The twisted equivariant Hecke operators provide a map

\[
\mathcal{T}_L^\alpha : \mathcal{L}_{g,h}^\alpha \longrightarrow (\mathcal{L}_{g,h}^\alpha) \otimes L
\]
The twisted equivariant Hecke operators provide a map

$$\mathcal{T}_L^\alpha : \mathcal{L}_{g,h}^\alpha \longrightarrow (\mathcal{L}_{g,h}^\alpha) \otimes L$$

Explicitly one can represent this action by

$$\mathcal{T}_L^\alpha \phi_{g,h}(\tau, z) := \frac{1}{L} \sum_{a, d > 0 \atop a \cdot d = L} \sum_{b=0}^{d-1} \chi_{g,h} \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \phi_{g^d, g^{-b}, h^a} \left( \frac{a \tau + b}{d}, a z \right)$$

This is a generalization of similar Hecke operators used in generalized monstrous moonshine by Ganter & Carnahan. (see also [Tuite][Govindarajan])
The twisted equivariant Hecke operators provide a map

\[ T_L^{\alpha} : \mathcal{L}_g,h \rightarrow \left( \mathcal{L}_g,h \right) \otimes L \]

Explicitly one can represent this action by

\[ T_L^{\alpha} \phi_{g,h}(\tau, z) := \frac{1}{L} \sum_{a, d > 0, \ a \cdot d = L} \sum_{b=0}^{d-1} \chi_{g,h} \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \phi_{g^d, g^{-b}, h^a} \left( \frac{a \tau + b}{d}, a z \right) \]

This is a generalization of similar Hecke operators used in generalized monstrous moonshine by Ganter & Carnahan. (see also [Tuite][Govindarajan])

multiplier phase determined by \([\alpha] \in H^3(M_{24}, U(1))\)
Example: for $g, h \in 2B$ we have

$$\mathcal{T}_2^\alpha \phi_{g,h}(\tau, z) = \frac{1}{2} \left[ -\phi_{g,e}(2\tau, 2z) + \phi_{e,h}(\frac{\tau}{2}, z) - \phi_{e,gh}(\frac{\tau+1}{2}, z) \right]$$
Example: for $g, h \in 2B$ we have

$$\mathcal{T}_2^{\alpha} \phi_{g,h}(\tau, z) = \frac{1}{2} \left[ - \phi_{g,e}(2\tau, 2z) + \phi_{e,h}(\frac{\tau}{2}, z) - \phi_{e,gh}(\frac{\tau+1}{2}, z) \right]$$

signs come from the multiplier system $\chi_{g,h}$
Example: for $g, h \in 2B$ we have

$$\mathcal{T}_2^\alpha \phi_{g,h}(\tau, z) = \frac{1}{2} \left[ -\phi_{g,e}(2\tau, 2z) + \phi_{e,h}(\frac{\tau}{2}, z) - \phi_{e,gh}(\frac{\tau+1}{2}, z) \right]$$

signs come from the multiplier system $\chi_{g,h}$

On the other hand, for $g, h \in 2B$ we in fact have

$$\phi_{g,h}(\tau, z) = 0$$

by cohomological obstructions from $H^3(M_{24}, U(1))$
Since

\[ \mathcal{T}_L^\alpha : \mathcal{L}_{g,h}^\alpha \rightarrow (\mathcal{L}_{g,h}^\alpha)^\otimes L \]

This implies that for sufficiently large \( L \) \( \mathcal{T}_L^\alpha \phi_{g,h} \) has trivial multiplier phase
Even if \( \phi_{g,h} \) vanishes by cohomological obstructions, all the second quantized twisted twining genera \( \Psi_{g,h} \) are unobstructed!

\[
\Psi_{g,h}(\sigma, \tau, z) := \exp \left[ \sum_{L=1}^{\infty} \frac{p^L}{L} \mathcal{T}_{L}^{\alpha} \phi_{g,h}(\tau, z) \right]
\]
Theorem (D.P., Volpato):

The second quantized twisted twining genera satisfy the following properties

- Infinite product formula

\[
\frac{1}{\Psi_{g,h}(\sigma, \tau, z)} = \prod_{d=1}^{\infty} \prod_{m=0}^{\infty} \prod_{\ell \in \mathbb{Z}} \prod_{t=0}^{M-1} \left( 1 - e^{\frac{2\pi it}{M} y^\ell p^d} \right) \hat{c}_{g,h}(d,m,\ell,t)
\]
Theorem (D.P., Volpato):

