POISSON STATISTICS FOR ANDERSON MODEL WITH SINGULAR RANDOMNESS

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ABSTRACT. In this work we consider the Anderson model on the d-dimensional lattice with the single site potential having singular distribution, mainly α-Hölder continuous ones and show that the eigenvalue statistics is Poisson in the region of exponential localization.

CONTENTS

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1. THE MODEL

We consider the Anderson model, namely the operators

$$H^\omega = H_0 + \sum_{j \in \mathbb{Z}^d} \omega_j P_j, \ \omega \in \Omega, \tag{1}$$

on $\ell^2(\mathbb{Z}^d)$ with P_j the orthogonal projection onto $\ell^2(\{j\})$. We take H_0 to commute with translations on $\mathbb{Z}^d$. Typically we have $H_0 = \Delta$, which is the discrete Laplacian with diagonal part dropped. $H_0 = 0$ is also included in the model.

We consider a cube Λ_L of side length L and cover the cube with smaller disjoint cubes C_p of side length l_L, so that $\Lambda_L = \cup_{p=1}^{N_L^d} C_p$. Given these we consider the matrices $H^\omega_{\Lambda_L}$ and $H^\omega_{C_p}$ obtained by compressing the operator H^ω to the finite dimensional subspaces $\ell^2(\Lambda_L)$ and $\ell^2(C_p)$ respectively.

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An explicit collection of such cubes C_p is obtained by we dividing $(-L-1, L]^d$ into N_L^d equal cubes C_p^* of the form $(c,d]^d$ for $p = 1, 2, \cdots, N_L^d$, with side length $\frac{2L+1}{N_L}$ and defining

$$C_p = C_p^* \cap \mathbb{Z}^d, \ int(C_p) = \{x \in C_p : dist(x, \partial C_p) > l_L\}, \tag{2}$$

where $\{N_L\}_L$ and $\{l_L\}_L$ are both increasing sequences of integers, which will be specified latter. For any cube $B \subset \mathbb{Z}^d$ the boundary of B is denoted by ∂B and is defined by

$$\partial B = \{x \in B : \exists\ x' \in B^c \ such\ that\ |x - x'| = 1\}.$$

Hypothesis 1.1. *We assume that the single site distribution μ is uniformly α-Hölder continuous for $0 < \alpha \leq 1$.*

We look in the region of exponential localization to study eigenvalue statistics. In this context by 'exponential localization' we mean that some appropriate fractional moments of the Green functions associated with the operators are exponentially decaying as shown by Aizenman-Molchanov [2] in the strong disorder case. For simplicity we assume complete localization and state their results as a hypothesis for the present work.

Hypothesis 1.2. *Let $\Lambda \subseteq \mathbb{Z}^d$ be any large cube, then the inequality*

$$\sup_{z \in \mathbb{C}^+, Re(z) \in [a,b]} \mathbb{E}\left(|\langle \delta_n, (H_\Lambda^\omega - z)^{-1} \delta_m \rangle|^s\right) \leq C e^{-\gamma |n-m|},$$

is valid for some $s > 0$ and $0 < \gamma < \infty$.

Henceforth the energy $E \in \sigma(H^\omega)$ appearing all the quantities below is assumed to lie in $[a,b]$ occurring in the above hypothesis.

Using the constant γ in the above hypothesis, we specify the numbers N_L, l_L of equation (2). We fix an $0 < \epsilon < 1$ and choose

$$N_L = O(L^{1-\epsilon}), \ l_L = \frac{5d}{\alpha\gamma} \ln(2L+1). \tag{3}$$

Given the operators in equation (1) satisfying the above hypotheses and an α, $0 < \alpha \leq 1$, we define the following random measures on $\mathbb{R}$, where we set $\beta_L = (2L+1)^{\frac{d}{\alpha}}$.

$$\begin{aligned} \xi_{L,E}^\omega(I) &= Tr(\chi_{\Lambda_L} E_{\beta_L(H_L^\omega - E)}(I)) \\ \eta_{p,E}^\omega(I) &= Tr(\chi_{\Lambda_L} E_{\beta_L(H_{C_p}^\omega - E)}(I)), \ p = 1, \ldots, N_L^d \\ \zeta_{L,E}^\omega(I) &= Tr(\chi_{\Lambda_L} E_{\beta_L(H^\omega - E)}(I)). \end{aligned} \tag{4}$$

The eigenvalue statistics was studied for random Schrödinger operators by Molchanov [22] followed by Minami [21], who obtained an estimate for ensuring that the Lévy measure of the eigenvalue point process is degenerate. Eigenvalue statistics was studied by Germinet-Klopp [11], Aizenman-Warzel [3] for the canopy graph. Combes-Germinet-Klein [7] obtained the Minami estimate in a more transparent form while extending the original estimate to more general single site distributions and obtaining estimates on probabilities associated with existence of multiple eigenvalues in an interval. The statistics results were extended to include localization centers by Nakano-Killip [17] and for the Schrödinger case Nakano [24] showed infinite divisibility of limiting point processes. In the case of Anderson model with higher rank random potentials Tautenhan-Veselić [28] obtained the Minami estimate leading to the Poisson statistics. Recently Hislop-Krishna [12] considered higher rank random potentials for Anderson models and showed the eigenvalue statistics to be compound Poisson. They also showed that in all the cases of random Schrödinger and Anderson models with higher rank i.i.d random potentials, the Wegner estimate and complete localization leads to a compound Poisson eigenvalue statistics in general.

