

AC spectrum for a class of random operators at small disorder

M Krishna
Institute of Mathematical Sciences
Taramani Chennai 600 113
India

Abstract

In this paper we present a class of Anderson type operators with independent, non-stationary (non-decaying) random potentials supported on a subset of positive density in the odd-dimensional lattice and prove the existence of pure absolutely continuous spectrum in the middle of the band for small disorder.

1 Introduction

Let the discrete Laplacian on $\ell^2(\mathbb{Z}^\nu)$ be defined by

$$(\Delta u)(n) = \sum_{i=1}^{\nu} (T_i + T_i^{-1})u(n)$$

where

$$(T_i)u(n) = u(n - e_i), e_i \in \mathbb{Z}^\nu, e_{ik} = \delta_{ik}, i, k = 1, \dots, \nu.$$

Given some infinite subset $\mathcal{N} \subset \mathbb{Z}^\nu$ consider the random potentials

$$V^\omega = \sum_{k \in \mathcal{N}} \omega_k \phi_k \tag{1}$$

where $\{\phi_k, k \in \mathcal{N}\}$ are real valued functions of compact and mutually disjoint supports and $\{\omega_k\}$ are independent identically distributed random variables. We then look at the model

$$H_\lambda^\omega = \Delta + \lambda V^\omega \quad (2)$$

and study its spectrum.

To specialize V^ω further we take the multiplication operators

$$(Q_i u)(n) = n_i u(n), i = 1, \dots, \nu \quad (3)$$

on $\ell^2(\mathbb{Z}^\nu)$. We note that T_i, T_i^{-1} are unitary for each i . We define

$$A = \frac{1}{2} \sum_{i=1}^{\nu} \{Q_i (T_i^{-1} - T_i) + (T_i^{-1} - T_i) Q_i\}. \quad (4)$$

The Q_i 's and A are self-adjoint on $\ell^2(\mathbb{Z}^\nu)$ with dense domains and the set of sequences in $\ell^2(\mathbb{Z}^\nu)$ of finite support forms a core for all of them.

Given a function ϕ we denote the operator of multiplication by ϕ on $\ell^2(\mathbb{Z}^\nu)$ also by the same symbol and for the following we set

$$\Lambda_s(n) = \{m \in \mathbb{Z}^\nu : |m - n| \leq s\}.$$

Hypothesis 1.1. *We assume that there is an infinite subset $\mathcal{N} \subset \mathbb{Z}^\nu$ and a collection of functions $\{\phi_k, k \in \mathcal{N}\}$ such that:*

1. $\{\phi_k, k \in \mathcal{N}\}$ are non-negative, bounded uniformly by 1, are of mutually disjoint supports and the quantities

$$[A, \phi_k], [A, [A, \phi_k]]$$

are uniformly bounded in k . We set $\| [A, \phi_k] \|_\infty = \sup_{k \in \mathcal{N}} \| [A, \phi_k] \|_\infty$.

2. For each $k \in \mathcal{N}$ there is an $n \in \mathbb{Z}^\nu$ such that

$$\Lambda_{|n|}^{\frac{1}{2}}(n) \subset \{n' \in \mathbb{Z}^\nu : \phi_k(n') = 1\}$$

3. The distribution μ of the random variables $\{\omega_k : k \in \mathcal{N}\}$ is compactly supported in \mathbb{R} and $0 \in \text{supp}(\mu)$. We set

$$E_+ = \sup(\text{supp}(\mu)), E_- = \inf(\text{supp}(\mu)), E_\infty = \max\{|E_+|, |E_-|\}$$

We then have the following theorem.