The second quantized twisted twining genera satisfy the following properties

1. **Infinite product formula**

\[
\frac{1}{\Psi_{g,h}(\sigma, \tau, z)} = \prod_{d=1}^{\infty} \prod_{m=0}^{\infty} \prod_{\ell \in \mathbb{Z}} \prod_{t=0}^{\infty} \left(1 - e^{-\frac{2\pi it}{M}} q^\frac{m}{N\lambda} y^{\ell} p^d \right) \hat{c}_{g,h}(d,m,\ell,t)
\]

\[M = O(h) \quad N = O(g)\]

\[\lambda \text{ length of the shortest cycle of } g \text{ in its 24-dim permutation reps}\]

\[
\hat{c}_{g,h}(d,m,\ell,t) := \sum_{k=0}^{M-1} \sum_{b=0}^{\lambda N - 1} e^{-\frac{2\pi i tk}{M}} e^{\frac{2\pi i bm}{\lambda N}} \chi_{g,h}(\begin{pmatrix} k & b \\ 0 & d \end{pmatrix}) \hat{c}_{g,d,g^{-b}h^k}(\frac{md}{N\lambda}, \ell)
\]
The second quantized twisted twining genera satisfy the following properties

- **Infinite product formula**

\[
\frac{1}{\Psi_{g,h}(\sigma, \tau, z)} = \prod_{d=1}^{\infty} \prod_{m=0}^{\infty} \prod_{\ell \in \mathbb{Z}} \prod_{t=0}^{M-1} (1 - e^{\frac{2\pi it}{M}} q^{\frac{m}{N\lambda}} y^{\ell} p^{\ell d}) \hat{c}_{g,h}(d,m,\ell,t)
\]

- **The ratio**

\[
\Phi_{g,h}(\sigma, \tau, z) := \frac{A_{g,h}(\sigma, \tau, z)}{\Psi_{g,h}(\sigma, \tau, z)}
\]

is a **Siegel modular form** for a subgroup \( \Gamma_{g,h}^{(2)} \subset Sp(4; \mathbb{R}) \)

For \( g = e \) this was conjectured by Cheng and partially proven by Raum.
Theorem (D.P., Volpato):
The second quantized twisted twining genera satisfy the following properties

- **Infinite product formula**

\[
\frac{1}{\Psi_{g,h}(\sigma, \tau, z)} = \prod_{d=1}^{\infty} \prod_{m=0}^{\infty} \prod_{\ell \in \mathbb{Z}} \prod_{t=0}^{M-1} \left(1 - e^{\frac{2\pi it}{M} \frac{m}{N\lambda} y^\ell p^d}\right) \hat{c}_{g,h}(d,m,\ell,t)
\]

- **The ratio**

\[
\Phi_{g,h}(\sigma, \tau, z) := \frac{A_{g,h}(\sigma, \tau, z)}{\Psi_{g,h}(\sigma, \tau, z)}
\]

is a **Siegel modular form** for a subgroup \( \Gamma_{g,h}^{(2)} \subset Sp(4; \mathbb{R}) \)

For \( g = e \) this was conjectured by Cheng and partially proven by Raum.

“Hodge anomaly”

\[
A_{g,h} = -p \vartheta_1(\tau, z)^2 \frac{\eta(\tau)^6}{\eta_g,h(\tau)}
\]

Mason’s generalized eta-products
The second quantized twisted twining genera satisfy the following properties:

- **Infinite product formula**

\[
\frac{1}{\Psi_{g,h}(\sigma, \tau, z)} = \prod_{d=1}^{\infty} \prod_{m=0}^{\infty} \prod_{\ell \in \mathbb{Z}} \prod_{t=0}^{M-1} \left( 1 - e^{2\pi i t} \frac{q^{m \ell \lambda} y^{\ell} p^d}{\eta(N \lambda)} \right) \hat{c}_{g,h}(d,m,\ell,t)
\]

- **The ratio**

\[
\Phi_{g,h}(\sigma, \tau, z) := \frac{A_{g,h}(\sigma, \tau, z)}{\Psi_{g,h}(\sigma, \tau, z)}
\]

is a **Siegel modular form** for a subgroup \( \Gamma_{g,h}^{(2)} \subset Sp(4; \mathbb{R}) \)

For \( g = e \) this was conjectured by Cheng and partially proven by Raum.