Level repulsion was proved for a class of Anderson models with decaying randomness by Dolai-Krishna [10], for a class of Schrödinger operators in one dimension Kotani-Nakano [19] obtained β-ensemble governing the statistics and for localization centers level repulsion was shown in the Anderson model by Nakano [23].

All these works, except that of Combes-Germinet-Klein [7], assumed that the single site distributions have an absolutely continuous bounded density, which amounts to taking $\alpha = 1$.

We show here that by changing the scale appropriately we can include more singular single site distributions. However this comes at a price. In view of the subtleties involved with singular measures, in particular the absence of a de la Valée Poussin type theorem, the results become weaker.

It is well known that once we have the Wegner estimate, limit points of the point processes in equation (4) are also point processes. We state this fact as a theorem below. The proof involves showing tightness of the family of measures $\xi^\omega_{L,E}$ using Wegner estimate and Chebyshev inequality, and the proof is given as part of Hislop-Krishna [12, Proposition 4.1], so we state it without proof.

Theorem 1.3. *Consider H^ω as in equation (1) satisfying Hypothesis 1.1 with $0 < \alpha \leq 1$. Then every limit point, in the sense of distributions, of $\xi^\omega_{L,E}$, is a point process.*

The main question is then to determine the nature of the limiting point processes which we do below for α-Hölder continuous measures.

To the best of our knowledge ours is the first instance where singular single site distributions are allowed to obtain eigenvalue statistics.

We use the same symbol $\mathcal{N}$ for the IDS and the measure associated with it, and it should be clear from the context the object used.

We define the α-derivative and the α-upper derivatives of the integrated density of states $\mathcal{N}$ of our model, by

$$d^\alpha_{\mathcal{N}}(E) = \lim_{\epsilon\to 0} \frac{\mathcal{N}((E-\epsilon, E+\epsilon))}{(2\epsilon)^\alpha}, D^\alpha_{\mathcal{N}}(E) = \limsup_{\epsilon\to 0} \frac{\mathcal{N}((E-\epsilon, E+\epsilon))}{(2\epsilon)^\alpha}.$$

We define the measures $\mathcal{L}_\alpha$ as

$$\mathcal{L}_\alpha(I) = \alpha 2^{\alpha-1} \int_I |y|^{\alpha-1} dy \tag{5}$$

for any bounded Borel subset $I \subset \mathbb{R}$.

Our main theorem is then to show that the limiting point processes give Poisson distributions for a class of intervals. For technical reasons, that have to do with the fact that we are dealing with singular measures, we consider only intervals symmetric about the origin below to obtain the parameters of the limiting Poisson distributions. In view of the fact that for singular measures ν, $d_\nu^\alpha(x)$ may not exist for almost all x w.r.t. ν, we have to deal with upper derivatives in which case we can only talk about limit points of the random measures we considered above.

Theorem 1.4. *Consider H^ω as in equation (1) satisfying Hypotheses 1.1 with $0 < \alpha \leq 1$ and Hypothesis 1.2 with E in the region of localization. For any bounded open interval I, suppose $\gamma_{E,I}$ is non-zero such that*

$$\gamma_{E,I} = \lim_{n\to\infty} \mathbb{E}(\xi^\omega_{L_n(I),E}(I)).$$

Then the random variables $\xi^\omega_{L_n(I),E}(I)$ converge in distribution to the Poisson random variable with parameter $\gamma_{E,I}$.

This theorem implies the following.

Corollary 1.5. *With the assumptions of Theorem 1.4, if $0 < D^\alpha_{\mathcal{N}}(E) < \infty$, then for each bounded open interval $I = -I$, there is a subsequence $L_n(I)$ such that $\xi^\omega_{L_n(I),E}(I)$ converges in distribution to a Poisson random variable with parameter $D^\alpha_{\mathcal{N}}(E)\mathcal{L}_\alpha(I)$.*

It is interesting to note that the measures $\mathcal{L}_\alpha$ occur in the theorem of Jensen-Krishna, [9, Theorem 1.3.2] dealing with continuous wavelet transforms of measures, where the constants c_α there are integrals of the function ψ, the function that generates the "continuous wavelet", with respect to $\mathcal{L}_\alpha$.

Since the limsup of a sequence is always a limit point of the sequence, the above theorem shows that when the upper derivative $D^\alpha_{\mathcal{N}}(E)$ is positive, there is at least one subsequence of $\xi^\omega_{L,E}(I)$ that converges in distribution to a Poisson random variable, the parameter of the corresponding Poisson distribution is then $D^\alpha_{\mathcal{N}}(E)\mathcal{L}_\alpha(I)$.

It was shown by Krishna [20, Corollary 1.7], Combes-Germinet-Klein [7], (and by Combes-Hislop-Klopp [6], Stollmann [27] for continuous models), that when the single site distribution is uniformly α-Hölder continuous (Hypothesis 1.1), the IDS, $\mathcal{N}$ is uniformly α-Hölder continuous. When $\mathcal{N}$ is α-Hölder continuous we can use the decomposition of Theorem 69, Rogers [26], with $h(x) = |x|^\alpha$, to obtain for any bounded Borel subset $E \subset \mathbb{R}$,

$$\mathcal{N}(E) = \int_E f(t) d\mu^h(x) + J(E),$$

with J being strongly h continuous. Accordingly if this f is non-zero a.e. μ^h then we will have non-zero $D^\alpha_{\mathcal{N}}(E)$ for those E for which $f(E)$ is non-zero.