Theorem 1.2. *Consider the random operators H_λ^ω given in equation (2). Suppose V^ω satisfies the hypothesis 1.1. Then*

1. *For each $\lambda \geq 0$, $\sigma_{\text{ess}}(H_\lambda^\omega) = [-2\nu, 2\nu] + \lambda \text{supp}(\mu)$ a.e. ω .*
2. *Let ν be odd and let I be a closed interval contained in $(-2, 2)$. Then there is a λ_I satisfying $\lambda_I E_\infty < 1$ such that for all $0 \leq \lambda < \lambda_I$,*

$$\sigma_s(H_\lambda^\omega) \cap I = \emptyset, \text{ a.e. } \omega.$$

Remark 1.3. • *The hypothesis 1.1(2) should not be necessary. The boundedness of the commutators stated in part (1) of the hypothesis should enable us to prove that $\phi_k \geq \alpha > 0$ on a cube $\Lambda_{|n|^{\frac{1}{2}}}(n)$ contained in the support of ϕ_k for large enough k and this should be sufficient to prove the first part of the theorem.*

The spectral theory of random operators of the form given in equation (2) is widely studied with various assumptions on V^ω . The spectrum is known to be pure point spectrum is well known for the Anderson model (which is the same as in equation (2) when $\mathcal{N} = \mathbb{Z}^\nu, \phi_k(n) = \delta_{kn}$) when λ is large or at the edges of the spectrum. We refer to the book of Carmona-Lacroix [3], Cycon-Froese-Kirsch-Simon [6], Figotin-Pastur [7] and Stollmann [22] for the already extensive literature on this aspect of the theory.

There is a rich literature on the a.c. spectrum for decaying random potentials on $\ell^2(\mathbb{Z})$ with many sharp results. A review of some of these models is given by Denisov-Kiselev [4].

For the Anderson model, however, absolutely continuous spectrum is proving to be elusive and the expected result that there is such spectrum for small λ and in higher (than 2) dimensions is far from being realized. The only higher dimensional result for such a model is on the Bethe lattice for which the absolutely continuous spectrum was shown by Klein [14] and by Froese-Hasler-Spitzer [9].

In $\nu \geq 2$, a slightly modified model with decaying randomness (where in the Anderson model one takes $a_n V^\omega(n)$ instead of $V^\omega(n)$ and requires $a_n \rightarrow 0, |n| \rightarrow \infty$ at some rate) has been considered by Krishna [15], Anne Boutet de Monvel-Sahbani [2] and Bourgain [1]. An alternative collection of models are those for which $V^\omega(n)$ is zero outside a "hyper surface" (of some thickness) in \mathbb{Z}^ν as done in Jakić - Last [11], [12]. Yet another collection of models assume that the $V^\omega(n)$ are zero outside a subset S of \mathbb{Z}^ν which are

”sparse” (i.e. of zero density in \mathbb{Z}^ν), these are by Krishna [16], Krutikov [17], Molchanov [19], Molchanov-Vainberg [20].

On the Bethe lattice Kupin [18] also considered decaying randomness and showed a.c. spectrum.

All these works show existence of absolutely continuous spectrum in some region or the other of the spectrum.

2 The proofs

We give the proof of the theorem 1.2 in this section. We present the ideas involved first.

To show the statement (1) on the essential spectrum we construct Weyl sequences for each point that is claimed to be in the essential spectrum. We follow closely the ideas in Kirsch-Krishna-Obermeit [13]

The part (2) of the theorem is an application of Mourre theory of the existence of local conjugate, which is possible in the given region of energy for small enough disorder parameter λ . We need the dimension to be odd and also the energy to be small here to show the positivity of commutators.

We start with a few technical lemmas first.

Lemma 2.1. *Let ψ be a smooth function of compact support and let H_λ^ω be as in equation (2) satisfying the hypothesis 1.1(1),(2). Then*

$$\|\psi(H_\lambda^\omega) - \psi(\Delta)\| \leq C|\lambda|$$

where $C \leq E_\infty \int |t| |\widehat{\psi}(t)| dt$.

