Matchings in Graphs

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We look at the linear programming method for the maximum matching and perfect matching problems. Given a graph $G = (V, E)$, an integer linear program (ILP) for the maximum matching problem can be written by defining a variable x_e for each edge $e \in E$ and a constraint for each vertex $u \in V$ as follows:

$$\begin{array}{rrcl} & \text{Maximize} \quad \sum_{e \in E} x_e & & \text{subject to} \\ \forall u \in V & \sum_{e \perp u} x_e & \leq & 1 \\ \forall e \in E & x_e & \in & \{0, 1\} \end{array}$$

Here $e \perp u$ denotes that e is incident on u. It can be seen that the optimum solution to this ILP is indeed a maximum matching in G. Therefore an algorithm to solve ILP can be used to get a maximum matching in G. However, ILP is known to be NP-complete and hence there is no polynomial-time algorithm known for it.

LP relaxation One way to deal with this is to relax the integrality constraints and allow $x_e \in [0, 1]$ to get a linear program, which can be solved in polynomial-time. However, this gives rise to fractional matchings. Characteristic vectors of matchings in G can be seen as points in $\mathbb{R}^m$ where $m = |E|$. The convex hull of all the matchings forms a polytope called *the matching polytope* M. However, the LP relaxation may give matchings that are outside M. Figure shows some examples. It can be seen that in examples (1) and (2) in Figure , the matching polytope contains all the fractional matchings which form the feasible region of the relaxed LP. However, in Example (3), the maximum value of the relaxed LP is attained at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which lies outside M. We will see next that the matching polytope contains all the fractional matchings if and only if the graph is bipartite. We first recall a useful property of convex polytopes: Let $\mathbb{P} \subseteq \mathbb{R}^m$ be a convex polytope. The following are equivalent:

1. A point x is a vertex of $\mathbb{P}$.
2. x cannot be written as a non-trivial convex combination of more than one point in $\mathbb{P}$.
3. There is a hyperplane H such that $\{x\} = H \cap \mathbb{P}$.
4. If $\mathbb{P}$ is given as an intersection of half-spaces, then x is a unique solution to a set of m linearly independent constraints met as equalities.

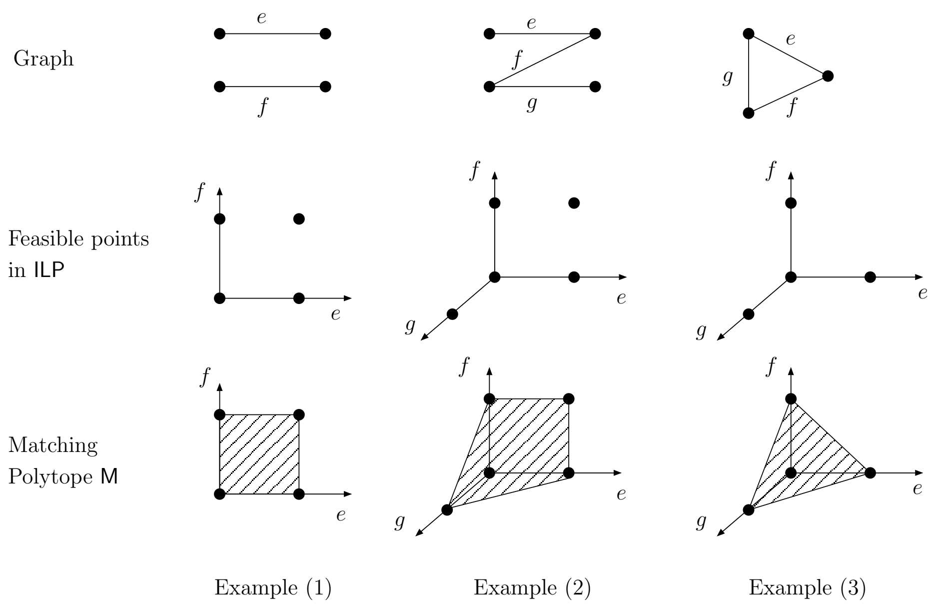

Figure 1: Some examples of LP relaxation

Definition 1 (Matching polytope) *For a given graph G, the* matching polytope M *is the convex hull of all the matchings in G. Thus*

$$\mathsf{M} = \left\{ \sum_i \lambda_i M_i \mid \forall i\ \lambda_i \geq 0, M_i \textit{ is a matching, and } \sum_i \lambda_i = 1 \right\}$$