However since the theorems of [20],[6], may not be optimal, in the sense that the $\mathcal{N}$ may have better modulus of continuity in some part of its support, (as seen in Kaminaga-Nakamura-Krishna [16], where the IDS is analytic in some region of the spectrum even for measures μ with singular component in them) even when μ is only α-Hölder continuous, we cannot be sure that the f is indeed non-zero for the given E.

We finally note that the subsequences of Theorem 1.3 and those in Corollary 1.5 may be different.

2. Ideas of Proofs

The strategy of proof is to first follow the procedure adopted by Minami [21] where one first shows that three classes of random measures considered above are asymptotically essentially the same in the sense that if one of the limits below exists then it does for all and they are all the same. A similar statement holds for any subsequence also.

Proposition 2.1. *Consider the processes defined in equation (4) associated with the operators H^ω satisfying Hypotheses 1.1,1.2. Then for any bounded interval $I \subset \mathbb{R}$ we have*

$$\lim_{L\to\infty} \zeta^\omega_{L,E}(I) = \lim_{L\to\infty} \xi^\omega_{L,E}(I) = \lim_{L\to\infty} \sum_p^{N_L^d} \eta^\omega_{p,E}(I). \tag{6}$$

In the above the limits are in the sense of convergence in distributions and the Proposition follows from Theorem 3.3 as in Minami's paper [21] and we omit the proof. An immediate Corollary of the above Proposition and Lemma 3.1 is the following.

Corollary 2.2. *For any bounded interval $I \subset \mathbb{R}$ we have*

$$\lim_{L\to\infty} \mathbb{E}\big(\zeta^\omega_{L,E}(I)\big) = \lim_{L\to\infty} \mathbb{E}\big(\xi^\omega_{L,E}(I)\big) = \lim_{L\to\infty} \mathbb{E}\bigg(\sum_p^{N_L^d} \eta^\omega_{p,E}(I)\bigg). \tag{7}$$

Once these results are established, our strategy is to use the the celebrated Lévy-Khintchine representation theorem for measures. The Lévy-Khintchine theorem (see Theorem 1.2.1, Applebaum [1]) says that a measure ν is infinitely divisible if and only if its characteristic function $\widehat{\nu}(t)$ is of the form

$$e^{iat+bt^2+\int_{|x|\le c}(e^{itx}-1-itx)dM(x)}$$

for some σ-finite measure M, which is called the Lévy measure associated with ν. In the case the measure M is finite we can absorb the linear term into the number a and rewrite this expression in the form

$$e^{iat+bt^2+\int_{|x|\le c}(e^{itx}-1)dM(x)}$$

It turns out that a distribution is Poisson iff $a = b = 0$ and M is supported on $\{1\}$ (notationally such a measure is written as a positive multiple of $\delta(x-1)dx$ by some authors). The parameter of the Poisson distribution is then the number $M(\{1\})$.

We emphasize here that to show Proposition 2.1 it is sufficient to have exponential localization (in the sense of Aizenman-Molchanov [2]) and the Wegner estimate for the IDS $\mathcal{N}$. The result that $M(\mathbb{R}\setminus\{1\}) = 0$ uses the Minami estimate.

Therefore the idea is to compute the Fourier transforms of the random variables

$$\sum_p \eta^{\omega}_{p,E}(I)$$

which are a sum of i.i.d random variables and show that the limit of the Fourier transform has the desired form.

In view of the Corollary 2.2, the value of the parameter of the Poisson distribution is computed using the fact that the parameter is the expectation of the Poisson distribution which in this case is obtained as the limit of $\mathbb{E}\left(\zeta^{\omega}_{L,E}(I)\right)$ either for the whole sequence or if the limit does not exits for some subsequences for which it does.

In the context of absolutely continuous single site distributions μ these limits exist at points in the spectrum where the density of states exists. In our context where we are dealing with singular single site distributions which have no density with respect to the Lebesgue measure we need to consider derivatives or upper derivatives with respect to Hausdorff measures to obtain these limit points.

3. Preliminaries

It was shown by Krishna [20, Corollary 1.7] that if the single site distribution μ is uniformly α-Hölder, $0 < \alpha \leq 1$ continuous , then the integrated density of states (IDS) is also uniformly α-Hölder continuous.
We state this fact in the form given by Combes-Germinet-Klein [7]. Given a probability measure μ let $S_\mu(s) := \sup_{a\in\mathbb{R}} \mu[a, a+s]$. Define

$$Q_\mu(s) := \begin{cases} \|\rho\|_\infty\ s & \text{if } \ \mu \textit{ has bounded density } \rho, \\ 8S_\mu(s) & \textit{otherwise.} \end{cases}$$

If μ is a Hölder continuous with exponent $\alpha \in (0,1]$ then $Q_\mu(s) \leq U s^\alpha$ for small $s > 0$, for some constant U.
In [7] Combes-Germinet-Klein prove the Wegner estimate and and Minami estimate for more general measure μ (single site distribution), we collect their results in the following lemma which immediately gives the following corollary. The inequality (8) is [7, inequality (2.2)], the inequality (9) is [7, Theorem 2.3] and the inequality (10) is [7, Theorem 2.1], so we omit the proofs.