Proof: Using the spectral theorem and the Fourier transform we have

$$\psi(H_\lambda^\omega) - \psi(\Delta) = \int (e^{itH_\lambda^\omega} - e^{it\Delta}) \widehat{\psi}(t) dt.$$

We also have by fundamental theorem of calculus

$$e^{itH_\lambda^\omega} - e^{it\Delta} = \int_0^t e^{isH_\lambda^\omega} \lambda \sum_k \omega_k \phi_k e^{i(t-s)\Delta} ds$$

Putting these two equations together and estimating the operators norms, using the fact that

$$\left\| \sum_k \omega_k \phi_k \right\|_\infty \leq \sup_k |\omega_k| \leq E_\infty,$$

from the hypothesis 1.1(1),(2), we get the lemma. □

For the following lemma we set

$$\Lambda(n) = \{m \in \mathbb{Z}^\nu : |m - n| \leq |n|^{\frac{1}{2}}\}.$$

Proof of Theorem 1.2 (1) : We essentially follow the ideas used in proving theorem 2.4 of Kirsch-Krishna-Obermeit [13] for doing this.

Fix a $\lambda > 0$ and an $r \in \text{supp}(\mu)$ and $E \in (-2\nu, 2\nu)$, we will show

$$E + \lambda r \in \sigma(H_\lambda^\omega), \text{ a.e. } \omega.$$

Given $\ell \in \mathbb{N}$, we have

$$\mu\left(\left(r - \frac{1}{\ell}, r + \frac{1}{\ell}\right)\right) > 0,$$

from the definition of support of μ . Now consider the events

$$A_{k,\ell} = \left\{\omega : \omega_k \in \left(r - \frac{1}{\ell}, r + \frac{1}{\ell}\right)\right\}.$$

All these (mutually independent) events have (the same) positive probability for each fixed ℓ as $k \in \mathcal{N}$ varies. Hence we have, for each fixed ℓ ,

$$\sum_{k \in \mathcal{N}} \text{Prob}(A_{k,\ell}) = \infty,$$

therefore by Borel-Cantelli lemma the events $\{A_{k,\ell}\}$ occur infinitely often with probability one. That is the set

$$\Omega_\ell = \bigcap_{r=1}^{\infty} \bigcup_{|k| \leq r} A_{k,\ell}$$

has measure 1. Therefore the set

$$\Omega_0 = \bigcap_{\ell \in \mathbb{N}} \Omega_\ell$$

also has measure 1, being a countable intersection of measure 1 sets.

Now since $E \in (-2\nu, 2\nu)$ which is the essential spectrum of Δ , there is a sequence (as seen for example using density of compactly supported functions in $\ell^2(\mathbb{Z}^\nu)$ together with Theorem 7.2, Weidman [23]) f_j of compactly supported functions in $\ell^2(\mathbb{Z}^\nu)$, with $\|f_j\| = 1$, such that

$$\|(\Delta - E)f_j\| \rightarrow 0, \text{ as } j \rightarrow \infty.$$

Since Δ commutes with translations, it is also true that for any $m \in \mathbb{Z}^\nu$ the translates $f_j(\cdot - m)$ also satisfy the above condition. Given $\epsilon > 0$, we find an ℓ such that $\frac{1}{\ell} < \epsilon$, and a $j(\ell)$ such that

$$\|(\Delta - E)f_{j(\ell)}(\cdot - m)\| \leq \epsilon,$$

with $f_{j(\ell)}$ having compact support and the size of this support being the same for all $f_{j(\ell)}(\cdot - m)$ as m varies. Now let $\omega \in \Omega_0$ be arbitrary but fixed, then the set

$$\mathcal{N}_\omega = \{k \in \mathcal{N} : \omega_k \in (r - \frac{1}{\ell}, r + \frac{1}{\ell})\},$$

is of infinite cardinality. The supports of ϕ_k are disjoint by hypothesis 1.1(1) so by hypothesis 1.1(2) there is an $m(k) \in \mathbb{Z}^\nu$ such that the sets

$$\Lambda_{m(k)\frac{1}{2}}(m(k)), \quad k \in \mathcal{N}_\omega$$

are mutually disjoint implying that the size of these sets goes to infinity as k goes to infinity in \mathcal{N}_ω . (Reason: $|m(k)| \rightarrow \infty$ as $|k| \rightarrow \infty$). Hence given $f_{j(\ell)}$ with compact support, we can find a $k_\ell \in \mathcal{N}_\omega$ and an associated $m(k_\ell) \in \mathbb{Z}^\nu$ such that

$$\text{supp}(f_{j(\ell)}(\cdot - m(k_\ell))) \subset \Lambda_{m(k_\ell)\frac{1}{2}}(m(k_\ell)), \quad \phi_{k_\ell} = 1 \text{ on } \text{supp}(f_{j(\ell)}(\cdot - m(k_\ell))).$$