- **“Wall-crossing formula”**

\[
\lim_{z \to 0} \frac{\Phi_{g,h}(\sigma, \tau, z)}{(2\pi i z)^2} = \eta_{g,h}(\tau) \eta_{g,h}(N \lambda \sigma)
\]
Automorphy of $\Phi_{g,h}$ follow from

- “Electric-magnetic duality”

$$\Phi_{g,h}(\sigma, \tau, z) = \Phi_{g,h'}\left(\frac{\tau}{N\lambda}, N\lambda\sigma, z\right)$$

where $h'$ is not necessarily in the same conjugacy class $[h]$

This generalizes the electric-magnetic duality in $\Phi_{10}$ [Dijkgraaf, Verlinde, Verlinde]
Automorphy of \( \Phi_{g,h} \) follow from

- “Electric-magnetic duality”

\[
\Phi_{g,h}(\sigma, \tau, z) = \Phi_{g,h'}\left(\frac{\tau}{N\lambda}, N\lambda\sigma, z\right)
\]

where \( h' \) is not necessarily in the same conjugacy class \([h]\)

This generalizes the electric-magnetic duality in \( \Phi_{10} \) \cite{Dijkgraaf,Verlinde,Verlinde}

- Using results of Gritsenko-Nikulin, one also has invariance under (an extension of) the para-modular group

\[
\Gamma_t(N) = \left\{ \begin{pmatrix}
* & t* & * & * \\
* & * & * & t^{-1}* \\
N* & Nt* & * & * \\
Nt* & Nt* & t* & *
\end{pmatrix} \in Sp(4, \mathbb{Q}), \ * \in \mathbb{Z}\right\}
\]
Every $\Phi_{g,h}$ is a modular function for some finite index subgroup $\Gamma_{g,h}^{(2)}$

of a para-modular group $\Gamma_t$ for some $t$
Every $\Phi_{g,h}$ is a modular function for some finite index subgroup $\Gamma_{g,h}^{(2)}$ of a para-modular group $\Gamma_t$ for some $t$.

We can therefore view this our construction as a \textbf{twisted equivariant} generalization of a multiplicative Borcherds lift

$$\text{Mult}_G[\phi_{g,h}] := A_{g,h}(\sigma, \tau, z) \exp \left[ - \sum_{L=1}^{\infty} p^L T^\alpha_L \phi_{g,h}(\tau, z) \right]$$
This resolves a puzzle about the connection with Mason’s old version of generalized $M_{24}$-moonshine for eta-products
(For $g = e$ this was observed previously by Cheng and Govindarajan.)

second-quantized twisted twining genera
(Siegel modular forms)
\[ \Phi_{g,h}(\sigma, \tau, z) \]

\[ z \to 0 \] ("wall-crossing")

\[ \eta_{g,h}(\tau)\eta_{g,h'}(N\lambda\sigma) \]

This diagram illustrates the relationship between twisted twining genera (weak Jacobi forms), generalized eta-products (modular forms), and the second-quantized twisted twining genera (Siegel modular forms).

- **Twisted Equivariant Multiplicative Lift**
  - "Second Quantization"

- **Generalized Eta-Products**
  - Modular Forms

- **Twisted Twining Genera**
  - Weak Jacobi Forms

For $g = e$, this was observed previously by Cheng and Govindarajan.
Physical interpretation: CHL-models

Can we interpret the second quantized twisted twining genera as counting spacetime BPS-states?
Physical interpretation: CHL-models

Can we interpret the second quantized twisted twining genera as counting spacetime BPS-states?

Suppose \((g, h)\) are commuting symmetries of the internal superconformal CFT of type II/(K3 \times T^2) or \(\text{Het}/T^6\)
Physical interpretation: CHL-models

Can we interpret the second quantized twisted twining genera as counting spacetime BPS-states?