Lemma 3.1. *For all bounded interval $I \subset \mathbb{R}$ and any finite volume $\Lambda \subset \mathbb{Z}^d$, we have*

$$\mathbb{E}\big(\langle \delta_n, E_{H^\omega}(I)\delta_n\big) \leq Q_\mu(|I|), \tag{8}$$

$$\mathbb{E}\big(Tr(E_{H^\omega_\Lambda}(I))\big) \leq Q_\mu(|I|)\ |\Lambda|, \tag{9}$$

$$\mathbb{E}\bigg(Tr(E_{H^\omega_\Lambda}(I))\big(Tr(E_{H^\omega_\Lambda}(I)) - 1\big)\bigg) \leq \bigg(Q_\mu(|I|)\ |\Lambda|\bigg)^2. \tag{10}$$

Corollary 3.2. *Consider $\mathcal{N}$, the IDS of the operators H^ω satisfying Hypothesis 1.1. Then for any $\psi \in C_c(\mathbb{R})$ and $n \in \mathbb{Z}^d$, we have*

$$\int_{\mathbb{R}} \psi(x)d\mathcal{N}(x) = \mathbb{E}\big(\langle \delta_n, \psi(H^\omega)\delta_n\big) \leq \|\psi\|_\infty\ Q_\mu(|s_\psi|),\ \ s_\psi = \text{supp}\ \psi. \tag{11}$$

$$\mathbb{E}\big(Tr(\psi(H^\omega_\Lambda))\big) \leq \|\psi\|_\infty\ Q_\mu(|s_\psi|)\ |\Lambda|. \tag{12}$$

Given any measure ν we denote notationally $\nu(f) = \int f(x)d\nu(x)$ below, where again the limits are to be understood as in the sense stated for Proposition 2.1.

Theorem 3.3. *Let H^ω satisfy the Hypotheses 1.1, 1.2 and let $E \in \sigma_p(H^\omega)$. Then for each $\psi \in C_c(\mathbb{R})$, we have*

$$\lim_{L\to\infty} \zeta^\omega_{L,E}(\psi) = \lim_{L\to\infty} \xi^\omega_{L,E}(\psi) = \lim_{L\to\infty} \sum_p^{N_L^d} \eta^\omega_{p,E}(\psi), \tag{13}$$

with convergence in the sense of distributions.

Proof: By general theory (see Kallenberg [14, Theorem 4.5]), the theorem follows if we show that

$$\lim_{L\to\infty} \mathbb{E}\left| e^{-\xi^\omega_{L,E}(\psi)} - e^{-\sum_p^{N_L^d} \eta^\omega_{p,E}(\psi)} \right| = 0, \tag{14}$$

$$\lim_{L\to\infty} \mathbb{E}\left| e^{-\zeta^\omega_{L,E}(\psi)} - e^{-\sum_p^{N_L^d} \eta^\omega_{p,E}(\psi)} \right| = 0. \tag{15}$$

Since the set of function $\phi_z(x) = Im\frac{1}{x-z},\ z \in \mathbb{C}^+$ are dense in $C_c(\mathbb{R})$ it is sufficient to verify (14) for such function, for more details we refer [13, Appendix: The Stone-Weierstrass Gavotte].
For $n \in int(C_p)$ and $z \in \mathbb{C}^+$ we have the well known perturbation formula, using the resolvent estimate,

$$G^{\Lambda_L}(z;n,n) = G^{C_p}(z;n,n) + \sum_{(m,k)\in\partial C_p} G^{C_p}(z;n,m)G^{\Lambda_L}(z;k,n) \tag{16}$$

where $(m,k) \in \partial C_p$ means $m \in C_p$, $k \in \mathbb{Z}^d \setminus C_p$ such that $|m-k| = 1$.
Denote $z_L = E + \beta_L^{-1}z$ then we have, proceeding as in the proof by Minami

[21],

$$\begin{aligned}
&\Big|\xi^{\omega}_{L,E}(\phi_z) - \sum_{p}^{N_L^d} \eta^{\omega}_{p,E}(\phi_z)\Big| \qquad (17)\\
&= \frac{1}{\beta_L}\Big| TrImG^{\Lambda_L}(z_L) - \sum_p TrImG^{C_p}(z_L)\Big|\\
&\leq \frac{1}{\beta_L}\sum_p \sum_{n\in C_p\setminus int(C_p)} \big\{ImG^{C_p}(z_L;n,n) + ImG^{\Lambda_L}(z_L;n,n)\big\}\\
&\qquad + \frac{1}{\beta_L}\sum_p \sum_{n\in int(C_p)} \sum_{(m,k)\in\partial C_p} |G^{C_p}(z_L;n,m)||G^{\Lambda_L}(z_L;k,n)|\\
&= A_L + B_L.
\end{aligned}$$

From Combes-Germinet-Klein [7, A.9] we have for given $k > 0$

$$Imz\ \mathbb{E}(G^{\Lambda}(z;n,n)) \leq \pi\Big(1+\frac{k}{2}\Big)S_\mu\Big(\frac{2\ Imz}{k}\Big). \qquad (18)$$

Since $Imz_L = \beta_L^{-1} Imz$ with $Imz > 0$ so using the α-Hölder continuity of μ we get

$$\begin{aligned}
\frac{1}{\beta_L}\mathbb{E}\Big(G^{\Lambda}(z_L;n,n)\Big) &\leq \frac{1}{Imz}\ \pi\Big(1+\frac{k}{2}\Big)S_\mu\Big(\frac{2\ Imz_L}{k}\Big),\ \Lambda = C_p,\ \Lambda_L \qquad (19)\\
&\leq C\ \Big(\frac{2\ \beta_L^{-1}\ Imz}{k}\Big)^{\alpha}\\
&\leq C\ (2L+1)^{-d},\ (\text{since } \beta_L = (2L+1)^{d/\alpha}).
\end{aligned}$$

From the inequality (17) and above we get

$$\mathbb{E}(A_L) \leq C\ (2L+1)^{-d} N_L^d \Big(\frac{2L+1}{N_L}\Big)^{d-1} l_L = O\big(L^{-\epsilon}\ ln(L)\big), \qquad (20)$$

in view of our choices for N_L, l_L in equation (3).