Therefore we have for this $m(k_\ell) \in \mathbb{Z}^\nu$,

$$\begin{aligned} \|H_\lambda^\omega - (E + \lambda r)f_{j(\ell)}(\cdot - m(k_\ell))\| &\leq \|(\Delta - E)f_{j(\ell)}(\cdot - m(k_\ell))\| \\ &\quad + \|(\lambda\omega_{k_\ell}\phi_{k_\ell} - \lambda r)f_{j(\ell)}(\cdot - m(k_\ell))\| \\ &\leq \epsilon + \frac{\lambda}{\ell} \leq (1 + \lambda)\epsilon. \end{aligned}$$

This exhibits a Weyl sequence $g_\ell = f_{j(\ell)}(\cdot - m(k_\ell))$ associated with the operators H_λ^ω for the point $E + \lambda r$ showing that this point is in the essential spectrum of H_λ^ω . This proves the theorem for each $\omega \in \Omega_0$ as we vary $E \in [-2\nu, 2\nu]$ and $r \in \text{supp}(\mu)$. \square

Proof of Theorem 1.2 (2) : We use Mourre theory for proving this. We show that the operator A defined in equation (4) is a local conjugate for H_λ^ω for all ω and $0 \leq \lambda < \lambda_I$.

We first verify Mourre's conditions (1) - (4) given in Definition 3.5.5 of [5], to see that the operator A is a local conjugate of H_λ^ω .

The conditions (1) - (3) of Definition 3.5.5 in [5] are easy in view of the fact that H_λ^ω is a bounded operator, $[A, H_\lambda^\omega] = [A, \Delta] + [A, \lambda V^\omega]$ and $[A, [A, H_\lambda^\omega]] = [A, [A, \Delta]] + [A, [A, \lambda V^\omega]]$ are bounded by a simple computation and by hypothesis 1.1(1) for any $\lambda \geq 0$ and any ω .

Therefore we are left only to verify the Mourre estimate (bound in (4) of Definition 3.5.5 in [5]) to conclude that $\sigma_{sc}(H_\lambda^\omega) = \emptyset$ for these λ, ω from the theorem of Mourre (see theorem 3.5.6, [5]).

Let $P_{H_\lambda^\omega}(I)$ denote the spectral projection of H_λ^ω associated with the interval $I = [a, b] \subset (-2, 2)$. Let $\delta = \min\{1 + a/2, 1 - b/2\}$. Since $0 \in \text{supp}(\mu)$ by the hypothesis 1.1(2), part (1) of the theorem ensures that $(-2, 2) \subset \sigma_{ess}(H_\lambda^\omega)$, so this spectral projection is non-trivial. We will show that the bound

$$P_{H_\lambda^\omega}(I)[A, H_\lambda^\omega]P_{H_\lambda^\omega}(I) \geq 3\delta P_{H_\lambda^\omega}(I) \quad (5)$$

is valid for all $0 \leq \lambda < \lambda_I$ for some λ_I .

