Szegö limit Theorem on the Lattice

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Abstract

In this paper, we prove a Szegö type limit theorem on $\ell^2(\mathbb{Z}^d)$. We take the self-adjoint operator $H = -\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where $(\Delta u)(\mathbf{n}) = \sum_{|\mathbf{n}-\mathbf{k}|=1} (u(\mathbf{k}) - u(\mathbf{n}))$ and the operator V is the multiplication by a positive sequence $\{V(\mathbf{n}), \mathbf{n} \in \mathbb{Z}^d\}$ with $V(\mathbf{n}) \to \infty$ as $|\mathbf{n}| \to \infty$. We take the orthogonal projection π_λ onto the subspace, in $\ell^2(\mathbb{Z}^d)$, spanned by eigenfunctions of H with eigenvalues $\leq \lambda$. Let B be a zeroth order self-adjoint pseudo-difference operator with symbol $b \in S_{1,0,\infty}(\mathbb{T}^d \times \mathbb{Z}^d)$. We then show for "nice functions" f, that

$$\lim_{\lambda\to\infty} \frac{Tr(f(\pi_\lambda B \pi_\lambda))}{Tr(\pi_\lambda)} = \lim_{\lambda\to\infty} \frac{1}{(2\pi)^d} \frac{\sum_{V(\mathbf{n})\leq\lambda} \int_{\mathbb{T}^d} f(b(\mathbf{x},\mathbf{n}))\, d\mathbf{x}}{\sum_{V(\mathbf{n})\leq\lambda} 1}.$$

1 Introduction

In 1952, G. Szegö considered a linear operator T_f on $L^2((0, 2\pi))$, of multiplication by f, associated with a positive function $f \in \mathcal{C}^{1+\alpha}[0, 2\pi], \alpha > 0$.

He took the orthogonal projections $\{P_n\}$ onto a linear subspace of $L^2[0,2\pi]$ spanned by the functions $\{e^{im\theta} : 0 \le m \le n; 0 \le \theta < 2\pi\}$ for each n. He then proved the following relation for such a triple $(f, T_f, \{P_n\})$.

$$\lim_{n\to\infty} \frac{1}{n+1} \log\det P_n T_f P_n = \frac{1}{2\pi}\int_0^{2\pi} \log f(\theta) d\theta. \tag{1}$$ szeg1

The orthogonal projections P_n coincide with the spectral projections π_λ of the self-adjoint operator $-\frac{d^2}{dx^2}$ on $L^2[0,2\pi]$, with a periodic boundary condition, corresponding to the interval $[0,\lambda)$ with $n < \lambda < n+1$. The result in equation (1) is the well known as Szegö limit theorem. We refer to [12, 6] for details and related results. More specifically, if f is a bounded real-valued integrable function then the eigenvalues $\{\lambda_i^n\}_{i=1}^n$ of $P_n T_f P_n$ are contained in $[\inf f, \sup f]$. This result was generalized to continuous functions F (instead of the logarithm in (1)) defined on $[\inf f, \sup f]$, in Sect. 5.3 of [6]. The result for such F is,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n F(\lambda_i^n) = \frac{1}{2\pi}\int_0^{2\pi} F(f(\theta)) d\theta.$$

Notice that the left hand side here can be seen to be the limit of

$$Tr(F(P_n T_f P_n))/Tr(P_n)$$

where $Tr(A)$ denotes the trace of the operator A, $e^{im\theta}$ is an eigenfunctions of $\Delta = -\frac{d^2}{dx^2}$, with a periodic boundary condition and the asymptotic of the functional

$$\rho_\lambda(F) = Tr(\pi_\lambda F(\pi_\lambda T_f \pi_\lambda)\pi_\lambda) = \sum_k F(\mu_k(\lambda))$$

is precisely the sum of Dirac measures located at the eigenvalues $\mu_k(\lambda)$ of the operator $\pi_\lambda T_f \pi_\lambda$. Similar results were obtained for various classes of differential and pseudo-differential operators in [2], [8], [9] and [15].

In [15], Zelditch considered a Schrödinger operator on $\mathbb{R}^n$ of the form $H = -\frac{1}{2}\Delta + V$, where V is a smooth positive function that grows like $V_0|x|^k$, $k > 0$ at infinity. He took a 0- th order self-adjoint pseudo-differential operator A associated with a symbol $a(x,\xi)$ relative to Beals -Fefferman weights $\varphi(x,\xi), \varphi(x,\xi) = (1 + |\xi|^2 + V(x))^{1/2}$ and proved the following generalization of Szegö type theorem: For any continuous function f,

$$\lim_{\lambda\to\infty} \frac{Trf(\pi_\lambda A \pi_\lambda)}{\text{rank } \pi_\lambda} = \lim_{\lambda\to\infty} \frac{\int_{\varphi(x,\xi)\le\lambda} f(a(x,\xi))\, dxd\xi}{\text{Vol}(\varphi(x,\xi)\le\lambda)},$$

assuming one of the limits exists. Such asymptotic spectral formulae expressing the relation between functions of pseudo-differential operators and their symbols is an important and interesting problem in mathematical analysis. In this paper we aim to develop similar formulae for some pseudo-difference operators on the lattice.

To establish a Szegö type theorem, it is clear that we need to consider ratios of distribution functions associated to different measures and their asymptotic behavior.

The asymptotic limit of such ratios is computed using a Tauberian theorem where some transform of these measures is considered and the limit taken for such transforms. For example, Zelditch [15] used the Laplace transform (via Karamata's Tauberian theorem ([14], p-192) whereas Robert [10] suggested the use of Stieltjes transform (via Keldysh Tauberian theorem) in [3]. The application of Keldysh's theorem requires one of the measures μ or ν to be absolutely continuous. We do not have this feature in our problem, so we use the Tauberian theorem of Grishin-Poedintseva theorem 3.4 (see [7]) and a theorem of Laptev-Safarov (see [5]), for estimating the errors, to prove our main theorem (Theorem 1.1).

There is extensive work on the Szegö's theorem associated with orthogonal polynomials in $L^2(\mathbb{T}, d\mu)$ with μ some probability measure on $\mathbb{T}$, we refer to the monumental work of Barry Simon [13] for the details.

We, however, concentrate on higher dimensions where not much is known and to our knowledge our results are new in the lattice case. We consider operators of the form

$$H = -\Delta + V \tag{2}$$ eqn0

on $\ell^2(\mathbb{Z}^d)$, where Δ is the self-adjoint operator

$$(\Delta u)(\mathbf{n}) = \sum_{|\mathbf{n}-\mathbf{k}|=1} (u(\mathbf{k}) - u(\mathbf{n}))$$

which is unitarily equivalent to multiplication by the function

$$2\sum_{j=1}^{d} \cos(\theta_j) - 2d, \quad \theta_j \in [0, 2\pi), \; j = 1, \dots, d$$

and V is multiplication by a positive sequence

$$V(\mathbf{n}) = \begin{cases} 1, & \mathbf{n} = 0 \\ |\mathbf{n}|^{\beta}, & \kappa \in (0,1) \quad \mathbf{n} \neq 0, \end{cases} \tag{3}$$ eqn1

Then we note that $-\Delta$ and V are positive operators and so H is a positive operator on $\ell^2(\mathbb{Z}^d)$ and the choice of V makes $(H-i)^{-1}$ compact showing that H has discrete spectrum. The eigenfunctions of H form a complete orthogonal basis for $\ell^2(\mathbb{Z})$. We denote the spectral projection of H by $E_H()$ and set $\pi_\lambda = E_H((0,\lambda])$.