On the other hand the term B_L is split as

$$\begin{aligned}
B_L &= \frac{1}{\beta_L}\sum_p \sum_{n\in int(C_p)} \sum_{(m,k)\in\partial C_p} |G^{C_p}(z_L;n,m)||G^{\Lambda_L}(z_L;k,n)| \qquad (21)\\
&= \frac{1}{\beta_L}\sum_p \sum_{n\in int(C_p)} \sum_{(m,k)\in\partial C_p} |G^{\Lambda_L}(z_L;k,n)|^s |G^{\Lambda_L}(z_L;k,n)|^{1-s} |G^{C_p}(z_L;n,m)|
\end{aligned}$$

Then using the fact that $(m,k) \in \partial C_p$ and $n \in int(C_p)$ so that $|n-k| > l_L$ for large enough L, the Hypothesis 1.2 with the number s chosen from there, to estimate,

$$|G^{\Lambda_L}(z_L;k,n)|^{1-s} \leq \frac{1}{|Imz_L|^{1-s}} \text{ and } |G^{C_p}(z_L;n,m)| \leq \frac{1}{|Imz_L|},$$

we obtain the following bound from taking expectation in the equality (21).

$$\mathbb{E}(B_L) \leq \frac{1}{\beta_L |Imz_L|^{2-s}} N_L^d \left(\frac{2L+1}{N_L}\right)^d \left(\frac{2L+1}{N_L}\right)^{d-1} l_L e^{-\gamma l_L}. \tag{22}$$

We simplify the right hand side of the above inequality to get

$$\mathbb{E}^\omega(B_L) \leq \frac{1}{\beta_L |Imz_L|^{2-s}} N_L^d \left(\frac{2L+1}{N_L}\right)^d \left(\frac{2L+1}{N_L}\right)^{d-1} l_L e^{-rl_L} \tag{23}$$

$$= O\big(L^{(1-s)\frac{d}{\alpha}} N_L^d \big(\frac{L}{N_L}\big)^{2d-1} e^{-\gamma \frac{5d}{\gamma\alpha}\ln(2L+1)} \ln(L)\big) \tag{24}$$

$$= O\big(L^{(\frac{d}{\alpha}(1-s)+(2d-1)+(d-1)(1-\epsilon)-\frac{5d}{\alpha})} \ln(L)\big) \tag{25}$$

$$= O(L^{-1}), \tag{26}$$

since $\alpha \leq 1$. In particular we have from (20) and (23)

$$\mathbb{E}(A_L + B_L) \to 0 \ as \ L \to \infty. \tag{27}$$

Finally the inequality $|e^{-x} - e^{-y}| \leq |x-y| \ for \ x, y > 0$ together with the bound (17) and above convergence gives the required vanishing of the limits equation (14).

Again using the resolvent equation for $G(z; n, n) = \langle \delta_n, (H^\omega - z)^{-1} \delta_n \rangle$ and the equality

$$\langle \delta_n, (\oplus H^\omega_{C_p} \oplus H^\omega_{\Lambda_L^c} - z)^{-1} \delta_n \rangle = \langle \delta_n, (\oplus H^\omega_{C_p} - z)^{-1} \delta_n \rangle$$

valid for each $n \in \Lambda_L$, gives us the relation

$$G(z; n, n) = G^{C_p}(z; n, n) + \sum_{(m,k) \in \partial C_p} G^{C_p}(z; n, m) G(z; k, n), \ n \in C_p,$$

for each $p = 1, \ldots, N_L^d$. The convergence in equation (15) is then obtained by essentially repeating the argument above. □

4. Proof of the main Theorem

We first prove the Theorem 1.4.

In the argument below we consider a subsequence L_n which converges to the limsup in equation (38) and use Proposition 2.1 to only consider $\xi_{L,E}$ instead of $\zeta_{L,E}$. We will show that

$$\lim_{n\to\infty} \mathbb{E}\big(e^{it\xi^\omega_{L_n,E}(I)}\big) = e^{(e^{it}-1)\gamma_{E,I}}.$$

This will then show, by Lévy-Khintchine theorem that $\xi^\omega_{L_n,E}(I)$ converge in distribution to the Poisson random variable with parameter $\gamma_{E,I}$. Since the convergence in distribution for a sequence of random variables is equivalent to the convergence of their Fourier transforms point wise combined with Theorem 3.3, it is enough to look at the limit with $\xi^\omega_{L_n,E}(I)$ replaced by $\sum_{p=1}^{N_{L_n}^d} \eta^\omega_{L_n,E}(I)$.