We consider a smooth function ψ , which is identically 1 on $I = [a, b]$ and zero outside $(-1 + a/2, 1 + b/2)$. Then it is clear that $\psi = 0$ outside $(-2 + \delta, 2 - \delta)$ for the δ given above. We then have as $\lambda \rightarrow 0$,

$$\begin{aligned} \psi(H_\lambda^\omega)[A, H_\lambda^\omega]\psi(H_\lambda^\omega) &= \psi(\Delta)[A, \Delta]\psi(\Delta) + (\psi(H_\lambda^\omega) - \psi(\Delta))[A, \Delta]\psi(H_\lambda^\omega) \\ &\quad + \psi(\Delta)[A, \Delta](\psi(H_\lambda^\omega) - \psi(\Delta)) \\ &\quad + \psi(H_\lambda^\omega)\lambda \sum_k \omega_k [A, \phi_k]\psi(H_\lambda^\omega) \\ &= \psi(\Delta)[A, \Delta]\psi(\Delta) + O(|\lambda|) \end{aligned} \quad (6)$$

where the last statement follows from Lemma 2.1, the uniform boundedness of

$$\left\| \sum_k \omega_k [A, \phi_k] \right\| \leq E_\infty \left\| [A, \phi_k] \right\|_\infty$$

coming from hypothesis 1.1(1),(2).

Computing the commutator $[A, \Delta]$ we get

$$\psi(\Delta)[A, \Delta]\psi(\Delta) = -\psi(\Delta)^2 \sum_j (T_j - T_j^{-1})^2.$$

Using the Fourier transform we see that the above operator is unitarily equiv-

alent to the operator of multiplication by the function

$$\left(\psi \left(2 \sum_{i=1}^{\nu} \cos(\theta_i) \right) \right)^2 4 \sum_{i=1}^{\nu} \sin^2(\theta_i)$$

on $L^2(\mathbb{T}^{\nu}, d\sigma)$, where σ is the normalized Lebesgue measure on the ν dimensional torus $\mathbb{R}^{\nu}/\mathbb{Z}^{\nu}$.

To estimate this quantity from below, let

$$W = \left\{ \theta \in \mathbb{T}^{\nu} : \psi \left(2 \sum_{i=1}^{\nu} \cos(\theta_i) \right) \neq 0 \right\}$$

Since ψ is zero outside $(-2 + \delta, 2 - \delta)$, we see that

$$W \subset \left\{ \theta \in \mathbb{T}^{\nu} : -1 + \delta/2 < \sum_i \cos(\theta_i) < 1 - \delta/2 \right\}.$$

Then we claim that for every $\theta \in W$, there is an index j such that

$$-1 + \delta/2 \leq \cos \theta_j \leq 1 - \delta/2.$$

Suppose this is not the case and there is a $\theta^0 \in W$ such that

$$\cos(\theta_i^0) > 1 - \delta/2 \text{ or } \cos(\theta_i^0) < -1 + \delta/2, \text{ for all } i = 1, \dots, \nu.$$

Let

$$K_{\pm} = \{i : \pm \cos(\theta_i^0) > 0\} \text{ and } n_{\pm} = \#K_{\pm}.$$

Then either $n_+ > n_-$ or $n_- > n_+$ since the dimension ν is odd. Consider the case $n_+ > n_-$ (the argument for the other case is similar). Then we have

$$\sum_i \cos(\theta_i^0) = \sum_{i \in K_+} \cos(\theta_i^0) + \sum_{i \in K_-} \cos(\theta_i^0) > n_+(1 - \delta/2) - n_- \geq 1 - \delta/2$$

contradicting the fact that $\theta^0 \in W$.

Therefore we have for every $\theta \in W$, for some j_{θ} ,

$$4 \sum_{i=1}^{\nu} \sin^2(\theta_i) \geq 4 \sin^2(\theta_{j_{\theta}}) = 4(1 - \cos^2(\theta_{j_{\theta}})) \geq 4(1 - (1 - \delta/2)^2) \geq 3\delta,$$

since $\delta < 1$. Thus we have as λ goes to zero, again using Lemma 2.1,

$$\psi(\Delta)[A, \Delta]\psi(\Delta) \geq 3\delta\psi(\Delta)^2 = 3\delta\psi(H_\lambda^\omega) + O(\lambda).$$

