Transforms of Measures

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(joint work with A. Jensen)

We look a the Wavelet transform of probability measures on $\mathbb{R}$ and recover the components of the measures in a Lebesgue decomposition from this transform. Our motivation for this work is that in the spectral theory of selfadjoint operators essentially only the resolvent operator and the unitary group generated by the operators are widely used and there are lots of interesting operators where the spectral theoretic questions are open where it seems hard to use these functions. Therefore there was a need to find out if there are other functions of the selfadjoint operators that could be used to study spectra and such a question translates the question we address in this work.

In the following ψ is a (complex valued) bounded continous even function with $\psi(0) \neq 0$, that satisfies the decay condition

$$|\psi(x)| + |x\psi'(x)| \leq C(1 + |x|)^{-\delta}, \ \delta > 1$$

for some $\delta > 1$. Given such a ψ, we define $\psi_a(x) = \psi(\frac{x}{a}),\ a > 0$ and let $\psi_a * \mu$ to be $\int \psi_a(\cdot - y) d\mu(y)$.

Then the theorems we prove are the following for which we normalize $\int \psi(y)dy = 1$.

Theorem 1. *Let ψ be as above and let μ be a probability measure on $\mathbb{R}$. Then*

(1) $\lim_{a \to 0} \psi_a * \mu(x) = \psi(0)\mu(\{x\})$.

(2) *For every continuous function f of compact support, the following is valid.*

$$\lim_{a \to 0} \int \left(\frac{1}{a}\psi_a * \mu\right)(x) f(x) dx = \int f(x) d\mu(x).$$

(3) *Let $d_\mu^\alpha(x) = \lim_{\epsilon \to 0} \frac{\mu((x-\epsilon, x+\epsilon))}{(2\epsilon)^\alpha}$ be finite, for some $0 < \alpha \leq 1$ and x, then*

$$\lim_{a \to 0} a^{-\alpha} \psi_a * \mu(x) = c_\alpha d_\mu^\alpha(x),$$

where $c_\alpha = \int_0^\infty \alpha 2^\alpha y^{\alpha-1} \psi(y)\ dy$.

The next theorem is the anologue of the theorems of Simon [4] for the case of Borel transforms.

Theorem 2. *Let μ be a probability measure on $\mathbb{R}$ and ψ be as above. Then for any bounded interval (c, d) the following are valid.*

(1) *Let*

$$C = \int_{\mathbb{R}} |\psi(x)|^2\ dx.$$

then

$$\lim_{a \to 0} \frac{1}{a} \int_c^d |\psi_a * \mu|^2(x)\ dx = C \left(\sum_{x \in (c,d)} \mu(\{x\})^2 + \frac{1}{2}\left[\mu(\{c\})^2 + \mu(\{d\})^2\right] \right).$$

(2) *Suppose that for some $p > 1$,*

$$\sup_{a > 0} \int_c^d \left| \frac{1}{a}(\psi_a * \mu)(x) \right|^p\ dx < \infty,$$

then μ *is purely absolutely continuous in* (c,d)*. In addition, for any compact subset* S *of* (c,d)

$$\frac{1}{a}\psi_a * \mu \to \frac{d\mu_{ac}}{dx}, \text{ in } L^p(S), \text{ as } a \to 0.$$

The converse that if μ *is purely absolutely continuous with the density* $\frac{d\mu_{ac}}{dx}$ *in* $L^p((c,d))$*, then the supremum above is finite, is also valid.*

(3) *For* $0 < p < 1$*, we have*

$$\lim_{a\to 0}\int_c^d \left|\frac{1}{a}\psi_a * \mu\right|^p (x)\ dx = \int_c^d \left|\frac{d\mu_{ac}}{dx}\right|^p\ dx.$$

And finally for the quantities

$$C_\mu^\alpha(x) = \limsup_{a\to 0}\frac{\psi_a * \mu}{a^\alpha}(x).\ D_\mu^\alpha(x) = \limsup_{\epsilon\to 0}\frac{\mu((x-\epsilon, x+\epsilon))}{(2\epsilon)^\alpha},$$

one has the following theorem.

Theorem 3. *Let* μ *be a probability measure and let* ψ *be as above. Then* $C_\mu^\alpha(x)$ *is finite for any* x*, whenever* $D_\mu^\alpha(x)$ *is finite for the same* x *and if* ψ *is positive then they are both finite or both infinite.*

There are lots of functions ψ satisfying the conditions we imposed in the above theorems. Therefore, there is hope that the above theorems will enlarge the useful techniques available to the spectral theory of random and deterministic Schrödinger operators.

References

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