We first note that from equation (7) we have,

$$\lim_{n\to\infty}\mathbb{E}\Big(\sum_{p}^{N_{L_n}^d}\eta_{p,E}^{\omega}(I)\Big)=\lim_{n\to\infty}\mathbb{E}\big(\zeta_{L_n,E}^{\omega}(I)\big)=\gamma_{E,I} \tag{28}$$

We now compute the limits of Fourier transforms

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\big(e^{it\xi_{L_n,E}^{\omega}(I)}\big)&=\lim_{n\to\infty}\mathbb{E}\Big(e^{it\sum_{p=1}^{N_{L_n}^d}\eta_{p,E}^{\omega}(I)}\Big) \qquad (29)\\ &=\lim_{n\to\infty}\prod_{p=1}^{N_{L_n}^d}\mathbb{E}\big(e^{it\eta_{p,E}^{\omega}(I)}\big)\\ &=\lim_{n\to\infty}\Big[\mathbb{E}\big(e^{it\eta_{p,E}^{\omega}(I)}\big)\Big]^{N_{L_n}^d}\end{aligned}$$

Now for $p=1,\dots,N_{L_n}^d$,

$$\begin{aligned}\mathbb{E}\big(e^{it\eta_{p,E}^{\omega}(I)}\big)&=\sum_{m=0}^{\infty}e^{itm}\mathbb{P}\big(\eta_{p,E}^{\omega}(I))=m\big) \qquad (30)\\ &=1+\mathbb{E}(\eta_{p,E}^{\omega}(I))[e^{it}-1]+R_{L_n}\end{aligned}$$

where R_{L_n} is given by

$$\begin{aligned}R_{L_n}&=\sum_{m=0}^{\infty}e^{itm}\mathbb{P}\big(\eta_{p,E}^{\omega}(I))=m\big)-1-\mathbb{E}(\eta_{p,E}^{\omega}(I))[e^{it}-1]\\ &=\sum_{m=0}^{\infty}e^{itm}\mathbb{P}\big(\eta_{p,E}^{\omega}(I))=m\big)-\sum_{m=0}^{\infty}\mathbb{P}\big(\eta_{p,E}^{\omega}(I)=m)\\ &-[e^{it}-1]\sum_{m=0}^{\infty}m\mathbb{P}\big(\eta_{p,E}^{\omega}(I))=m\big)\\ &=\sum_{m=2}^{\infty}\big(e^{itm}-me^{it}+m-1\big)\mathbb{P}\big(\eta_{p,E}^{\omega}(I))=m\big).\end{aligned}$$

Then using the inequality

$$|e^{itm}-me^{it}+m-1|\le(m+1)+(m-1)\le 2m,\ \text{when } m\ge 2$$

and setting $J_{L,E} = E + \beta_L^{-1} I$ we get,

$$\begin{aligned}
|R_{L_n}| &\leq \sum_{m=2}^{\infty} \big(|e^{itm} - me^{it}| + (m-1)\big)\ \mathbb{P}\big(\eta^{\omega}_{p,E}(I)) = m\big) \\
&\leq \sum_{m=2}^{\infty} \big((m+1) + (m-1)\big)\ \mathbb{P}\big(\eta^{\omega}_{p,E}(I)) = m\big) \\
&\leq 2\sum_{m=2}^{\infty} m\mathbb{P}\big(\eta^{\omega}_{p,E}(I)) = m\big) \\
&\leq 2\sum_{m=2}^{\infty} m(m-1)\mathbb{P}\big(\eta^{\omega}_{p,E}(I)) = m\big) \\
&\leq 2\mathbb{E}\Bigg(Tr(E_{H^{\omega}_{C_p}}(J_{L_n,E})\big(Tr(E_{H^{\omega}_{C_p}}(J_{L_n,E}) - 1\big)\Bigg),
\end{aligned}$$

Now from the Minami estimate of Lemma 3.1 inequality (10) we have

$$\begin{aligned}
&N^d_{L_n}\ \mathbb{E}\Bigg(Tr(E_{H^{\omega}_{C_p}}(J_{L_n,E})\big(Tr(E_{H^{\omega}_{C_p}}(J_{L_n,E}) - 1\big)\Bigg) \qquad (31)\\
&\leq N^d_{L_n}\Bigg(Q_\mu(|J_{L_n,E}|)\ |C_p|\Bigg)^2 \\
&\leq N^d_{L_n}\big(|J_{L_n,E}|^{\alpha}|C_p|\big)^2 \\
&= O\Bigg(\beta_{L_n}^{-2\alpha} N^d_{L_n}\bigg(\frac{2L_n+1}{N_{L_n}}\bigg)^{2d}\Bigg).
\end{aligned}$$

The above calculation together with (31) estimate will give

$$N^d_{L_n} R_{L_n} \to 0\ as\ n \to \infty.$$

¿From the above computation we get

$$N^d_{L_n}\Bigg[\mathbb{E}(\eta^{\omega}_{p,E}(I))[e^{it}-1] + R_{L_n}\Bigg] \xrightarrow{n\to\infty} \gamma_{E,I}[e^{it}-1].$$

We use the equations (29) and (30) to obtain the equality

$$\mathbb{E}\big(e^{it\xi^{\omega}_{L_n,E}(I)}\big) = \left[1 + \frac{N^d_{L_n}\big[\mathbb{E}(\eta^{\omega}_{p,E}(I))[e^{it}-1] + R_{L_n}\big]}{N^d_{L_n}}\right]^{N^d_{L_n}}. \qquad (32)$$

which combined with the convergence of $\big(1 + \frac{z_n}{n}\big)^n$ to e^z, whenever $z_n \to z$ as $n \to \infty$ gives us finally the limit

$$\mathbb{E}\big(e^{it\xi^{\omega}_{L_n,E}(I)}\big) \xrightarrow{n\to\infty} e^{\gamma_{E,I}(e^{it}-1)}.$$

□

Proof of Corollary 1.5:

We first note that, if we denote

$$D^{\alpha}_{\mu}(E) < \infty \text{ iff } \limsup \frac{\mathcal{N}(E+\epsilon I)}{(2\epsilon)^{\alpha}} < \infty, \text{ for all bounded symmetric intervals } I.$$