Putting this inequality together with the inequality (6) we get that for sufficiently small λ ,

$$\psi(H_\lambda^\omega)[A, H_\lambda^\omega]\psi(H_\lambda^\omega) \geq 3\delta\psi(H_\lambda^\omega)^2 + O(\lambda).$$

Now we multiply either side on the inequality by $P_{H_\lambda^\omega}(I)$ and choose λ_δ such that the terms $O(\lambda)$ is smaller than δ for each $\lambda < \lambda_\delta$, to get

$$P_{H_\lambda^\omega}(I)[A, H_\lambda^\omega]P_{H_\lambda^\omega}(I) \geq 2\delta P_{H_\lambda^\omega}(I),$$

obtaining the inequality (5) with $\lambda_I = \lambda_\delta$.

The absence of point spectrum in I follows from the Virial theorem (see a version given in Proposition 2.1, [15], the condition (1) there is easy since $[A, H_\lambda^\omega]$ extends to a bounded operator from $D(A)$ and (2) there holds for all vectors in $\ell^2(\mathbb{Z}^\nu)$ of finite support, which are mapped into domain of A by $(H_\lambda^\omega \pm z)^{-1}$ for $|z| > \|H_\lambda^\omega\|$ as can be seen from a use of Neumann series expansion). \square

3 Examples

We give examples of random potentials that satisfy the hypothesis 1.1, our examples are adapted from the continuum case given in Krishna [15] which are extensions of the Rodnianski-Schlag models [21]. We note that we use below the ℓ^2 -norm on \mathbb{R}^ν while we use the ℓ^∞ norm on \mathbb{Z}^ν , so that

$$|x| = \sqrt{\sum_{i=1}^{\nu} x_i^2}, \quad x \in \mathbb{R}^\nu \text{ and } |n| = \max\{|n_i|, i = 1, \dots, \nu\}, \quad n \in \mathbb{Z}^\nu.$$

Recall the definition

$$\Lambda_r(n) = \{m \in \mathbb{Z}^\nu : |m - n| \leq r\}.$$

We define a function $\phi_{r,n}$ associated with a set $\Lambda_r(n)$ on \mathbb{Z}^ν as follows. Let ϕ be a smooth bump function on \mathbb{R}^ν with (full) support in $\{x : |x| < 1\}$ vanishing on the sphere $\{x : |x| = 1\}$. Let

$$\widetilde{\phi}_{r,n}(x) = \phi\left(\frac{x - n}{r}\right), \quad x \in \mathbb{R}^\nu$$

and let $\phi_{r,n}$ be its restriction to \mathbb{Z}^ν . For any $n \in \mathbb{Z}^\nu$, let $r(n)$ be a positive number satisfying

$$c_1|n| \leq r(n) \leq c_2|n|, \quad 0 < c_1 \leq c_2 < \infty,$$

for any $n \in \mathbb{Z}^\nu$. Then for any n we claim that $[A, \phi_{r(n),n}]$ and $[A, [A, \phi_{r(n),n}]]$ are uniformly bounded in n . We will show the boundedness of the second commutator and the computation for the first commutator becomes clear in the process. We will again show the boundedness of

$$\left[\sum_{j=1}^{\nu} Q_j T_j, \left[\sum_{k=1}^{\nu} Q_k T_k, \phi_{r(n),n} \right] \right],$$

the other terms in the expansion of $[A, [A, \phi_{r(n),n}]]$ are similarly computed and shown to be bounded.

$$\begin{aligned} & \left[\sum_{j=1}^{\nu} Q_j T_j, \left[\sum_{k=1}^{\nu} Q_k T_k, \phi_{r(n),n} \right] \right] \\ &= \sum_{j,k=1}^{\nu} Q_j [T_j, Q_k [T_k, \phi_{r(n),n}]] \\ &= \sum_{j,k=1}^{\nu} Q_j [T_j, Q_k] [T_k, \phi_{r(n),n}] + Q_j Q_k [T_j, [T_k, \phi_{r(n),n}]] \quad (7) \\ &= \sum_{j,k=1}^{\nu} Q_j \delta_{jk} [T_k, \phi_{r(n),n}] + Q_j Q_k [T_j, [T_k, \phi_{r(n),n}]] \\ &= \sum_{k=1}^{\nu} Q_k [T_k, \phi_{r(n),n}] + \sum_{k,j=1}^{\nu} Q_j Q_k [T_j, [T_k, \phi_{r(n),n}]] \end{aligned}$$

We note that the vectors $e_k \in \mathbb{Z}^\nu \subset \mathbb{R}^\nu$, $k = 1, \dots, \nu$ are also unit vectors along the co-ordinate axes and we use this fact without further comment in the calculations below.