For a bounded self-adjoint operator B we set $K = [-\|B\|, \|B\|] \subset \mathbb{R}$, so that the spectra of operators B and $\pi_\lambda B \pi_\lambda$ lie in K for all λ and set $L^2(\mathbb{T}^d) = L^2\left(\mathbb{T}^d, \frac{dx}{(2\pi)^d}\right)$. Then our main theorems are the following.

szego **Theorem 1.1.** *Let H and V be as in equation ([2, 3]). Let b be a bounded real valued measurable function on $\mathbb{T}^d$, let M_b be the operator of multiplication by b on $L^2(\mathbb{T}^d)$ and B its unitary equivalent on $\ell^2(\mathbb{Z}^d)$ under the Fourier Series.*

Then for all $f \in C(K)$, we have

$$\lim_{\lambda\to\infty} \frac{Tr\left(f(\pi_\lambda B \pi_\lambda)\right)}{Tr(\pi_\lambda)} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(b(\mathbf{x}))\, d\mathbf{x}. \tag{4}$$

We recollect some facts on toroidal symbols from Ruzhanski-Turunen [11] below. A linear operator A on $L^2(\mathbb{T}^d)$ associated with symbols $\sigma(\mathbf{x},\mathbf{n})$, $(\mathbf{x},\mathbf{n}) \in \mathbb{T}^d \times \mathbb{Z}^d$, (the reader should note that the lattice variable ξ appearing in [11] should be replaced by $\mathbf{m}, \mathbf{n}$ etc in our notation) is defined by

$$(A\phi)(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}} \sigma(\mathbf{x},\mathbf{n})\phi(\mathbf{y})\, d\mathbf{y} \tag{5}$$ symbol

where $\phi \in C^\infty(\mathbb{T}^d)$ and the symbol $\sigma \in C^\infty(\mathbb{T}^d \times \mathbb{Z}^d)$, ($a \in C^\infty(\mathbb{T}^d \times \mathbb{Z}^d)$ means $a(\cdot,\mathbf{n}) \in C^\infty(\mathbb{T}^d)$ for all $\mathbf{n} \in \mathbb{Z}^d$). Then A extends to a bounded linear operator and hence (via the unitary isomorphism implemented by the Fourier series, call it U^*, between $L^2(\mathbb{T}^d)$ and $\ell^2(\mathbb{Z}^d)$) also a bounded operator on $\ell^2(\mathbb{Z}^d)$.

We will say that $b(\mathbf{n},\mathbf{x})$ is the symbol of a bounded linear operator B on $\ell^2(\mathbb{Z}^d)$ if it is the symbol of a bounded linear operator on $L^2(\mathbb{T}^d)$ unitarily equivalent to B. It is then clear that every symbol $\sigma \in C^\infty(\mathbb{T}^d \times \mathbb{Z}^d)$ gives rise to a bounded operator B on $\ell^2(\mathbb{Z}^d)$.

We take the partial difference operator Δ_{n_j} given by $(\Delta_{n_j}\phi)(\mathbf{m}) = \phi(\mathbf{m}+e_j) - \phi(\mathbf{m})$, e_j being the unit vector in the j^{th} direction in $\mathbb{Z}^d$. Denoting $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and the multi index $\alpha = (\alpha_1,\ldots,\alpha_d)$, we define the difference operator $\Delta_{\mathbf{n}}^\alpha = \Delta_{n_1}^{\alpha_1}\Delta_{n_2}^{\alpha_2}\cdots\Delta_{n_d}^{\alpha_d}$ for $\alpha_j \in \mathbb{N}_0, j = 1,\ldots,d$. Let $\langle\mathbf{n}\rangle = (1+|\mathbf{n}|^2)^{\frac{1}{2}}$. The class of rapidly decreasing sequences is given by

$$\mathcal{S}(\mathbb{Z}^d) = \{\phi(\mathbf{n}) : |\phi(\mathbf{n})| \leq C_{\phi,M}\langle \mathbf{n}\rangle^{-M},\ \forall\ M \in \mathbb{N}\}.$$

Let $m \in \mathbb{R}, 0 \leq \rho, \delta \leq 1$. The *toroidal symbol class* $S^m_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$ is defined as all $\sigma \in C^\infty(\mathbb{T}^d \times \mathbb{Z}^d)$ such that

$$|\Delta^\alpha_{\mathbf{n}} D^\beta_{\mathbf{x}} \sigma(\mathbf{x},\mathbf{n})| \leq C_{\sigma\alpha\beta m}\langle \mathbf{n}\rangle^{m-\rho|\alpha|+\delta|\beta|},\ \forall (\mathbf{x},\mathbf{n}) \in \mathbb{T}^d \times \mathbb{Z}^d \text{ and } \alpha, \beta \in \mathbb{N}^d_0,$$

where $D^\beta_{\mathbf{x}} = D^{\beta_1}_{x_1} D^{\beta_2}_{x_2} \cdots D^{\beta_d}_{x_d},\ D^{\beta_j}_{x_j} = \frac{\partial^{\beta_j}}{\partial x^{\beta_j}}$. The class $S^0_{1,0}$ is denoted simply by $S_{1,0}$. Let us define a subclass of symbols where all the derivatives in $\mathbf{x}$ also have uniform bounds.

$$S^m_{1,0,\infty}(\mathbb{T}^d \times \mathbb{Z}^d) = \{\sigma \in S^m_{1,0}(\mathbb{T}^d \times \mathbb{Z}^d) : C_{\sigma\alpha\beta m} \text{ are independent of } \beta\}.$$

For example the symbol $\cos(\mathbf{x}\cdot\mathbf{x} + \gamma_{\mathbf{n}})$ with $\gamma_{\mathbf{n}} \to 0$ as $|\mathbf{n}| \to \infty$ is in $S_{1,0,\infty}(\mathbb{T}^d \times \mathbb{Z}^d)$. We denote $S^0_{1,0,\infty}$ by $S_{1,0,\infty}$.

The Theorem 4.4 of [11] gives an expression for the symbol of the adjoint B^* of such a B, namely,

$$\sigma^*(\mathbf{x},\mathbf{n}) = \sum_{\alpha\geq 0} \frac{1}{\alpha!} \Delta^\alpha_{\mathbf{n}} \Delta^{(\alpha)}_{\mathbf{x}} \overline{\sigma(\mathbf{x},\mathbf{n})}, \tag{6}$$

and asserts that $\sigma_{B^*} \in S^m_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$ whenever $\sigma \in S^m_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$. Therefore, it is clear from Theorem 4.4 of [11] that given a symbol $\sigma \in S^m_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$, we can form the symbol σ^* also in $S^m_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$ and consider a self-adjoint operator associated with the symbol $\sigma + \sigma^*$. It is also clear from equation (6) that $\sigma^* \in S_{1,0,\infty}(\mathbb{T}^d \times \mathbb{Z}^d)$ whenever σ is. Therefore we are justified in assuming that there are symbols in this class giving rise to self-adjoint operators.