We will show that

$$\limsup \frac{\big(\beta_L^\alpha \mathbb{E}\big(\langle \delta_0, E_{H^\omega}(E+\beta_L^{-1}I)\delta_0\rangle\big)}{|I|^\alpha} \geq \frac{1}{2^d} D_{\mathcal{N}}^\alpha(E).$$

Then by the assumption of theorem the right hand side is positive, so a limit point of $\mathbb{E}(\zeta_L(I))$ is positive. We recall that $\beta_L^\alpha = (2L+1)^d$. Let I be a bounded open interval and choose $\epsilon \in (\beta_{L+1}^{-1}, \beta_L^{-1}]$,then we have

$$E + \beta_L^{-1} I \supseteq E + \epsilon I \text{ and } \mathcal{N}\big(E+\beta_L^{-1}I\big) \geq \mathcal{N}\big(E+\epsilon I\big).$$

Therefore we have, since $\beta_{L+1}^\alpha \epsilon^\alpha \geq 1$,

$$\begin{aligned} \frac{\beta_L^\alpha \mathcal{N}\big(E+\beta_L^{-1}I\big)}{|I|^\alpha} &\geq \left(\frac{\beta_L}{\beta_{L+1}}\right)^\alpha \frac{\mathcal{N}\big(E+\epsilon I\big)}{(\epsilon|I|)^\alpha} \\ &\geq \left(\frac{\beta_L}{\beta_{L+1}}\right)^\alpha \frac{\mathcal{N}(E-c\epsilon, E+c\epsilon)}{(\epsilon|I|)^\alpha}, \end{aligned} \tag{33}$$

These inequalities imply that

$$\sup_{L\geq M} \frac{\beta_L^\alpha \mathcal{N}\big(E+\beta_L^{-1}I\big)}{|I|^\alpha} \geq \left(\frac{1}{1+\frac{2}{2M+1}}\right)^d \sup_{\epsilon\in(\beta_{L+1}^{-1},\beta_L^{-1}],\ L\geq M} \frac{\mathcal{N}(E+\epsilon I)}{(\epsilon|I|)^\alpha} \tag{34}$$

$$\geq \left(\frac{1}{1+\frac{2}{2M+1}}\right)^d \sup_{\epsilon\in(0,\beta_M^{-1}]} \frac{\mathcal{N}(E+\epsilon I)}{(\epsilon|I|)^\alpha}, \tag{35}$$

where we used the facts that

$$\bigcup_{L\geq M} \big(\beta_{L+1}^{-1}, \beta_L^{-1}\big] = \big(0, \beta_M^{-1}\big] \text{ and } \left(\frac{\beta_L}{\beta_{L+1}}\right)^\alpha \geq \left(\frac{1}{1+\frac{2}{2M+1}}\right)^d, \text{ for } L \geq M.$$

We now let $M \to \infty$ in both side of above then from the definition of limsup we get

$$\overline{\lim_{L\to\infty}} \frac{\beta_L^\alpha \mathcal{N}\big(E+\beta_L^{-1}I\big)}{|I|^\alpha} \geq D_{\mathcal{N}}^\alpha(E). \tag{36}$$

Similarly starting with $\epsilon \in (\beta_{L+1}^{-1}, \beta_L^{-1}]$ we get the inequality

$$\frac{\beta_{L+1}^\alpha \mathcal{N}\big(E+\beta_{L+1}^{-1}I\big)}{|I|^\alpha} \leq \left(\frac{\beta_{L+1}}{\beta_L}\right)^\alpha \frac{\mathcal{N}\big(E+\epsilon I\big)}{(\epsilon|I|)^\alpha}$$

and proceed as in the above argument, with upper bounds now, to get

$$\overline{\lim_{L\to\infty}} \frac{\beta_L^\alpha \mathcal{N}\big(E+\beta_L^{-1}I\big)}{|I|^\alpha} \leq D_{\mathcal{N}}^\alpha(E). \tag{37}$$

Putting the inequalities (36) and (37) we get

$$\overline{\lim_{L\to\infty}} \frac{\beta_L^\alpha \mathcal{N}\big(E+\beta_L^{-1}I\big)}{|I|^\alpha} = D_{\mathcal{N}}^\alpha(E).$$

The above inequality shows that, noting again that $\beta_L^\alpha = (2L+1)^d$,

$$\begin{aligned} \gamma_{E,I} &= \overline{\lim_{L\to\infty}} \mathbb{E}\big(\zeta_{L,E}^\omega(I)\big) \\ &= \overline{\lim_{L\to\infty}} \mathbb{E}\big(\sum_{n\in\Lambda_L} \langle \delta_n, E_{H^\omega}(I)\delta_n\rangle\big) \\ &= \overline{\lim_{L\to\infty}} \beta_L^\alpha \mathcal{N}\big(E+\beta_L^{-1}I\big) \\ &= D_{\mathcal{N}}^\alpha(E)|I|^\alpha = D_{\mathcal{N}}^\alpha(E)\mathcal{L}_\alpha(I), \end{aligned} \tag{38}$$

where to pass to the third line we used the fact that $\mathbb{E}\big(\langle \delta_n, E_{H^\omega}(I)\delta_n\rangle\big)$ does not depend on n. Since the limsup above is a limit point of the sequence considered, we have the corollary. □

5. Example

Examples 5.1. *We now give an example of random operators that have singular density of states and for which the local eigenvalue statistics is Poisson. We note while this example may appear trivial, it is one for which none of the existing theorems can show Poisson eigenvalue statistics.*

Consider the operator

$$H^\omega = \sum_{n\in\mathbb{Z}^d} \omega_n P_n$$

P_n *is projection onto* $\ell^2(\{n\})$ *as in the model (1) with* $\{\omega_n\}$ *i.i.d random variable distributed by a measure* μ*. Then the IDS agrees with the distribution of the measure* μ*, so if we choose a singular* α*-continuous measure* μ *(such as the Cantor measure, for which* $\alpha = \log(2)/\log(3)$*), then the conditions of our theorem are valid for* $H_0 = 0$ *(which is in some sense infinite disorder limit of the large disorder Anderson model).*

Therefore Poisson eigenvalue statistics holds for points in the spectrum.