Let $u \in \ell^2(\mathbb{Z}^\nu)$, then

$$\begin{aligned} ([T_k, \phi_{r(n),n}]u)(x) &= (\phi_{r(n),n}(x - e_k) - \phi_{r(n),n}(x)) (T_k u)(x) \\ &= \left(\phi\left(\frac{x - n - e_k}{r(n)}\right) - \phi\left(\frac{x - n}{r(n)}\right) \right) (T_k u)(x) \quad (8) \\ &= \left(\frac{-1}{r(n)} \frac{\partial}{\partial t} \phi\left(\frac{x - n}{r(n)} - t_0 e_k\right) \right) (T_k u)(x) \end{aligned}$$

for some $t_0 \in (0, \frac{1}{r(n)})$, using the mean value theorem for the function ϕ on the line segment in the direction of $\{\frac{x-n}{r(n)} - te_k, 0 \leq t \leq \frac{1}{r(n)}\}$ in \mathbb{R}^{ν} . From this we see that

$$\begin{aligned} |Q_k([T_k, \phi_{r(n),n}]u)(x)| &= \left| \frac{x_k - e_k}{r(n)} \left(\frac{1}{r(n)} \frac{\partial}{\partial t} \phi \left(\frac{x-n}{r(n)} - t_0 e_k \right) \right) (T_k u)(x) \right| \\ &\leq C(T_k u)(x), \end{aligned} \quad (9)$$

the bound coming from the uniform bound on the partial derivative of ϕ in the unit ball and from the uniform boundedness of $(x-n+e_k)/r(n)$ coming from the assumption on $r(n)$. Similarly we find using the mean value theorem

$$\begin{aligned} &([T_j, [T_k, \phi_{r(n),n}]u)(x) \\ &= ((\phi_{r(n),n}(x - e_k - e_j) - \phi_{r(n),n}(x - e_j)) \\ &\quad - (\phi_{r(n),n}(x - e_k) - \phi_{r(n),n}(x))) (T_j T_k u)(x) \\ &= \left(\frac{-1}{r(n)} \left(\frac{\partial}{\partial t} \phi \right) \left(\frac{x-n-e_j}{r(n)} - t_0 e_k \right) + \frac{1}{r(n)} \left(\frac{\partial}{\partial t} \phi \right) \left(\frac{x-n}{r(n)} - t_1 e_k \right) \right) (T_j T_k u)(x) \\ &= \left(\left(\frac{-1}{r(n)} \left(\frac{\partial}{\partial t} \phi \right) \left(\frac{x-n}{r(n)} - \frac{1}{r(n)} e_j - t_0 e_k \right) + \frac{1}{r(n)} \left(\frac{\partial}{\partial t} \phi \right) \left(\frac{x-n}{r(n)} - t_0 e_k \right) \right) \right. \\ &\quad \left. + \left(\frac{-1}{r(n)} \left(\frac{\partial}{\partial t} \phi \right) \left(\frac{x-n+}{r(n)} - t_0 e_k \right) + \frac{1}{r(n)} \left(\frac{\partial}{\partial t} \phi \right) \left(\frac{x-n}{r(n)} - t_1 e_k \right) \right) \right) (T_j T_k u)(x), \end{aligned} \quad (10)$$

for some $t_0, t_1 \in (0, \frac{1}{r(n)})$. Applying the mean value theorem one more time we get