Now we are in a position to state our next theorem.

Theorem 1.2. *Let H and V be as in equation (2, 3). Consider a bounded self-adjoint operator B on $\ell^2(\mathbb{Z}^d)$ associated with a symbol $b \in S_{1,0,\infty}(\mathbb{T}^d \times \mathbb{Z}^d)$. Then for all $f \in C(K)$, we have*

$$\lim_{\lambda\to\infty} \frac{Tr\,(f(\pi_\lambda B \pi_\lambda))}{Tr(\pi_\lambda)} = \lim_{\lambda\to\infty} \frac{\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \sum_{V(\mathbf{n})\leq\lambda} f(b(\mathbf{x},\mathbf{n}))\ d\mathbf{x}}{\sum_{V(\mathbf{n})\leq\lambda} 1}. \tag{7}$$

2 The Proofs:

Let $\#S$ denote the cardinality of the set S. Consider H, V as in equations (2, 3). Since the operators $(H+\lambda)^{-1}$ and $(V+\lambda)^{-1}$ are compact operators for $\lambda > 0$, choose a suitable $m \in \mathbb{N}$ such that $(V+\lambda)^{-m}$ and $(H+\lambda)^{-m}$ are both trace class operators on $\ell^2(\mathbb{Z}^d)$.

lem1 **Lemma 2.1.** *Consider the self-adjoint operators V and H as given in equations (2,3). Then for a suitable $m \in \mathbb{N}$ for which $(V+\lambda)^{-m}$ and $(H+\lambda)^{-m}$ are trace class operators on $\ell^2(\mathbb{Z}^d)$ we have,*

$$\left| \frac{Tr((H+\lambda)^{-m})}{Tr((V+\lambda)^{-m})} - 1 \right| \to 0$$

as $\lambda \to \infty$.

Proof: Since the operators $-\Delta$ and $(V+\lambda)^{-1}$ are bounded and positive we write

$$(H+\lambda) = (V+\lambda)^{\frac{1}{2}}((V+\lambda)^{-\frac{1}{2}}(-\Delta)(V+\lambda)^{-\frac{1}{2}} + 1)(V+\lambda)^{\frac{1}{2}}.$$

Therefore

$$(H+\lambda)^{-m} = (V+\lambda)^{-m} + (V+\lambda)^{-\frac{m}{2}} \left((1+K_\lambda)^{-m} - 1\right) (V+\lambda)^{-\frac{m}{2}}, \quad (8)$$ expand

where $K_\lambda = (V+\lambda)^{-\frac{1}{2}}(-\Delta)(V+\lambda)^{-\frac{1}{2}}$. Henceforth we denote by I, the identity operator. Since K_λ is a positive operator the expression in (8) makes sense and we also have $\|(I+K_\lambda)^{-1}\| \leq 1$ for any $\lambda > 0$. Taking trace on both sides of the above equation and using the inequality $|Tr(XCX)| \leq \|C\| Tr(X^2)$, when X is a positive trace class operator and C is a bounded operator on $\ell^2(\mathbb{Z}^d)$, we get

$$\begin{aligned}
& |\, Tr((H+\lambda)^{-m}) - Tr((V+\lambda)^{-m}) \,| \\
= \ & \left| Tr \left((V+\lambda)^{-\frac{m}{2}} \left((I+K_\lambda)^{-m} - 1\right) (V+\lambda)^{-\frac{m}{2}} \right) \right| \\
\leq \ & Tr((V+\lambda)^{-m}) \| \left((1+K_\lambda)^{-m} - 1\right) \| \\
\leq \ & m \|K_\lambda\| Tr((V+\lambda)^{-m}) \\
\leq \ & m \|\Delta\| \|(V+\lambda)^{-1}\| Tr((V+\lambda)^{-m}).
\end{aligned}$$

Therefore,

$$\left| \frac{Tr((H+\lambda)^{-m})}{Tr((V+\lambda)^{-m})} - 1 \right| \leq 4dm \|(V+\lambda)^{-1}\| \to 0 \ \text{ as } \ \lambda \to \infty.$$

$\square$

Denote the distribution functions of the measures $Tr(E_H(\cdot))$ and $Tr(E_V(\cdot))$ respectively by ϕ_H and ϕ_V. Then we have

$$\phi_H(\lambda) = Tr(\pi_\lambda), \quad \phi_V(\lambda) = \#\{\mathbf{n} : V(\mathbf{n}) \in (0, \lambda]\}.$$

Then Lemma 2.1 immediately gives us the Weyl formula for the functions $Tr(\pi_\lambda)$ as the following corollary. In the following $[r]$ denotes the largest integer smaller than or equal to r.

Corollary 2.2. *Consider V and H self-adjoint operators as given in equations (2,3). We have the following asymptotic :*

1. *ϕ_V is multiplicatively continuous.*
2. $Tr(\pi_\lambda) = \#\{\mathbf{n} : V(\mathbf{n}) \in (0, \lambda]\}$ as $\lambda \to \infty$.
3. $Tr(\pi_\lambda) = 2^d[\lambda]^{\frac{d}{\kappa}} + o(\lambda^{\frac{d}{\kappa}})$ as $\lambda \to \infty$.
4. $\sup_{\mu \leq \lambda} \, (Tr(\pi_{\mu+r}) - Tr(\pi_\mu)) \leq Tr(\pi_\lambda) \left(\frac{d}{\kappa} \frac{r-1}{\lambda - 1} + O\left(\frac{1}{[\lambda - 1]}\right)\right)$, *as* $\lambda \to \infty$.