Suppose \((g, h)\) are commuting symmetries of the internal superconformal CFT of type II/\((K3 \times T^2)\) or Het/\(T^6\)

- Consider the orbifold of this theory by \(g\)

\[\text{new } \mathcal{N} = 4 \text{ theory} \]
\[\text{“CHL-model”} \]
\[\text{[Chaudhuri, Hockney, Lykken]}\]
Physical interpretation: CHL-models

Can we interpret the second quantized twisted twining genera as counting spacetime BPS-states?

Suppose \((g, h)\) are commuting symmetries of the internal superconformal CFT of type \(\text{II}/(K3 \times T^2)\) or \(\text{Het}/T^6\)

Consider the orbifold of this theory by \(g\) → “CHL-model”

[Chaudhuri, Hockney, Lykken]

In this orbifold theory we have “twisted” dyon states counted by the twisted BPS-index

\[
B_{6;g,h}(P,Q) := \frac{1}{6!} \text{Tr} \mathcal{H}^g_{Q,P} (h(-1)^{2J} (2J)^6)
\]

[Sen]

Computed for some pairs of symmetries [Dabholkar, Gaiotto][Dabholkar, Nampuri][Jatkar, Sen][David][Dabholkar, Cheng][Govindarajan][Sen]...
Expanding the second quantized twisted twining genera

\[
\frac{1}{\Phi_{g,h}(\sigma, \tau, z)} = \sum_{m,n,\ell} d_{g,h}(m, n, \ell) q^n p^m y^\ell
\]

we find that

\[
B_{6;g,h}(P, Q) = d_{g,h}\left(\frac{Q^2}{2}, \frac{P^2}{2}, Q \cdot P\right)
\]
Expanding the second quantized twisted twining genera

\[
\frac{1}{\Phi_{g,h}(\sigma, \tau, z)} = \sum_{m,n,\ell} d_{g,h}(m, n, \ell) q^n p^m y^\ell
\]

we find that

\[
B_{6;g,h}(P, Q) = d_{g,h} \left( \frac{Q^2}{2}, \frac{P^2}{2}, Q \cdot P \right)
\]

**Coincides with Fourier coefficients of** \( \Phi_{g,h} \)** for some pairs \((g, h)\)**

**Could it be that all of the** \( \Phi_{g,h} \)** have interpretations as partition functions for BPS-dyons?**
4. Connection with umbral moonshine
Umbral moonshine

Cheng, Duncan, Harvey proposed a generalization of Mathieu moonshine involving 23 examples labelled by ADE-type root systems.

Here we focus on the 6 cases corresponding to pure A-type root systems.

\((G^{(\ell)}, Z^{(\ell)})\quad \ell \in \{2, 3, 4, 5, 7, 13\}\)
Umbral moonshine

Cheng, Duncan, Harvey proposed a generalization of Mathieu moonshine involving 23 examples labelled by ADE-type root systems.

Here we focus on the 6 cases corresponding to pure A-type root systems.

\[
(G^{(\ell)}, Z^{(\ell)}) \quad \ell \in \{2, 3, 4, 5, 7, 13\}
\]
Umbral moonshine

Cheng, Duncan, Harvey proposed a generalization of Mathieu moonshine involving 23 examples labelled by ADE-type root systems.

Here we focus on the 6 cases corresponding to pure A-type root systems.

\[(G^{(\ell)}, Z^{(\ell)}) \quad \ell \in \{2, 3, 4, 5, 7, 13\}\]

(\(G^{(2)}, Z^{(2)}\) = (\(M_{24}, \chi(K3; \tau, z)\))

Mathieu moonshine corresponds to \(\ell = 2\)

We shall now see that there appears to be a relation between umbral moonshine and generalized Mathieu moonshine.
Let us consider the case when \( g, h \in 2A \) in \( M_{24} \):

\[
\phi_{g,h} = 0 \quad \text{but} \quad \mathcal{T}_2^\alpha \phi_{g,h} \in J_{0,2}^{weak}
\]
Let us consider the case when $g, h \in 2A$ in $M_{24}$

$\implies \phi_{g,h} = 0$ but $\mathcal{T}_2^\alpha \phi_{g,h} \in J^{weak}_{0,2}$