References

[1] Applebaum, David: *Lévy processes and stochastic calculus*, Cambridge Studies in Advances Mathematics **116**, second edition, Cambridge: Cambridge University Press, 2009.

[2] Aizenman, Michael; Molchanov, Stanislav: *Localization at large disorder and at extreme energies: an elementary derivation*, Commun. Math. Phys. **157(2)** (1993), 245-278.

[3] Aizenman, Michael; Warzel, Simone: *The Canopy Graph and Level Statistics for Random Operators on Trees*, Mathematical Physics, Analysis and Geometry, **9(4)** (2006), 291-333.

[4] Carmona, René; Lacroix, Jean: *Spectral theory of random Schrodinger operators*, Boston, Birkhauser, 1990.

[5] Carmona, René; Klein, Abel; Martinelli, Fabio: *Anderson localization for Bernoulli and other singular potentials* , Comm. Math. Phys. **108(1)** (1987), 41-66.

[6] Combes, Jean-Michel; Hislop, Peter D; Klopp, Frédéric: *An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators*, Duke Math. J. **140(3)** (2007), 469-498.

[7] Combes, Jean-Michel; Germinet, François; Klein, Abel: *Generalized Eigenvalue-Counting Estimates for the Anderson Model*, J Stat Physics **135(2)** (2009), 201-216.
[8] Daley, D.J; Vere-Jones: *An Introduction to the Theory of Point Processes* **II**, *General theory and structure*, Springer, New York, 2008.
[9] Demuth, Michael; Krishna, M: *Determining Spectra in Quantum Theory*, Progress in Mathematical Physics. **44**, Birkhäuser, Boston, 2004.
[10] Dolai, Dhriti; Krishna, M: *Level Repulsion for a class of decaying random potentials*, Markov Processes and Related Fields (to be appear), arXiv:1305.5619[math.SP].
[11] Germinet, François; Klopp, Frédéric: *Spectral statistics for the discrete Anderson model in the localized regime*, Spectra of random operators and related topics, (2011), 11-24, RIMS Kôkyûroku Bessatsu, B27, Res. Inst. Math. Sci. (RIMS), Kyoto.
[12] Hislop, Peter D; Krishna. M: *Eigenvalue statistics for random Shroödinger operators with non-rank one perturbation*, preprint (2014).
[13] H. L. Cycon; R. G. Froese; W. Kirsch; B. Simon: *Schrödinger operators, with application to quantum mechanics and global geometry*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987.
[14] Kallenberg, Olav: *Lectures on Random Measures*, Institute of Statistics Mimeo Series No. 963, 1974.
[15] Kallenberg, Olav: *Foundations of Modern Probability*, Springer, New York 2002.
[16] Kaminaga, M.; Krishna, M.; Nakamura, S: *A note on the analyticity of density of states*, Journal of Statistical Physics, **149**(3), (2013), 496-504.
[17] Killip, Rowan; Nakano, Fumihiko: *Eigenfunction Statistics in the Localized Anderson Model*, Ann. Henri Poincare **8(1)** (2007), 27-36.
[18] Kirsch, Werner: *An Invitation to Random Schrödinger Operators, (With an appendix by Frédéric Klopp)* Panor. Synthèses, **25**, Random Schrödinger operators, **1** 1-119, Soc. Math. France, Paris, 2008.
[19] Kotani, S; Nakano, Fumihiko: *Level statistics of one-dimensional Schrödinger operators with random decaying potential*, Preprint, 2012.
[20] Krishna, M: *Continuity of intregrated density of states-independent randomness*, Proc. Ind. Acad. Sci. **117(3)** (2007), 401-410.
[21] Minami, Nariyuki: *Local Fluctuation of the Spectrum of a Multidimensional Anderson Tight Binding Model*, Commun. Math. Phys. **177(3)** (1996), 709-725.
[22] Molchanov, S. A. *The local structure of spectrum of a random one-dimensional Schrödinger operator*, Trudy Sem. Petrovsk, **8** (1982), 195-210.
[23] Nakano, Fumihiko: *The repulsion between localizatgion centers in the Anderson model*, J. Stat. Phys. **123** (2006), 803-810.
[24] Nakano, Fumihiko: *Infinite divisibility of random measures associated to some random Schrödinger operators*, Osaka J. Math, **46** (2009), 845-862.
[25] Reed, Michael; Simon, Barry: *Method of modern mathematical physics* **I**, *Functional Analysis*, Academic Press, 1978.
[26] Rogers, C, A : *Hausdorff Measures*, Cambridge University Press, 1970.
[27] Stollmann, Peter: *From uncertainty principles to Wegner estimates*, Math. Phys. Anal. Geom. **13**, (2010), 145-157.
[28] Tautenhahn, M; Veselić, I: *Minami's estimate: Beyond rank one perturbation and monotonicity*, Ann. Henri Poincaré **15** (2014), 737-754.

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