Proof: (1) The function ϕ_V is given by

$$\begin{aligned} \phi_V(\lambda) &= \#\{\mathbf{n} : V(\mathbf{n}) \leq \lambda\} = \#\{\mathbf{n} : |\mathbf{n}|^\kappa \leq \lambda\} \\ &= \#\{\mathbf{n} : |\mathbf{n}| \leq [\lambda]^{1/\kappa}\} = (2[\lambda]^{\frac{1}{\kappa}} + 1)^d. \end{aligned} \tag{9}$$

Clearly $\lim_{\lambda \to \infty} \lim_{\tau \to 1} \phi_V(\tau\lambda)/\phi_V(\lambda) = 1$. On the other hand, using the notation (r) for the fractional part of r, we see from equation (9) that

$$\begin{aligned} \frac{\phi_V(\tau\lambda)}{\phi_V(\lambda)} &= \frac{(2[\tau\lambda]^{\frac{1}{\kappa}} + 1)^d}{(2[\lambda]^{\frac{1}{\kappa}} + 1)^d} = \frac{(2(\tau\lambda - (\tau\lambda))^{\frac{1}{\kappa}} + 1)^d}{(2(\lambda - (\lambda))^{\frac{1}{\kappa}} + 1)^d} \\ &= \tau^{d/\kappa} \frac{\left(2\left(1 - \frac{(\tau\lambda)}{\tau\lambda}\right)^{1/\kappa} + \frac{1}{|\tau\lambda|^{1/\kappa}}\right)^d}{\left(2\left(1 - \frac{(\lambda)}{\lambda}\right)^{1/\kappa} + \frac{1}{\lambda^{1/\kappa}}\right)^d}. \end{aligned} \tag{10}$$

Taking the limit over λ first and then over τ we see that

$$\lim_{\tau \to 1} \lim_{\lambda \to \infty} \frac{\phi_V(\tau\lambda)}{\phi_V(\lambda)} = 1.$$

Lemma 2.1 implies that

$$\int_0^\infty \frac{\lambda^m}{(\lambda+u)^m}\, d\phi_H(u) \Big/ \int_0^\infty \frac{\lambda^m}{(\lambda+u)^m}\, d\phi_V(u) \to 1, \text{ as } \lambda \to \infty.$$

Applying Theorem 3.4 of Grishin-Poedintseva we get

$$\phi_H(\lambda)/\phi_V(\lambda) \to 1, \text{ as } \lambda \to \infty. \tag{11}$$ eqn-ratio

This proves (2). Part (3) follows directly from (2) and equation (9). Using the asymptotic (3), bounding the terms in the ratio $\frac{Tr(\pi_{\mu+r})}{Tr(\pi_\lambda)}$ we get

$$\begin{aligned} 2^d \sup_{\mu \le \lambda}([\mu]^{\frac{d}{\kappa}} - [\mu - r]^{\frac{d}{\kappa}}) &\le 2^d[\lambda]^{\frac{d}{\kappa}} \left(1 - \left(1 - \frac{r-1}{\lambda-1}\right)^{\frac{d}{\kappa}}\right) \\ &\approx 2^d[\lambda]^{\frac{d}{\kappa}} \left(\frac{d}{\kappa}\frac{r-1}{\lambda-1} + O\left(\frac{1}{[\lambda-1]}\right)\right) \end{aligned}$$

the estimate in (4). □

This corollary implies that ϕ_H is also a multiplicatively continuous function from the following Lemma.

lem3 **Lemma 2.3.** *The function ϕ_H considered above is multiplicatively continuous at infinity.*

Proof: We will show that if φ, χ are two distribution functions satisfying

$$\lim_{r\to\infty} \frac{\varphi(r)}{\chi(r)} = 1,$$

then φ is multiplicatively continuous whenever χ is. Clearly

$$\lim_{r\to\infty} \lim_{\tau\to 1} \frac{\varphi(\tau r)}{\varphi(r)} = \lim_{r\to\infty} \frac{\varphi(r)}{\varphi(r)} = 1.$$

Now consider

$$\begin{aligned} \lim_{\tau\to 1} \lim_{r\to\infty} \frac{\varphi(\tau r))}{\varphi(r)} &= \lim_{\tau\to 1} \lim_{r\to\infty} \frac{\frac{\varphi(\tau r)}{\chi(\tau r)}}{\frac{\varphi(r)}{\chi(r)}} \\ &= \lim_{\tau\to 1} 1 = 1, \end{aligned} \tag{12}$$

where in the last step we used the assumption on ϕ/χ and the fact that χ is multiplicatively continuous. Since ϕ_V is multiplicatively continuous, the above result together with equation (11) now shows that ϕ_H is multiplicatively continuous. □

lem2 **Lemma 2.4.** *Suppose B is a bounded positive operator on $\ell^2(\mathbb{Z}^d)$, then, for $m \in \mathbb{N}$ for which $(V+\lambda)^{-m}$ (and hence $(H+\lambda)^{-m}$) is trace class, we have,*

$$\left| \frac{Tr(B(H+\lambda)^{-m})}{Tr(B(V+\lambda)^{-m})} - 1 \right| \to 0$$

as $\lambda \to \infty$.

Proof: The proof is similar to that in the above lemma, except that we have to do a bit more of algebra in handling the error term, namely, using equation (8) we write

$$\begin{aligned} Tr(B(H+\lambda)^{-m}) &= Tr(B(V+\lambda)^{-m}) \\ &\quad + Tr\left(B(V+\lambda)^{-\frac{m}{2}}\left((1+K_\lambda)^{-m} - 1\right)\right)(V+\lambda)^{-\frac{m}{2}}\right). \end{aligned} \tag{13}$$

we set $W_\lambda = (V+\lambda)^{-\frac{m}{2}} B (V+\lambda)^{-\frac{m}{2}}$ which is a positive trace class, so we rewrite the error term as

$$\begin{aligned} & \left|Tr\left(B(V+\lambda)^{-\frac{m}{2}}\left((1+K_\lambda)^{-m} - 1\right)\right)(V+\lambda)^{-\frac{m}{2}}\right)\right| \\ \leq\ & \left|Tr\left(W_\lambda(1+K_\lambda)^{-m} - 1)\right)\right| \\ =\ & \left|Tr\left(W_\lambda^{\frac{1}{2}}\left((1+K_\lambda)^{-m} - 1)W_\lambda^{\frac{1}{2}}\right)\right)\right| \\ \leq\ & m\|\Delta\|\|(V+\lambda)^{-1}\| Tr(B(V+\lambda)^{-m}). \end{aligned}$$

The rest of the proof is as in the Lemma 2.1 using the above estimate. □

prop1 **Proposition 2.5.** *Consider V, H as in equations (2, 3). Then for any bounded positive operator B and $m \in \mathbb{N}$ be such that $(V+\lambda)^{-m}$ is trace class. Then we have*

(i) The following equality is valid in the sense that if one of the limits exists then the other also does and the limits are the same.

$$\lim_{\lambda\to\infty} \frac{Tr(B(H+\lambda)^{-m})}{Tr((H+\lambda)^{-m})} = \lim_{\lambda\to\infty} \frac{Tr(B(V+\lambda)^{-m})}{Tr((V+\lambda)^{-m})}.$$

(ii) If in addition B comes from an operator of multiplication by a function b on $L^2(\mathbb{T}^d)$, then the limits in (i) exist and equal

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} b(\mathbf{x})d\mathbf{x}.$$

Proof: (i) For each $\lambda > 0$ we have the equality

$$\frac{\left(\frac{Tr(B(H+\lambda)^{-m})}{Tr(B(V+\lambda)^{-m})}\right)}{\left(\frac{Tr((H+\lambda)^{-m})}{Tr((V+\lambda)^{-m})}\right)} = \frac{\left(\frac{Tr(B(H+\lambda)^{-m})}{Tr((H+\lambda)^{-m})}\right)}{\left(\frac{Tr(B(V+\lambda)^{-m})}{Tr((V+\lambda)^{-m})}\right)}. \qquad (14)$$

Since the left hand side has limit 1 by Lemma 2.4 and Lemma 2.1, the right hand side limit in (14) exists and equals to 1. Therefore if either the numerator or the denominator in the fraction in the right hand side has a limit in (14), then the other also has a limit and they both agree which implies the proposition.