In fact, this is nothing but the **umbral Jacobi form** for $\ell = 3$

$$\mathcal{T}_2^\alpha \phi_{g,h} = Z^{(3)}(\tau, z)$$
Let us consider the case when \( g, h \in 2A \) in \( M_{24} \)

\[ \phi_{g,h} = 0 \quad \text{but} \quad \mathcal{T}_2^\alpha \phi_{g,h} \in J_{0,2}^{\text{weak}} \]

In fact, this is nothing but the **umbral Jacobi form** for \( \ell = 3 \)

\[ \mathcal{T}_2^\alpha \phi_{g,h} = Z^{(3)}(\tau, z) \]

The same holds for a few other conjugacy classes in \( M_{24} \) that we checked

\[ (3A, 3A) \quad \mathcal{T}_3^\alpha \phi_{g,h} = Z^{(4)}(\tau, z) \]

\[ (4B, 4B) \quad \mathcal{T}_4^\alpha \phi_{g,h} = Z^{(5)}(\tau, z) \]
Starting from the umbral Jacobi forms Cheng-Duncan-Harvey constructed a class of Siegel modular forms using a standard **Borcherds lift:**

$$
\Phi^{(\ell)} = \text{Mult}[Z^{(\ell)}] = p^{A(\ell)} q^{B(\ell)} y^{C(\ell)} \prod_{(m,n,r) > 0} (1 - p^m q^n y^r)^{c^{(\ell)}(mn,r)}
$$
Starting from the umbral Jacobi forms Cheng-Duncan-Harvey constructed a class of Siegel modular forms using a standard **Borcherds lift:**

\[
\Phi^{(\ell)} = \text{Mult}[Z^{(\ell)}] = p^{A(\ell)} q^{B(\ell)} y^{C(\ell)} \prod_{(m,n,r)>0} (1 - p^m q^n y^r)^{c^{(\ell)}(mn,r)}
\]

For \( \ell \in \{2, 3, 4, 5\} \) one has \( \Phi^{(\ell)} = (\Delta_k)^2 \quad k = \frac{7-\ell}{\ell-1} \)

\( \Delta_k \) = weight \( k \) **Siegel modular forms** constructed by Gritsenko-Nikulin
Starting from the umbral Jacobi forms Cheng-Duncan-Harvey constructed a class of Siegel modular forms using a standard \textbf{Borcherds lift}:

\[ \Phi^{(\ell)} = \text{Mult}[Z^{(\ell)}] = p^{A(\ell)} q^{B(\ell)} y^{C(\ell)} \prod_{(m,n,r)>0} (1 - p^m q^n y^r)^{c^{(\ell)}(mn,r)} \]

For \( \ell \in \{2, 3, 4, 5\} \) one has \( \Phi^{(\ell)} = (\Delta_k)^2 \quad k = \frac{7 - \ell}{\ell - 1} \)

\( \Delta_k \) = weight \( k \) \textbf{Siegel modular forms} constructed by Gritsenko-Nikulin

We observe that these Siegel modular forms coincide with some of the second quantized twisted twining genera in generalized Mathieu moonshine:

\( (2A, 2A) : \Phi_{g,h} = (\Delta_2)^2 = \Phi^{(3)} \)

\( (3A, 3A) : \Phi_{g,h} = (\Delta_1)^2 = \Phi^{(4)} \)

\( (4B, 4B) : \Phi_{g,h} = (\Delta_{1/2})^2 = \Phi^{(5)} \)
Starting from the umbral Jacobi forms Cheng-Duncan-Harvey constructed a class of Siegel modular forms using a standard **Borcherds lift:**

\[ \Phi^{(\ell)} = \text{Mult}[Z^{(\ell)}] = p^{A(\ell)} q^{B(\ell)} y^{C(\ell)} \prod_{(m,n,r) > 0} (1 - p^m q^n y^r)^{c^{(\ell)}(mn,r)} \]

For \( \ell \in \{2, 3, 4, 5\} \) one has \( \Phi^{(\ell)} = (\Delta_k)^2 \)

\( \Delta_k = \text{weight } k \) **Siegel modular forms** constructed by Gritsenko-Nikulin

We observe that these Siegel modular forms coincide with some of the second quantized twisted twining genera in generalized Mathieu moonshine:

\( (2A, 2A) : \Phi_{g,h} = (\Delta_2)^2 = \Phi^{(3)} \)