Definition 2 (Fractional matching polytope) *The* fractional matching polytope FM *is defined as the feasible region for the* LP

$$\forall u \in V : \sum_{e \perp u} x_e \leq 1, \qquad \forall e \in E : x_e \in [0, 1]$$

For a graph G, M and FM may not be the same. But they are the same for bipartite graphs.

Claim 3 *For bipartite graphs, the* LP *relaxation gives a matching as an optimal solution.*

Proof: The proof follows from the fact that the optimum of an LP is attained at a vertex of the polytope, and that the vertices of FM are the same as those of M for a bipartite graph, as proved in Claim 6 below. ∎

We define the *perfect matchings polytope* PM and the *fractional perfect matchings polytope* FPM.

Definition 4 (Perfect Matching Polytope) *For a given graph G, the* perfect matching polytope PM *is the convex hull of all the perfect matchings in G. Thus*

$$\mathsf{FM} = \left\{ \sum_i \lambda_i M_i \mid \forall i \ \lambda_i \geq 0, M_i \textit{ is a perfect matching, and } \sum_i \lambda_i = 1 \right\}.$$

Definition 5 (Fractional Perfect Matching Polytope) *The* fractional perfect matching polytope FPM *is defined as the feasible region for the* LP

$$\forall u \in V : \sum_{e \perp u} x_e = 1, \qquad \forall e \in E : x_e \in [0,1]$$

For M and PM, we know the extremal points of the polytopes, but we do not yet have a set of halfspaces whose intersection defines the polytope. For FM and FPM, on the other hand, we know the defining half-spaces but do not yet know the extremal points. For bipartite graphs, fortunately, the two coincide.

Claim 6

1. *$\mathsf{M} \subseteq \mathsf{FM}$, $\mathsf{PM} \subseteq \mathsf{FPM}$.*
2. *If G is bipartite, then $\mathsf{M} = \mathsf{FM}$ and $\mathsf{PM} = \mathsf{FPM}$.*
3. *If G is non-bipartite, then $\mathsf{M} \neq \mathsf{FM}$.*

Proof:

Proof of 1: It can be seen that any matching in G satisfies the constraints for FM. Thus the extreme points of M are all contained in FM and hence $\mathsf{M} \subseteq \mathsf{FM}$. Similarly, $\mathsf{PM} \subseteq \mathsf{FPM}$.

Proof of 2: Note that all the vertices of M are in $\{0,1\}^m$, where $m = |E|$. Further, any vertex in $\mathsf{FM} \cap \{0,1\}^m$ is also in M. Thus, it suffices to prove that all the vertices of FM are integral. Assume that there is a non-integral vertex x of FM. We will show that x can be written as a non-trivial convex combination of two points in FM, which contradicts the assumption that it is a vertex.

Define $G_x = (V, E_x)$ where $E_x = \{e \mid x_e \notin \{0,1\}\}$. Suppose G_x has a cycle C. Since G is bipartite, C is of even length. Let a and b be the minimum and maximum values of x_e for $e \in C$ respectively. We refer to x_e as the weight on e. Define $\epsilon = \min\{1-b, a\}$. Define two matchings x^+ and x^- obtained from x by adding and subtracting ϵ from the weights on alternate edges of C; see Figure 2 for an example. Then $x = \frac{1}{2}(x^+ + x^-)$.

If G_x has no cycle, pick any maximal path in G_x. The end-points of the path have weights strictly less than 1 on the edges incident on them. Define a, b, ϵ and construct x^+ and x^- as before, which again contradicts the extremality of x.

The same argument works for $\mathsf{PM} = \mathsf{FPM}$, except that the case G_x being acyclic does not arise.