(ii) Let B be an operator of multiplication by a function b on $L^2(\mathbb{T}^d)$. For $\mathbf{n}, \mathbf{k} \in \mathbb{Z}^d$ define $\delta_{\mathbf{n}}(\mathbf{k}) = 1$ if $\mathbf{n} = \mathbf{k}$ and 0 otherwise. The sequence $\{\delta_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ forms an orthonormal basis for $\ell^2(\mathbb{Z}^d)$. So

$$\begin{aligned} Tr(B(V+\lambda)^{-m}) &= \sum_{n\in\mathbb{Z}^d} \langle \delta_{\mathbf{n}}, B\delta_{\mathbf{n}}\rangle (V(\mathbf{n})+\lambda)^{-m} \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} b(\mathbf{x})d\mathbf{x} \sum_{\mathbf{n}\in\mathbb{Z}^d} (V(\mathbf{n})+\lambda)^{-m} \qquad (15) \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} b(\mathbf{x})d\mathbf{x} \ \cdot \ Tr((V+\lambda)^{-m}). \end{aligned}$$

Therefore we have for each $\lambda > 0$,

$$\frac{Tr(B(V+\lambda)^{-m})}{Tr((V+\lambda)^{-m})} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} b(\mathbf{x})d\mathbf{x}.$$

Thus the limit of the left hand side exists as λ goes to infinity. $\square$

The proof of Theorem 1.1 turns out to be simple using the above Proposition but for proving Theorem 1.2 we use the Tauberian theorem of Grishin-Poedintseva (see Appendix, Theorem 3.4). We start with the necessary preliminaries.

Lemma 2.6. *(a) Let B, V be the operators as in Theorem 1.2 with $\kappa \in (0,1)$ given after equation 3. Then the operator $[H, B]$ is bounded on $\ell^2(\mathbb{Z}^d)$.*

(b) Under the assumption of Theorem 1.2, we have

$$\left| \frac{Tr\left(f(\pi_\lambda B \pi_\lambda)\right)}{Tr(\pi_\lambda)} - \frac{Tr\left(\pi_\lambda f(B)\pi_\lambda\right)}{Tr(\pi_\lambda)} \right| \to 0 \quad as \quad \lambda \to \infty. \qquad (16)$$

Proof: It suffices to show that $[V,B]$ is bounded, since $[H,B]=[-\Delta,B]+[V,B]$ and $[-\Delta,B]$ is bounded since both $-\Delta$ and B are bounded.

Let $N>d+1$. Observe that $e^{i(\mathbf{m}-\mathbf{n})\cdot\mathbf{x}}=\frac{(1-L_{\mathbf{x}})^N}{(1+|\mathbf{m}-\mathbf{n}|^2)^N}e^{i(\mathbf{m}-\mathbf{n})\cdot\mathbf{x}}$, where $L_{\mathbf{x}}$ denotes the Laplacian on $L^2(\mathbb{T}^d)$. Using the definition U^* and the action of B in terms of its symbol given in equation (5) and $[V,B]=VB-BV$, we have for $u\in\ell^2(\mathbb{Z}^d)$,

$$(U^*[V,B]Uu)(\mathbf{n})$$
$$=\sum_{m\in\mathbb{Z}}\frac{(|\mathbf{n}|^\kappa-|\mathbf{m}|^\kappa|)}{(1+|\mathbf{m}-\mathbf{n}|^2)^N}\frac{(-1)^N}{(2\pi)^d}\int_{\mathbb{T}^d}e^{i(\mathbf{m}-\mathbf{n})\cdot\mathbf{x}}(1-L_{\mathbf{x}})^N b(\mathbf{m},\mathbf{x})\ u(\mathbf{m})\,d\mathbf{x},\quad |\mathbf{n}|\neq 0$$

where we transferred the Laplacian term to act on b by N integrations by parts. We use the inequality, which is trivial when $\mathbf{m}$ or $\mathbf{n}$ is zero or $\mathbf{m}=\mathbf{n}$. So taking w.l.g $|\mathbf{n}|>|\mathbf{m}|$,

$$||\mathbf{n}|^\kappa-|\mathbf{m}|^\kappa|\leq|\mathbf{m}|^\kappa(\frac{|\mathbf{n}|}{|\mathbf{m}|}-1)^\kappa\leq|\mathbf{m}|(\frac{|\mathbf{n}|}{|\mathbf{m}|}-1)\leq|\mathbf{n}-\mathbf{m}|$$

in the second line below.

$$\begin{aligned}
&\|[V,B]Uu\|_2\\
=&\ \left(\sum_{\mathbf{n}\in\mathbb{Z}^d}\left|\sum_{\mathbf{m}\in\mathbb{Z}^d}u(\mathbf{m})\frac{|\mathbf{n}|^\kappa-|\mathbf{m}|^\kappa}{(1+|\mathbf{m}-\mathbf{n}|^2)^N}\frac{(-1)^N}{(2\pi)^d}\int_{\mathbb{T}^d}(1-L_{\mathbf{x}})^N b(\mathbf{m},\mathbf{x})e^{i(\mathbf{m}-\mathbf{n})\cdot\mathbf{x}}\right|^2\right)^{\frac{1}{2}}\\
\leq&\ C_N\left(\sum_{\mathbf{n}\in\mathbb{Z}^d}\left|\sum_{\mathbf{m}\in\mathbb{Z}^d}u(\mathbf{m})\frac{(|\mathbf{n}-\mathbf{m}|)}{(1+|\mathbf{m}-\mathbf{n}|^2)^N}\right|^2\right)^{\frac{1}{2}}, \qquad (17)\\
\leq&\ C_N\left(\sum_{\mathbf{n}\in\mathbb{Z}^d}\left|\sum_{\mathbf{m}\in\mathbb{Z}^d}\frac{1}{(1+|\mathbf{m}-\mathbf{n}|^2)^{N-1}}u(\mathbf{m})\right|^2\right)^{\frac{1}{2}}=\|K*u\|_2,
\end{aligned}$$

where $C_N:=\sup_{\mathbf{m}\in\mathbb{Z}^d}\sup_{(\mathbf{n},\mathbf{x})\in\mathbb{Z}^d\times\mathbb{T}^d}\left|\frac{(-1)^N}{(2\pi)^d}\int_{\mathbb{T}^d}e^{i(\mathbf{m}-\mathbf{n})\cdot\mathbf{x}}(1-L_{\mathbf{x}})^N b(\mathbf{m},\mathbf{x})\,d\mathbf{x}\right|$ and K is the function

$$K(\mathbf{m})=\frac{1}{(1+|\mathbf{m}|^2)^{N-1}}$$

For $N-1 > d/2$, the Kernel K is in $\ell^1(\mathbb{Z}^d)$, so we get by and application of Minkowski's inequality gives the bound

$$\|[V,B]Uu\|_2 \le C\|K\|_1\|u\|_2.$$

This proves part (a).