$$\begin{aligned} &([T_j, [T_k, \phi_{r(n),n}]u)(x) \\ &= \left(\frac{1}{r(n)^2} \left(\frac{\partial^2 \phi}{\partial t \partial s} \left(\frac{x-n}{r(n)} - s_1 e_j - t_0 e_k \right) \right) \right. \\ &\quad \left. + \frac{(t_0 - t_1)}{r(n)} \left(\frac{\partial^2 \phi}{\partial t^2} \left(\frac{x-n}{r(n)} - t_2 e_k \right) \right) \right) (T_j T_k u)(x), \end{aligned} \quad (11)$$

where $t_2 \in (t_1, t_0)$ (taking w.l.g. $t_1 < t_0$) with the notation that the variable s, t denotes taking derivatives along the directions e_k, e_j respectively. Denoting below these partial derivatives by $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial x_k}$ respectively we get the bound

$$\|([T_j, [T_k, \phi_{r(n),n}]u)\| \leq \frac{2}{r(n)^2} \left(\left\| \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} (\phi) \right\|_{\infty} + \left\| \frac{\partial^2}{\partial x_k^2} (\phi) \right\|_{\infty} \right),$$

where we used the fact that $|t_1 - t_0| \leq 2/r(n)$. From this bound and noting that the coordinates x_j, x_k are bounded in modulus by $cr(n)$ when $x = (x_1, \dots, x_\nu)$ is in the support of $\phi_{r(n),n}$ and its derivatives. Therefore the quantity

$$(Q_j Q_k [T_j, [T_k, \phi_{r(n),n}]]u)(x) = x_j x_k [T_j, [T_k, \phi_{r(n),n}]]u(x)$$

has the bound

$$\|(Q_j Q_k [T_j, [T_k, \phi_{r(n),n}]]u)\| \leq c^2 r(n)^2 \frac{2}{r(n)^2} (\|\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k}(\phi)\|_\infty + \|\frac{\partial^2}{\partial x_k^2}(\phi)\|_\infty) \|u\|.$$

This gives the boundedness of $[A, [A, \phi_{r(n),n}]]$ uniformly in n .

Using these facts we construct a random potential as follows. Let M be a large integer and let $A_0 = \{m \in \mathbb{Z}^\nu : |m| \leq M\}$ and consider the annuli

$$A_{k+1} = \{m \in \mathbb{Z}^\nu : 2^k M < |m| \leq 2^{k+1} M\}, \quad k = 0, 1, \dots$$

We define

$$r(m) = 2^{k-2} M, \quad \forall m \in A_k, \quad k = 1, 2, \dots,$$

to get the uniform bounds

$$4 \leq \frac{2^k M}{2^{k-2} M} \leq \frac{|n|}{r(n)} \leq \frac{2^{k+1} M}{2^{k-2} M} \leq 8, \quad \text{for all } n \in A_k, \quad k = 1, 2, \dots$$

Then consider the sets \mathcal{N}_k , $k = 1, 2, \dots$, such that

$$\mathcal{N}_k \subset A_k, \quad n, m \in \mathcal{N}_k \implies \Lambda_{r(n)}(n) \cap \Lambda_{r(m)}(m) = \emptyset.$$

It is clear that the cardinality of \mathcal{N}_k is at least 2ν , since one can fit a cube $\Lambda_{r(n)}(n)$ alongside of each face of the cube $\Lambda_{2^k M}(0)$ in A_{k+1} .

Let $\mathcal{N} = \cup_{k=1}^\infty \mathcal{N}_k$ and let $\{\omega_\ell : \ell \in \mathcal{N}\}$ be i.i.d. random variables with distribution μ . Then by the definition of A_k and $r(n)$ it is clear that $c_1 |n| \leq r(n) \leq c_2 |n|$ is valid. The random potential

$$V^\omega = \sum_{n \in \mathcal{N}} \omega_n \phi_{r(n),n}$$

then satisfies the hypothesis (1.1).

The distribution μ can be atomic, singularly continuous or absolutely continuous or a mixture of all these.

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