Proof of 3: Let G be non-bipartite. Take an odd cycle C in G. Consider a fractional matching x that has weights $\frac{1}{2}$ on each of the edges in C. Clearly, this matching is in FM. It can be seen that it cannot be written as a convex combination of any two or more matchings, and so it is not in M. ■

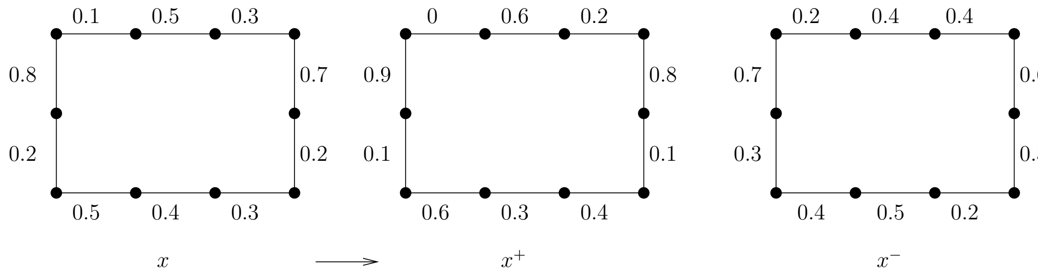

Figure 2: Expressing a cycle in x as a convex combination of x^+ and x^-

Despite the third point in Claim 6, we can say something about FM even for non-bipartite graphs.

Claim 7 *All the vertices of* FM *are half-integral; that is, in* $\{0, \frac{1}{2}, 1\}^m$.

We will prove the following stronger version, which implies the above claim:

Theorem 8 *For any graph G, $x \in$* FPM *if and only if x satisfies the following conditions:*

1. $\forall e \in E$, $x_e \in \{0, 1, \frac{1}{2}\}$.
2. *Let $G_0 = (V, E_0)$ where $E_0 = \{e \in E \mid x_e = \frac{1}{2}\}$. Then G_0 is a collection of vertex-disjoint odd cycles.*

Proof of Theorem 8: First we prove that if x satisfies the two conditions, then x is a vertex. Thus assume that $x \in \{0, 1, \frac{1}{2}\}^m$. Define

$$\begin{aligned} E_0 &= \left\{e \in E \mid x_e = \frac{1}{2}\right\} \\ E_1 &= \{e \in E \mid x_e = 1\} \\ E_2 &= \{e \in E \mid x_e = 0\} \end{aligned}$$

To show that x is a vertex, we will show that there is a hyperplane H such that $\{x\} = H \cap \mathsf{FPM}$. For any $w \in \mathbb{R}^m$ and any $a \in \mathbb{R}$, let $H_{w,a}$ denote the hyperplane $\{y \mid \sum_e w_e y_e = a\}$. To construct H, we set $w_e = -1$ if $x_e = 0$ and $w_e = 0$ if $x_e > 0$. To ensure that x lies on H, we need $a = 0$, since $w^T x = 0$. Now we need to show that no other point in FPM lies on H. Suppose $y \in H \cap \mathsf{FPM}$. Then $y \in \mathbb{R}^{\geq 0}$, and $w|_{E_2} = -1$, so $y^T w = y^T|_{E_2} w|_{E_2}$, so it must be that $y|_{E_2} = 0$. Thus $y_e = 0$ whenever $x_e = 0$. Since we have only odd cycles in the graph restricted to E_0, we can see from the constraints of FPM $y|_{E_0} = \frac{1}{2}$, and so also $y|_{E_1} = 1$. Thus y cannot be different from x. Hence $\{x\} = H \cap \mathsf{FPM}$ and so x is a vertex.

Now we prove the other direction. Let x be a vertex of FPM. We will prove that x satisfies the two conditions. Let $H = \{z \mid w^T z = a\}$ be the witnessing hyperplane such that $\{x\} = H \cap \mathsf{FPM}$. Without loss of generality, assume that FPM lies on the side of H satisfying $w^T z \leq a$.