Since B is bounded self-adjoint, its spectrum is real and compact, so functions $f(B)$ can be approximated in the sup norm by smooth functions so, w.l.g we assume that $f \in C^2(\sigma(B))$. Then by Theorem 1.6 of Laptev-Safarov [4], we get setting $A = H, B = B, \chi = 0, \psi = f, \pi_\lambda = P_\lambda$ in their Theorem,

$$\begin{aligned} &|Tr\,(\pi_\lambda f(B)\pi_\lambda) - \pi_\lambda f(\pi_\lambda B\pi_\lambda)\pi_\lambda)\,| \\ &\le \tfrac{1}{2}\|f^{(2)}\|_\infty N_r(\lambda)\left(\|(\pi_\lambda - \pi_{\lambda-r})B\|^2 + \tfrac{\pi^2}{6r^2}\|\pi_{\lambda-r}[A,B]\|^2\right)\end{aligned}$$

Dividing both the sides by $Tr(\pi_\lambda)$ and setting $r = \lambda^\alpha,\ \alpha \in (0,1)$ and the using the boundedness of $B, [H,B]$, we see that

$$|Tr\,(\pi_\lambda f(B)\pi_\lambda - \pi_\lambda f(\pi_\lambda B\pi_\lambda)\pi_\lambda) \le C\frac{N_{\lambda^\alpha}(\lambda)}{Tr(\pi_\lambda)},$$

where $N_r(\lambda) = sup_{\mu\le\lambda}(Tr(\pi_\mu - \pi_{\mu-r}))$. The part (b) now follows from part (4) of Corollary (2.2) and (??) as $\lambda \to \infty$. □

rem **Lemma 2.7.** *If M_b is the operator of multiplication defined as in Theorem 1.1, then $Tr\ f(\pi_\lambda B\pi_\lambda) = Tr\ \pi_\lambda f(B)\pi_\lambda$.*

Proof. Note that $\|(I-\pi_\lambda)B\pi_\lambda\|_{HS}^2 = Tr\ (\pi_\lambda B(I-\pi_\lambda)B\pi_\lambda) = Tr\ (\pi_\lambda B^2\pi_\lambda) - Tr\ (\pi_\lambda B\pi_\lambda)^2$. The operators $(\pi_\lambda B\pi_\lambda)^2$ and $\pi_\lambda B^2\pi_\lambda$ are operators on $\ell^2(\mathbb{Z}^d)$ with kernels $K_1(\mathbf{m},\mathbf{n}) = \sum_{\mathbf{r}\in\mathbb{Z}}\left(\int_{\mathbb{T}^d} b(\mathbf{x})e^{i(\mathbf{m}-\mathbf{r})\cdot\mathbf{x}}d\mathbf{x}\right)\left(\int_{\mathbb{T}^d} b(\mathbf{y})e^{i(\mathbf{r}-\mathbf{n})\cdot\mathbf{y}}d\mathbf{y}\right)$ and $K_2(\mathbf{m},\mathbf{n}) = \int_{\mathbb{T}^d} b^2(\mathbf{x})e^{i(\mathbf{m}-\mathbf{n})\cdot\mathbf{x}}d\mathbf{x}$ respectively. Therefore $Tr\ (\pi_\lambda B^2\pi_\lambda) = \displaystyle\sum_{|\mathbf{n}|\le M(\lambda)} K_1(\mathbf{n},\mathbf{n}) = \sum_{|\mathbf{n}|\le M(\lambda)}\int_{\mathbb{T}^d} b^2(\mathbf{x})d\mathbf{x} = \sum_{|\mathbf{n}|\le M(\lambda)} K_2(\mathbf{n},\mathbf{n}) = Tr\ (\pi_\lambda B\pi_\lambda)^2$, since π_λ is given by

$$\pi_\lambda\phi(\mathbf{n}) = \begin{cases} 0, & |\mathbf{n}| > M(\lambda) \\ \phi(\mathbf{n}), & |\mathbf{n}| \le M(\lambda),\end{cases}$$

for $\phi \in \ell^2(\mathbb{Z}^d)$. □

Proof of Theorem 1.1:

Let $\pi_{V,\lambda}$ denote the spectral projection $E_V((0,\lambda])$, then

$$Tr(\pi_{V,\lambda} f(B) \pi_{V,\lambda}) = \sum_{V(\mathbf{n})\le\lambda} \langle \delta_{\mathbf{n}}, f(B)\delta_{\mathbf{n}}\rangle, \text{ and } Tr(\pi_{V,\lambda}) = \sum_{V(\mathbf{n})\le\lambda} 1. \tag{18}$$

Under the Fourier series the basis vectors $|\delta_{\mathbf{n}}\rangle$ go over to the basis $e^{i\mathbf{n}\cdot\mathbf{x}}$ in $L^2(\mathbb{T}^d)$ and B is an operator of multiplication by a bounded positive function $b(\mathbf{x})$ there, we have $\langle \delta_{\mathbf{n}}, f(B)\delta_{\mathbf{n}}\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(b(\mathbf{x}))d\mathbf{x}$. Further, using (18) for each λ we have

$$\frac{Tr(\pi_{V,\lambda} f(B)\pi_{V,\lambda})}{Tr(\pi_{V,\lambda})} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(b(\mathbf{x}))d\mathbf{x}.$$

Without loss of generality add a suitable constant to make the function f positive. Then $f(B)$ is a positive operator and $f(b(x))$ is a positive function on $\mathbb{T}^d$.

We set

$$\phi_{H,f}(\lambda) = Tr(\pi_\lambda f(B) \pi_\lambda) = Tr(f(B)^{\frac{1}{2}}\pi_\lambda f(B)^{\frac{1}{2}}) \tag{19}$$

and

$$\phi_{V,f}(\lambda) = Tr(\pi_{V,\lambda} f(B) \pi_{V,\lambda}) = Tr(f(B)^{\frac{1}{2}}\pi_{V,\lambda} f(B)^{\frac{1}{2}}). \tag{20}$$

We apply the spectral theorem for H and V and write the traces in the above two equations as integrals, perform an integration by parts and obtain the equality below, where we use Proposition 2.5 for the middle equality and Theorem 3.4 for the extreme left and right equalities.

$$\begin{aligned}\lim_{r\to\infty}\frac{\displaystyle\int_0^\infty \frac{\phi_{H,f}(u)}{(1+\frac{u}{r})^{m+1}}\,du}{\displaystyle\int_0^\infty \frac{\phi_{H}(u)}{(1+\frac{u}{r})^{m+1}}\,du} &= \lim_{\lambda\to\infty}\frac{\phi_{H,f}(\lambda)}{\phi_H(\lambda)} = \lim_{\lambda\to\infty}\frac{\phi_{V,f}(\lambda)}{\phi_V(\lambda)} \\ &= \lim_{r\to\infty}\frac{\displaystyle\int_0^\infty \frac{\phi_{V,f}(u)}{(1+\frac{u}{r})^{m+1}}\,du}{\displaystyle\int_0^\infty \frac{\phi_{V}(u)}{(1+\frac{u}{r})^{m+1}}\,du}. \end{aligned} \tag{21}$$

