Succinct Hitting Sets and Barriers to Proving Algebraic Circuits Lower Bounds

Ben Lee Volk

Joint with

Michael A. Forbes
Amir Shpilka
How NOT to Prove Algebraic Circuits Lower Bounds

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Alternate Title:
WHY IS IT HARD TO PROVE CIRCUIT LOWER BOUNDS?

(One) Answer: natural proofs barrier [Razborov-Rudich]:

“A computationally-bounded observer cannot distinguish between the truth table of a random function with small circuit and that of a truly random function (assuming some crypto). So every lower bound proof attempt which yields such an algorithm cannot work.”
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Def: A property $S$ of boolean functions is **natural** if it is:

1. **Useful:** if $f$ has $S$ then $f$ doesn't have a small ckt.
2. **Large:** random functions have $S$ with large probability.
3. **Constructive:** Given truth table of $f$ of size $N = 2^n$, there is an algorithm for deciding whether $f \in S$ with running time $\text{poly}(N) = 2^{O(n)}$.

**Natural proofs:** a lower bound proof which exhibits a natural property. 

[Razborov-Rudich]: Most known lower bounds are natural and if there's a pseudorandom function in $C$ then no natural lower bound against $C$. 

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ALGEBRAIC CIRCUITS

\[ f(x_1, x_2, x_3) \in \mathbb{F}[x_1, x_2, x_3] \]
LOWER BOUNDS FOR ALGEBRAIC CIRCUITS

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Can we identify formal barriers?

(also asked by [Aaronson-Drucker] and [Grochow])
ALGEBRAICALLY NATURAL LOWER BOUNDS

Many lower bounds for restricted models of algebraic circuits have this form:

1. Given $f$, construct some matrix $M_f$ whose entries are coefficients of $f$.
2. Argue that if $f$ is computed by a small circuit, $\text{rank} (M_f) = \text{small}$.
3. Show some explicit $f_0$ with $\text{rank} (M_{f_0}) = \text{large}$.
   (Examples: evaluation dimension, partial derivatives, shifted partial derivatives, ...)
   Equivalently: for some $r$, submatrix, $\det (M'_{f_0}) \neq 0$ while $\det (M'_f) = 0$ for all simple $f$.

Therefore, the property $f g : \det (M'_g) \neq 0$ is useful, constructive (determinant is efficiently computable) and large.
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Def: A (distinguisher) polynomial $D \neq 0$ is an algebraic natural proof against a class $\mathcal{C}$ if

1. (Usefulness) $D(\text{coeff}(f)) = 0$ for all $f \in \mathcal{C}$
2. (Constructiveness) $D$ has a small algebraic circuit (largeness comes for free)

Important: $D$ is an $N$-variate polynomial for $N = n + d$ (if we deal with $n$-variate degree $d$ polynomials) or $N = 2^n$ (if we deal with multilinear polynomials)

Toy example: $\mathcal{C}$ is the class of perfect squares among polynomials of the form $ax^2 + bx + c$, and $D(a, b, c) = b^2 - 4ac$.

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GEOMETRIC COMPLEXITY THEORY

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VP is a zero set of a set of polynomials $T$. What is the complexity of $T$?
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POLYNOMIAL IDENTITY TESTING

Given an algebraic circuit $C$, decide deterministically whether $C$ computes the zero polynomial.
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$\implies \{\text{coeff}(f) : f \in C\}$ is not a hitting set for $D$.

In other words: if $\{\text{coeff}(f) : f \in C\}$ is a hitting set for a class $D$, then no natural proof for $C$ with the distinguisher coming from $D$. 
**Succinct Hitting Sets**

**Def:** Let $\mathcal{C} \subseteq \mathbb{F}[x_1, \ldots, x_n]$ be a class of degree $d$ polynomials, and $\mathcal{D} \subseteq \mathbb{F}[X_1, \ldots, X_N]$ for $N = \binom{n+d}{d}$. $\mathcal{C}$ is a **succinct hitting set** for $\mathcal{D}$ if $\mathcal{H} := \{\text{coeff}(f) : f \in \mathcal{C}\}$ is a hitting set for $\mathcal{D}$.

**Thm:** If $\mathcal{C}$ is a succinct hitting set for $\mathcal{D}$, no algebraically natural proof against $\mathcal{C}$ with the distinguisher coming from $\mathcal{D}$.

**Proof:** If $\mathcal{D} \neq 0$ then $\mathcal{D}$ does not vanish on $\mathcal{H}$. (also observed independently by [Grochow-Kumar-Saraf-Saks](#)).