Interpret the adjacency matrix of G as the bipartite adjacency matrix of a graph G'. Equivalently, define a bipartite graph $G' = (V', V'', E')$ such that $V' = V'' = V$. Thus each $u \in V$ has a copy $u' \in V'$ and $u'' \in V''$. Further, for each edge, make two copies as follows: $E' = \{(u', v''), (v', u'') \mid (u, v) \in E\}$. Thus $|E'| = 2m$. Each perfect matching M in G also has a corresponding perfect matching M' in G' such that $(u, v) \in M \Rightarrow (u', v''), (v', u'') \in M'$. This holds for fractional perfect matchings as well; if $x \in \mathsf{FPM}$ and y is defined as $x_e = y_{e_1} = y_{e_2}$ where for each edge e, e_1 and e_2 are the copies of E in G', then $y \in \mathsf{FPM}(G')$. As G' is bipartite, by Claim 6, we also have $y \in \mathsf{PM}(G')$.

Now define the hyperplane $H' = \{z \mid w'^T z = 2a\}$ where w' is obtained by concatenating the vector w with itself. Clearly, since $x \in H$, we have $y \in H'$. Thus the following LP on $2m$ variables has a non-empty feasible region, and the optimum is at least $2a$.

$$\text{maximise } w'^T Y \text{ subject to } Y \in PM(G').$$

In particular, the optimum is achieved at a vertex, say z. Since z is a vertex of $\mathsf{PM}(G')$ for bipartite G', z is in $\{0,1\}^{2m}$. Define a new vector $x' \in \mathbb{R}^m$ such that $\forall e \in E$, $x'_e = \frac{z_{e_1}+z_{e_2}}{2}$. Clearly, $w^T x' = \frac{1}{2}(w'^T z) \geq a$. But it is easy to see that $x' \in \mathsf{FPM}(G)$, so $w^T x' \leq a$. Therefore $w^T x' = a$; and so $x' \in H \cap \mathsf{FPM} = \{x\}$, implying $x' = x$. Therefore for each edge e, $\frac{z_{e_1}+z_{e_2}}{2} = x'_e = x_e$. This implies that x is half-integral, which proves the first condition of the theorem.

For the second condition, let x have an even cycle with weights $\frac{1}{2}$ on all its edges. Then x can be written as a convex combination of two matchings which contain alternate edges from the cycle. Thus the presence of an even cycle in G_0 implies that x is not a vertex. Therefore G_0 contains only disjoint odd cycles. ∎

As we have seen, for non-bipartite graphs, $\mathsf{FPM} \not\subseteq \mathsf{PM}$. This happens precisely because an odd subset of vertices can have a fractional perfect matching but not an integral one. Hence, to get a set of defining half-spaces for PM, we need to introduce more constraints in the FPM polytope, which essentially require that at least one vertex of each odd subset be matched outside the subset. Alternatively, for each odd subset S, the total weight of edges within S should be at most $\frac{|S|-1}{2}$. With this additional constraint, we define a new perfect matching polytope $\mathsf{P}(G)$ as the set of points $x \in \mathbb{R}^m$ satisfying

$$\forall e \in E : x_e \geq 0; \qquad \forall u \in V : \sum_{e \perp u} x_e = 1; \qquad \forall S \subseteq V : |S| \text{ odd } \Rightarrow \sum_{e \in E(S, \bar{S})} x_e \geq 1$$

Similarly define a new fractional matching polytope $\mathsf{MP}(G)$ as the set of points $x \in \mathbb{R}^m$ satisfying

$$\forall e \in E : x_e \geq 0; \qquad \forall u \in V : \sum_{e \perp u} x_e \leq 1; \qquad \forall S \subseteq V : |S| \text{ odd } \Rightarrow \sum_{e \in E(S)} x_e \leq \frac{|S|-1}{2}$$

Now we show that these polytopes, defined by their bounding halfspaces, correspond to the polytopes defined by having (perfect) matchings as extremal points. It is straightforward to see that $\mathsf{M}(G) \subseteq \mathsf{MP}(G)$ and $\mathsf{PM}(G) \subseteq \mathsf{P}(G)$, since every vertex of $\mathsf{M}(G)$ is easily seen to be in $\mathsf{MP}(G)$, and every vertex of $\mathsf{PM}(G)$ is in $\mathsf{P}(G)$. We show the converse below.