From the above equation we see that $\left|\frac{Tr(\pi_\lambda f(B)\pi_\lambda)}{Tr(\pi_\lambda)} - \frac{Tr(\pi_{V,\lambda} f(B)\pi_{V,\lambda})}{Tr(\pi_{V,\lambda})}\right| \to 0$ as $\lambda\to\infty$. This fact together with Lemma 2.7 completes the proof of Theorem 1.1. □

We need the following lemma before proceeding to prove Theorem 1.2.

lem4 **Lemma 2.8.** *Consider a symbol $a(x,n) \in S_{1,0,\infty}(\mathbb{T}^d \times \mathbb{Z}^d)$ and let A be the pseudo-difference operator on $L^2(\mathbb{T}^d)$ associated with it. Then for any $\ell \in \mathbb{N}$, the symbol $a_\ell(\mathbf{x},\mathbf{n})$ of the operator A^ℓ has the asymptotic behavior*

$$a_\ell(\mathbf{x},\mathbf{n}) \approx (a(\mathbf{x},\mathbf{n}))^\ell + E_\ell(\mathbf{x},\mathbf{n}), \tag{22}$$ compo

with $E_\ell(\mathbf{x},\mathbf{n}) \in S^{-1}_{1,0,\infty}$.

From Theorem 4.3 [11], we see that if $a,b \in S^0_{1,0}(\mathbb{T}^d \times \mathbb{Z}^d)$ then their composition $Op(a) \circ Op(b)$ is a pseudo-difference operator on $L^2(\mathbb{T}^d)$ with a symbol $\sigma(x,n) \in S^0_{1,0}(\mathbb{T}^d \times \mathbb{Z}^d)$, which has an expression as an asymptotic sum, namely,

$$\sigma(\mathbf{x},\mathbf{n}) \approx \sum_{\alpha \geq 0} \frac{1}{\alpha!} (\Delta^\alpha_{\mathbf{n}} a(\mathbf{x},\mathbf{n})) D^{(\alpha)}_{\mathbf{x}} b(\mathbf{x},\mathbf{n}), \tag{23}$$ compose

where $Op(a)$ is the operator with symbol a as defined in (5). Using (23) and $\ell \in \mathbb{N}$, we prove that the symbol of $\ell-$fold composition of $Op(a)$ with itself is asymptotically equal to $Op(a^\ell)$.

Proof of Lemma 2.8:

We will prove this by induction on ℓ. For $\ell = 1$, $E_1(\mathbf{x},\mathbf{n}) = 0$ and (22) is trivially true. We assume that the Lemma is valid for $a_{\ell-1}(\mathbf{x},\mathbf{n})$, so we have

$$a_{\ell-1}(\mathbf{x},\mathbf{n}) \approx (a(\mathbf{x},\mathbf{n}))^{\ell-1} + E_{\ell-1}(\mathbf{x},\mathbf{n}), \text{ with } E_{\ell-1} \in S^{-1}_{1,0,\infty}.$$

We use the composition rule in equation (23) to get

$$\begin{aligned} a_\ell(\mathbf{x},\mathbf{n}) &\approx \sum_{\alpha \geq 0} \frac{1}{\alpha!} (\Delta^\alpha_{\mathbf{n}} a(\mathbf{x},\mathbf{n})) D^{(\alpha)}_{\mathbf{x}} a_{k-1}(\mathbf{x},\mathbf{n}) \\ &\approx \sum_{\alpha \geq 0} \frac{1}{\alpha!} (\Delta^\alpha_{\mathbf{n}} a(\mathbf{x},\mathbf{n})) D^{(\alpha)}_{\mathbf{x}} (a(\mathbf{x},\mathbf{n}))^{\ell-1} + \sum_{\alpha \geq 0} \frac{1}{\alpha!} (\Delta^\alpha_{n} a(x,n)) D^{(\alpha)}_{x} E_{\ell-1}(\mathbf{x},\mathbf{n}) \\ &\approx (a(\mathbf{x},\mathbf{n}))^\ell + T_1(\mathbf{x},\mathbf{n}) + T_2(\mathbf{x},\mathbf{n}), \end{aligned} \tag{24}$$ comp2

where

$$\begin{aligned} T_1(\mathbf{x},\mathbf{n}) &= \sum_{|\alpha| \geq 1} \frac{1}{\alpha!} (\Delta^\alpha_{\mathbf{n}} a(\mathbf{x},\mathbf{n})) D^{(\alpha)}_{\mathbf{x}} (a(\mathbf{x},\mathbf{n}))^{\ell-1}, \\ T_2(x,n) &= \sum_{\alpha \geq 0} \frac{1}{\alpha!} (\Delta^\alpha_{\mathbf{n}} a(\mathbf{x},\mathbf{n})) D^{(\alpha)}_{\mathbf{x}} E_{\ell-1}(\mathbf{x},\mathbf{n}). \end{aligned} \tag{25}$$

We set $E_\ell(\mathbf{x},\mathbf{n}) = T_1(\mathbf{x},\mathbf{n}) + T_2(\mathbf{x},\mathbf{n})$ and recall the relation

$$\Delta_{\mathbf{n}}^{\alpha}\sigma(\mathbf{n}) = \sum_{\beta\le\alpha}(-1)^{|\alpha-\beta|}\binom{\alpha}{\beta}\sigma(\mathbf{n}+\beta).$$

from Proposition 3.1 in [11]. Using this and the facts that $a \in S_{1,0,\infty}$, $E_{\ell-1} \in S^{-1}_{1,0,\infty}$ we estimate

$$|\Delta_{\mathbf{n}}^{\alpha} a(\mathbf{x},\mathbf{n})| \le C2^{|\alpha|}, |D_{\mathbf{x}}^{\alpha}E_{\ell-1}(\mathbf{x},\mathbf{n})| \le C\langle\mathbf{n}\rangle^{-1},$$

so that each of the terms in the sum defining $T_2(\mathbf{x},\mathbf{n}) \in S^{-1}_{1,0,\infty}$ and

$$|T_2(\mathbf{x},\mathbf{n})| \le C\langle\mathbf{n}\rangle^{-1}.$$

so that $T_2(\mathbf{x},\mathbf{n}) \in S^{-1}_{1,0,\infty}$. To estimate $T_1(\mathbf{x},\mathbf{n})$ we define multi-indices $\alpha^{(j)}$ to be $\alpha^{(j)}_r = \delta_{rj}, r = 1,\ldots,d$, and split T_1 as

$$T_1(\mathbf{x},\mathbf{n}) = \sum_{j=1}^{d}\sum_{\alpha\ge1,\alpha_j\ge1}\frac{1}{\alpha!}(\Delta_{\mathbf{n}}^{\alpha-\alpha^{(j)}}\Delta_{\mathbf{n}}^{\alpha^{(j)}}a(\mathbf{x},\mathbf{n}))D_{\mathbf{x}}^{(\alpha)}(a(\mathbf{x},\mathbf{n}))^{\ell-1}.$$