**Question:** are poly$(n)$ size and poly$(n)$ degree circuits a hitting sets for poly$(N)$ size and poly$(N)$ degree circuits? ("does VP hit VP?")
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Proof: If $D \in \mathcal{D}$ is non-zero then $D$ does not vanish on $\mathcal{H}$. □
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Let $\mathcal{H} = \{\text{coeff}(f) : f \in \text{VP}(n)\}$. If $\mathcal{H}$ is a hitting set for $\text{VP}(N)$ then there are no $\text{VP}$-algebraic natural proofs against $\text{VP}$. 
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(note: $\mathcal{H}$ may a-priori be infinite but we’ll soon see that this actually implies there exists some small $\mathcal{H}'$)
Generators

**Def:** A polynomial map $G : \mathbb{F}^\ell \to \mathbb{F}^N$ is a generator for a class $\mathcal{C}$ if for every non-zero $F \in \mathcal{C}$, $F(G(y)) \neq 0$. 
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(want: $\text{deg poly}(N)$, $\ell$ as small as possible)
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(\( \iff \) : evaluate. \( \iff \) : interpolate.)

succinct hitting sets \( \Rightarrow \) ?
Def: A polynomial $G(x, y)$ is a $C$-succinct generator for $D$ if:

1. For every $x, y, G(x, y) \leq C$.
2. The polynomial map $G = \text{coeff} x (G(x, y))$ is a generator for $D$. 
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$$G(x, y) = 1 \cdot (y_1 + y_2) + x_1 \cdot (y_1 y_2^3) + x_2 \cdot (y_1^2 + y_2) + x_1 x_2 \cdot 1$$

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$$G(y) = (y_1 + y_2, y_1 y_2^3, y_1^2 + y_2, 1)$$

$\{G(x, \alpha) : \alpha \in \mathbb{F}^\ell\}$ is a $C$-succinct hitting set against $\mathcal{D}$.

So succinct generator $\implies$ succinct hitting set (and even a “uniform” one).
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Aside: why not just require $G(x, y) \in \mathcal{C}$?
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We can, but

1. unnecessary for succinct hitting sets which imply barriers
2. $G(x, \alpha)$ might be in even smaller class (e.g., if $y$ has high deg)
Recall: succinct generator $\implies$ succinct hitting sets.

Other direction? Interpolating has complexity $\text{poly}(jH_j)$ which is not succinct. But still true because of the existence of universal circuits. In particular, if $H = \text{coeff}(f) : f \in \text{VP}(n)$ $g$ is an (infinite) hitting set for $\text{VP}(N)$, there is a $N \text{poly log}(N)$ size hitting set ($H$ is in the image of a universal circuit).
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Succinct generators and hitting sets

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But still true because of the existence of universal circuits.

In particular, if $\mathcal{H} := \{\text{coeff}(f) : f \in \text{VP}(n)\}$ is an (infinite) hitting set for $\text{VP}(N)$, there is a $N^{\text{polylog}(N)}$ size hitting set ($\mathcal{H}$ is in the image of a universal circuit).
Conjecture: VP hits VP.

Challenge: establish this under some crypto hardness assumption.
**EVIDENCE?**

**Conjecture:** VP hits VP.

How to obtain evidence to support this conjecture?
**Evidence?**

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We can’t make all known hitting sets succinct but we have some excuses (more on that later).
Toy example: $\mathcal{C} \subseteq \mathbb{F}[X_1, \ldots, X_N]$ is the class of polynomials with monomials of support $\leq \text{polylog}(N)$. 
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Q: Is this succinct?

A: Yes. Each \( v \) is a coefficient vector of a \( \text{poly}(n) \) \( \Sigma \Pi \) circuit in \( x_1, \ldots, x_n \) (only \( \text{poly}(n) \) monomials with non-zero coefficient). \( \Box \)
Less-of-a-toy example: $\mathcal{C} \subseteq \mathbb{F}[X_1, \ldots, X_N]$ is the class of polynomials of sparsity at most $s$. 
Sparse Polynomials

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**Cor:** \( \text{poly} (\log s, n) - \Sigma \Pi \Sigma \) succinct hitting set for sparse polynomials.
MORE SUCCINCT HITTING SETS

In the paper: many other succinct generators, most of them follow from various combinations of basic constructs such as Shpilka-Volkovich generator and Gabizon-Raz’s rank condenser, which we make succinct.
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Builds on a lot of previous work: [Dvir-Shpilka, Karnin-Shpilka, Kayal-Saraf, Saxena-Seshadhri, Shpilka-Volkovich, Forbes-Shpilka, Forbes-Shpilka-Saptharishi, Beecken-Mittmann-Saxena, Agrawal-Saha-Saxena-Saptharishi,...].
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Cor: Super-polynomial lower bounds on defining equations of \( \overline{\text{VP}} \) in many models.
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\( y \) new var, \( k \) integer and \( p \) prime chosen from sufficiently large set.
WHAT WE CAN’T DO

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**Challenge:** make it succinct, i.e., find a small circuit in \(\{x_1, \ldots, x_n, y\}\) such that the coefficient of \(x_S\) is \(y^{k_{\text{bin}(S)}} \mod p\), where \(\text{bin}(S) = \text{integer whose binary expansion is the characteristic vector of } S\).
USEFULNESS OF KS

The generator $X_i \mapsto y_1^{k_i} \mod p_1 \cdots y_m^{k_i} \mod p_m$ for $m = O(\log n)$ hits roABPs (in any order) and read-once determinants (polys of the form $\det(M)$ where each entry in $M$ contains a var or a constant and each var appears at most once)

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Our constructions are all small multilinear formulas, so we might need new ideas.
MORE OPEN PROBLEMS

More models for which we know of hitting sets but not succinct ones:

- roABPs in any order
- read-
k oblivious ABPs
- bounded-depth multilinear formulas

Also: pseudorandom polynomials? Is [Aaronson-Drucker]'s construction pseudorandom?
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THANK YOU