Theorem 9 (Edmonds Theorem) $\mathsf{PM}(G) = \mathsf{P}(G)$.

Before proving this theorem, we prove that it implies the following:

Theorem 10 $\mathsf{MP}(G) = \mathsf{M}(G)$.

Proof: We need to show that $\mathsf{MP}(G) \subseteq \mathsf{M}(G)$. Construct a new graph $H = (V', E')$ as follows: Take two disjoint copies of G, say G_1, G_2 and for each $u \in V$, add edge (u_1, u_2) where u_1 and u_2 are the copies of u in G_1 and G_2 respectively. For $x \in \mathsf{MP}(G)$, we construct $y \in \mathsf{P}(H)$ such that both the copies of the edge $e \in E$ have weight x_e in y. Moreover, for each vertex $u \in V$, we set the weight of the edge (u_1, u_2) in y to be the deficit of u in x. (By deficit we mean the difference between 1 and the total weights of edges incident on u.)

We show that y satisfies the constraints of $\mathsf{P}(H)$. It is easy to see that y satisfies the first two constraints of $\mathsf{P}(H)$. To see that y satisfies the third constraint as well, consider an odd cardinality subset S of vertices in H. Let $S = X_1 \cup Y_2$ where $X_1 \subseteq V(G_1)$ and $Y_2 \subseteq V(G_2)$. For subset T of vertices, let $\delta(T)$ denote $\sum_{e \in E'(T,\bar{T})} y_e$. Then

$$\delta(S) \geq \delta(X_1 \setminus Y_1) + \delta(Y_2 \setminus X_2)$$

where Y_1 and X_2 are copies of Y_2 in G_1 and of X_1 in G_2 respectively. Hence, without loss of generality, we can assume that the underlying sets X and Y in G are disjoint. Further, without loss of generality we can also assume that $Y \neq \emptyset$ and that $|X|$ is odd. Since $|X|$ is odd, there is a deficit of at least $|S| - \frac{|S|-1}{2}$, which is added to the edges going out of X. Thus y satisfies the third constraint too and hence $y \in \mathsf{P}(H)$. But $\mathsf{P}(H) = \mathsf{PM}(H)$ by Theorem 9. Therefore we can write y as a convex combination of perfect matchings in H. From this, a convex combination of matchings in G yielding x can be computed by loking at just one copy of G in H. So $x \in \mathsf{M}(G)$. ■

Now we can restrict ourselves to perfect matchings and prove Edmonds' theorem:
Proof of Theorem 9: For odd n, $\mathsf{PM}(G) = \emptyset$. So is $\mathsf{P}(G)$, because for $S = V$, the third constraint cannot be satisfied. So there is nothing to prove.

For even n, we prove the theorem by contradiction. Let G be a graph such that $\exists x \in \mathsf{P}(G) \setminus \mathsf{PM}(G)$. Take the smallest such G *i.e.* a graph with minimum number of vertices and breaking ties by picking a graph with minimum number of edges, which satisfies this condition. Consider x as defined above for G. By minimality of G, we have the following:

1. n is even.

2. $\forall e \in E$, $0 < x_e < 1$.
   If there is an edge e with $x_e = 0$, we can discard it, contradicting the minimality of G. If there is an edge $e = (u, v)$ with $x_e = 1$, we can discard u and v, again contradicting minimality of G.

3. There are no isolated or pendant vertices in G.
   At pendant vertices, the single incident edge would have weight 1, contradicting the above. Isolated vertices contradict minimality of G.

4. $\exists v \in V$ with degree greater than 2. Hence $m > n$.
   If all vertices have degree 2, then G is a collection of vertex-disjoint cycles. The cycles cannot be odd as G satisfies the constraints of $\mathsf{P}(G)$. So G is bipartite and hence by Claim 6 cannot be a counter example.