Then clearly $\Delta_{\mathbf{n}}^{\alpha^{(j)}}a(\mathbf{x},\mathbf{n}) \in S^{-1}_{1,0,\infty}$. If $a, b \in S^0_{1,0}$,then $ab \in S^0_{1,0}$, however the same is not true for $S^0_{1,0,\infty}$ in view of Leibniz rule for derivatives. Therefore using the property of $a \in S_{1,0,\infty}$, we estimate

$$|(\Delta_{\mathbf{n}}^{\alpha-\alpha^{(j)}}\Delta_{\mathbf{n}}^{\alpha^{(j)}}a(\mathbf{x},\mathbf{n}))| \le C2^{|\alpha|}\langle\mathbf{n}\rangle^{-1},\ \frac{1}{\alpha!}|D_{\mathbf{x}}^{(\alpha)}(a(\mathbf{x},\mathbf{n}))^{\ell-1}| \le C^{\ell-1}\prod_{j=1}^{d}\theta_j,$$

where

$$\theta_j = \begin{cases} 1, & \alpha_j \le \ell-1, \\ \frac{1}{(\alpha_j-\ell+1)!}, & \alpha_j-\ell+1>0 \end{cases}\quad,\quad j = 1,\ldots,d,$$

Then arguing as done for the term $T_2(\mathbf{x},\mathbf{n})$ we see that $T_1(\mathbf{x},\mathbf{n}) \in S^{-1}_{1,0,\infty}$ and

$$|T_1(\mathbf{x},\mathbf{n})| \le C\langle\mathbf{n}\rangle^{-1},$$

This show that $E_\ell(x,n) \in S^{-1}_{1,0,\infty}$, proving the Lemma. □

Proof of Theorem 1.2:

If $\sigma_{B^k}(x,n)$ is the symbol associated with B^k then,

$$\langle \delta_n, B^\ell \delta_{\mathbf{n}} \rangle = \langle U\delta_n, (UBU^*)^\ell U\delta_{\mathbf{n}} \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \sigma_{B^\ell}(\mathbf{x}, \mathbf{n})\, d\mathbf{x}.$$

Using Lemma 2.8 we see that

$$\langle \delta_{\mathbf{n}}, B^\ell \delta_{\mathbf{n}} \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} b(\mathbf{x}, \mathbf{n})^\ell \, d\mathbf{x} + \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} E_\ell(\mathbf{x}, \mathbf{n})\, d\mathbf{x}$$

with $\sup_{\mathbf{x}} |E_\ell(\mathbf{x}, \mathbf{n})| \to 0$ as $|\mathbf{n}| \to \infty$.

We compute the limits using an $\frac{\epsilon}{3}$ argument and the fact that $r_{\mathbf{n}} \to 0, \ \ |\mathbf{n}| \to \infty$ implies

$$\lim_{\lambda \to \infty} \ \frac{\left(\sum_{V(\mathbf{n}) \leq \lambda} r_{\mathbf{n}} \right)}{\sum_{V(\mathbf{n}) \leq \lambda} 1} = 0.$$

An application of Lebesgue dominated convergence theorem and the properties of $E_\ell(\mathbf{x}, \mathbf{n})$ gives

$$\lim_{\lambda \to \infty} \frac{\sum_{\mathbf{n}: V(\mathbf{n}) \leq \lambda} \langle \delta_{\mathbf{n}}, B^\ell \delta_{\mathbf{n}} \rangle}{\sum_{V(\mathbf{n}) \leq \lambda} 1} = \lim_{\lambda \to \infty} \left(\frac{\sum_{\mathbf{n}: V(\mathbf{n}) \leq \lambda} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (b(\mathbf{x}, \mathbf{n}))^\ell \, d\mathbf{x}}{\sum_{V(\mathbf{n}) \leq \lambda} 1} + E(\lambda) \right),$$

where $E(\lambda) = \dfrac{\sum_{\mathbf{n}: V(\mathbf{n}) \leq \lambda} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} E_\ell(\mathbf{x}, \mathbf{n})\, d\mathbf{x}}{\sum_{V(\mathbf{n}) \leq \lambda} 1} \to 0$ as $\lambda \to \infty$. □

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3 Appendix

In this appendix we collect two theorems we use in our paper for the reader's convenience.

The first one is a Tauberian theorem of Grishin-Poedintseva from [7].

Definition 3.1. *Let ϕ be a positive function on the half line $[0, \infty)$. Let*

$$S = \{\alpha : \exists M, R \text{ with } \phi(tr) \leq M t^\alpha, \text{ for all } t \geq 1, r \geq R\}$$

and

$$G = \{\alpha : \exists M, R \text{ with } \phi(tr) \geq M t^{\alpha}, \text{ for all } t \geq 1, r \geq R\}$$

Then $\alpha(\phi) = \inf S$ *and* $\beta(\phi) = \sup G$ *are called the* **upper and lower Matushevskaya index** *of* ϕ *respectively.*

Theorem 3.2. *([7], Theorem 2)*

Let $m > -1$. *Assume that* φ *is positive measurable function on* $[0, \infty)$ *that does not vanish identically in any neighbourhood of infinity. Let* $\Phi(r) = \int_0^{\infty} \frac{\varphi(rt)}{(1+t)^{m+1}} dt$ *be finite. Then the functions* φ *and* Φ *have same growth at infinity if and only if* $\beta(\varphi) > -1$ *and* $\alpha(\varphi) < m$.

Definition 3.3. *A function* φ *is said to be multiplicatively continuous at infinity if it satisfies* $\lim_{\substack{r\to\infty \\ \tau\to 1}} \frac{\varphi(\tau r)}{\varphi(r)} = 1$.

Theorem 3.4. *([7], Theorem 8) Let* φ *and* ψ *be positive functions on* $[0, \infty)$ *satisfying the following conditions:*

1. *the functions* φ *and* ψ *do not vanish identically in any neighbourhood of infinity;*

2. *the function* φ *is multiplicatively continuous at infinity and* $\beta(\varphi) > -1$*;*

3. *the function* ψ *is increasing;*

4. *at least one of the inequalities* $\alpha(\varphi) < m$ *and* $\alpha(\psi) < m$ *holds, where* $m > -1$*;*

5. *the functions*

$$\Phi(r) = \int_0^{\infty} \frac{\varphi(ru)}{(1+u)^{m+1}} du \quad \text{and} \quad \Psi(r) = \int_0^{\infty} \frac{\psi(ru)}{(1+u)^{m+1}} du$$

are finite and $\lim_{r\to\infty} \frac{\Psi(r)}{\Phi(r)} = 1$ *then* $\lim_{r\to\infty} \frac{\psi(r)}{\varphi(r)} = 1$.

The above theorem derives asymptotic behavior of φ, ψ from the asymptotic behavior of ϕ, Ψ by assuming additional conditions on φ and ψ.

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