\( (3A, 3A) : \Phi_{g,h} = (\Delta_1)^2 = \Phi^{(4)} \)

\( (4B, 4B) : \Phi_{g,h} = (\Delta_{1/2})^2 = \Phi^{(5)} \)

**Overlap between umbral moonshine and generalized Mathieu moonshine!**
conjugacy classes in $M_{24}$

$(2A, 2A) : \Phi_{g,h} = (\Delta_2)^2 = \Phi^{(3)}$

$(3A, 3A) : \Phi_{g,h} = (\Delta_1)^2 = \Phi^{(4)}$

$(4B, 4B) : \Phi_{g,h} = (\Delta_{1/2})^2 = \Phi^{(5)}$

Overlap between umbral moonshine and generalized Mathieu moonshine!
(2A, 2A): \( \Phi_{g,h} = (\Delta_2)^2 = \Phi^{(3)} \)

(3A, 3A): \( \Phi_{g,h} = (\Delta_1)^2 = \Phi^{(4)} \)

(4B, 4B): \( \Phi_{g,h} = (\Delta_{1/2})^2 = \Phi^{(5)} \)

Overlap between umbral moonshine and generalized Mathieu moonshine!

Note that this is non-trivial since the LHS is constructed using an **equivariant lift** while the RHS is constructed using a **standard Borcherds lift**:

\[
\text{Mult}_G[\phi_{g,h}] = \text{Mult}[Z^{(\ell)}]
\]

These Siegel modular forms also appear in CHL-models. [Sen][Govindarajan]

In fact, following an observation by Govindarajan, for these cases one can also show that the same functions can be obtained using an **additive lift** from the “Hodge anomaly“ \( A_{g,h}(\sigma, \tau, z) \)
conjugacy classes in $M_{24}$

\[(2A, 2A) : \Phi_{g,h} = (\Delta_2)^2 = \Phi^{(3)}\]
\[(3A, 3A) : \Phi_{g,h} = (\Delta_1)^2 = \Phi^{(4)}\]
\[(4B, 4B) : \Phi_{g,h} = (\Delta_{1/2})^2 = \Phi^{(5)}\]

Overlap between umbral moonshine and generalized Mathieu moonshine!

Note that this is non-trivial since the LHS is constructed using an **equivariant lift** while the RHS is constructed using a **standard Borcherds lift**:

$$\text{Mult}_G[\phi_{g,h}] = \text{Mult}[Z^{(\ell)}]$$

These Siegel modular forms also appear in CHL-models. [Sen][Govindarajan]

In fact, following an observation by Govindarajan, for these cases one can also show that the same functions can be obtained using an **additive lift** from the “Hodge anomaly” $A_{g,h}(\sigma, \tau, z)$

**A modular coincidence or an indication of some deeper relation?**
5. Summary and outlook
We have established that generalised Mathieu moonshine holds by computing all twisted twining genera $\phi_{g,h}$.

Twisted twining genera can be expanded in projective characters of $C_{M_{24}}(g)$.

A key role is played by the third cohomology group $H^3(M_{24}, U(1))$.

All the second quantized twisted twining genera found and verified to be Siegel modular forms.

Some of these correspond to partition functions of twisted dyons in CHL-models.

Intriguing connection with umbral moonshine.
Outlook

- Can one construct a generalised Kac-Moody algebra for each conjugacy class $[g] \in M_{24}$? (c.f. [Borcherds][Carnahan])

- Twisted equivariant additive lifts: $\text{Add}_G[A_{g,h}]$? (see also [Eguchi, Hikami])

- Relation with BPS-algebras à la Harvey Moore...?

- Generalised Umbral Moonshine...? [Cheng, Duncan, Harvey]

- Recent interesting results indicate that there is are $N=2$ and $N=1$ versions of Mathieu Moonshine in heterotic string theory. [Cheng, Dong, Duncan, Harvey, Kachru, Wrase][Harrison, Kachru, Paquette][Wrase]

- Does $M_{24}$ play a role in mirror symmetry?

- Can one construct an action of $M_{24}$ on the (cohomology) of the chiral de Rham complex of K3? See Katrin’s talk!
What does $M_{24}$ act on?

Our results strongly suggest that there is something like a holomorphic vertex operator algebra underlying Mathieu Moonshine

...but which one remains a mystery...