Without loss of generality, assume x to be a vertex of $\mathsf{P}(G)$. Therefore it is the unique solution of m linearly independent constraints satisfied as equalities. By 2 above, the constraints can not be of the form $x_e \geq 0$. There are only n constraints of the form $\sum_{e \perp u} x_e = 1$. Therefore at least one of these m constraints should be of the form $\sum_{e \in E(S,\bar{S})} x_e \geq 1$. Let S be a subset where such a constraint is satisfied by x with equality. Further, $|S|, |\bar{S}| > 1$, otherwise the constraint will be of the form $\sum_{e \perp u} x_e = 1$. Since S is odd, $3 \leq |S|, |\bar{S}| \leq n-3$.

Construct G_1 and G_2 as follows: In G_1, $\bar{S}$ is contracted to a single vertex $\bar{s}$, S remains the same as in G, and multiple edges obtained are replaced by single edges. Similarly, define G_2 by contracting S to s and leaving $\bar{S}$ as in G. Let $x^{(1)}$ be the restriction of x to G_1 such that the edges incident on $\bar{s}$ get a weight equal to the sum of the weights of the corresponding edges in x. Similarly construct $x^{(2)}$ as restriction of x to G_2. Clearly $x^{(1)} \in \mathsf{P}(G_1)$. But G_1 is smaller than G and hence $\mathsf{P}(G_1) = \mathsf{PM}(G_1)$. Similarly, $\mathsf{P}(G_2) = \mathsf{PM}(G_2)$. Therefore $x^{(1)}$ and $x^{(2)}$ can be written as convex combinations of perfect matchings in G_1 and G_2 respectively:

$$x^{(1)} = \sum_i \alpha_i \chi_{L_i} \qquad x^{(2)} = \sum_j \beta_j \chi_{N_j}$$

where χ_M is the characteristic vector of a matching M, L_i is a perfect matching in G_1 and N_j in G_2, the coefficients α_i, β_j are non-negative, and $\sum_i \alpha_i = \sum_j \beta_j = 1$. The idea is to express x as a convex combination of matchings in G by patching together the L_i and N_j. Consider perfect matchings L in G_1 and N in G_2 from above convex combinations. Let u_L (respectively v_N) be the vertex in G_1 (G_2) which is matched to $\bar{s}$ (s) in L (N). If $(u_L, v_N) \in E$, then $L \setminus \{(u_L, \bar{s})\} \cup N \setminus \{(s, v_N)\} \cup \{(u_L, v_N)\}$ is a perfect matching in G. Construct such perfect matchings M_{ij} for all the pairs (L_i, N_j) wherever $(u_{L_i}, v_{N_j}) \in E$. (Otherwise set $M_{ij} = \emptyset$.) We claim that x can be written as a convex combination of matchings of the form M_{ij}.

Let u_i denote the vertex matched to $\bar{s}$ in L_i, that is u_{L_i}. Similarly, let v_j denote the vertex v_{N_j} that is, matched by N_j to s. For vertex $u \in S$, define $g(u)$ as $g(u) = \sum_{v \in \bar{S}} x_{uv}$. Then $g(u) = x^{(1)}_{u\bar{s}} = \sum_{i:u_i=u} \alpha_i$. Similarly, for vertex $v \in \bar{S}$, define $g(v)$ as $g(v) = \sum_{u \in S} x_{uv}$; then $g(v) = x^{(2)}_{sv} = \sum_{j:v_j=v} \beta_j$. Now set the γ weights as follows:

$$\gamma_{ij} = \left(\frac{\alpha_i}{g(u_i)}\right)\left(\frac{\beta_j}{g(v_j)}\right) x_{u_i v_j}$$

Claim 11 $x = \sum_{i,j} \gamma_{ij} \chi_{M_{ij}}$. *That is, for each $f \in E$,*

$$x_f = \sum_{ij:\ M_{ij} \text{ contains } f} \gamma_{ij}.$$

The claim, whose proof is left as an exercise, contradicts the assumption that x is a vertex of $\mathsf{P}(G)$. ∎