New criteria to identify spectrum

A Jensen
Department of Mathematical Sciences and MaPhySto*
Aalborg University, Fr. Bajers Vej 7G
DK-9220 Aalborg Ø, Denmark
and
M Krishna
Institute of Mathematical Sciences
Taramani, Chennai 600 113, India

July 14, 2004

Abstract

In this paper we give some new criteria for identifying the components of a probability measure, in its Lebesgue decomposition. This enables us to give new criteria to identify spectral types of self adjoint operators on Hilbert spaces, especially those of interest.

1 Introduction

In the spectral theory of self adjoint operators it is of interest to identify the type of the spectrum. This problem is equivalent to identifying the components of the spectral measures. The components of a probability measure can be identified via a transform of the measure. Two of these are well known, viz. the Fourier transform and the Borel transform. In this paper we address the question of identifying the components using a more general transform. We give results using a general approximate identity, and an associated continuous wavelet transform.

Concerning the literature, the connection between an approximate identity and the continuous wavelet transform was discussed in the book by

*MaPhySto—A Network in Mathematical Physics and Stochastics, funded by the Danish National Research Foundation

Holschneider [1], while wavelet coefficients of fractal measures were studied by Strichartz in [4]. In the theory of selfadjoint operators finer decomposition of spectra with respect to Hausdorff measures was first used by Last [2] and general criteria for recovering a measure from its Borel transform was done by Simon [3].

2 The criteria

We need to introduce conditions on our function ψ. Several of these can be relaxed in some of the results. We use the standard notation $\langle x\rangle = (1+x^2)^{1/2}$.

Assumption 2.1. *Assume that $\psi \in C^1(\mathbf{R})$, $\psi(0) = 1$, ψ is even, and there exist $C > 0$ and $\delta > 1$, such that*

$$|\psi(x)| + |x\psi'(x)| \leq C\langle x\rangle^{-\delta}, \quad x \in \mathbf{R}. \tag{2.1}$$

We set $A_\psi = \int_{\mathbf{R}} \psi(x)\, dx$ and assume that $A_\psi \neq 0$.

In the sequel we always impose this assumption on ψ. We introduce the notation

$$\psi_a(x) = \psi(x/a) \quad \text{and} \quad \tilde{\psi}_a(x) = \frac{1}{a}\psi_a(x), \quad a > 0. \tag{2.2}$$

In particular, the family $\{A_\psi^{-1}\tilde{\psi}_a\}$ is an approximate identity. Let μ be a probability measure on $\mathbf{R}$ in what follows, with Lebesgue decomposition $\mu = \mu_{\mathrm{s}} + \mu_{\mathrm{ac}}$. Let f be a function. We recall that the convolution $(f*\mu)(x) = \int f(x-y)d\mu(y)$ is defined, when the integral converges. Since ψ is bounded, the convolution $\psi_a * \mu$ is defined for all $a > 0$.

For $0 \leq \alpha \leq 1$ we define

$$(d_\alpha\mu)(x) = \lim_{\varepsilon\downarrow 0} \frac{\mu((x-\varepsilon, x+\varepsilon))}{(2\varepsilon)^\alpha}, \tag{2.3}$$

whenever the limit on the right hand side exists.

We can now state the results. We first give results based on ψ_a and $\tilde{\psi}_a$, and then on an associated continuous wavelet transform.

Theorem 2.2. *Let μ be a probability measure. Then*

1. *Let ψ satisfy Assumption 2.1. Then for every continuous function f of compact support, the following is valid.*

$$\lim_{a\to 0} \int (\tilde{\psi}_a * \mu)(x) f(x) dx = A_\psi \int f(x) d\mu(x).$$

2. $\lim_{a\to 0}(\psi_a * \mu)(x) = \mu(\{x\})$.

3. *Assume* $0 < \alpha \le 1$ *and* $(d_\alpha\mu)(x)$ *finite. Then we have*

$$\lim_{a\to 0} a^{-\alpha}(\psi_a * \mu)(x) = c_\alpha (d_\alpha \mu)(x), \tag{2.4}$$

where $c_\alpha = \int_0^\infty \alpha 2^\alpha y^{\alpha-1}\psi(y)dy$.

Remark 2.3. (1) Equation (2.4) implies that if μ is purely singular, then the limit of $\tilde{\psi}_a * \mu(x)$ is zero almost everywhere with respect to the Lebesgue measure, since the derivative $(d_1\mu)(x) = 0$ almost everywhere for purely singular μ.
(2) If x is not in the topological support of μ, then for each $0 \le \alpha \le 1$,

$$\lim_{a\to 0} a^{-\alpha}\psi_a * \mu(x) = 0.$$

Our next theorem is a bit more and the first part is analogous to Wiener's theorem and its extension by Simon [3].

Theorem 2.4. *Let* μ *be a probability measure. Then for any bounded interval* (c, d) *the following are valid.*

1. *Let*

$$C = \int_{\mathbf{R}} |\psi(x)|^2 dx,$$

then

$$\lim_{a\to 0}\frac{1}{a}\int_c^d |(\psi_a * \mu)(x)|^2 dx = C\Big(\sum_{x\in(c,d)} \mu(\{x\})^2 + \tfrac{1}{2}\left[\mu(\{c\})^2 + \mu(\{d\})^2\right]\Big). \tag{2.5}$$

2. *For* $0 < p < 1$, *we have*

$$\lim_{a\to 0}\int_c^d |(\tilde{\psi}_a * \mu)(x)|^p dx = |A_\psi|^p \int_c^d \Big|\frac{d\mu_{\text{ac}}}{dx}(x)\Big|^p dx. \tag{2.6}$$

This theorem has the following corollary.

Corollary 2.5. *Let* μ *be a probability measure. Then we have the following results*

1. *μ has no point part in $[c, d]$, if and only if*

$$\liminf_{a\to 0} \frac{1}{a}\int_c^d |(\psi_a * \mu)(x)|^2 dx = 0. \tag{2.7}$$

2. *If μ has no absolutely continuous part in (c, d), if and only if for some p, $0 < p < 1$,*

$$\liminf_{a\to 0} \int_c^d |(\tilde{\psi}_a * \mu)(x)|^p dx = 0. \tag{2.8}$$

Now to state the results in terms of the continuous wavelet transform, we introduce

$$h(x) = \psi(x) + x\psi'(x). \tag{2.9}$$

Under Assumption 2.1 we clearly have

$$|h(x)| \le C\langle x\rangle^{-\delta}, \tag{2.10}$$

with the δ from the assumption. Integration by parts and (2.9) imply that h satisfies the admissibility condition for a continuous wavelet, i.e. $\int_{-\infty}^{\infty} h(x)dx = 0$.

Thus we can define the continuous wavelet transform of a probability measure μ as

$$W_h(\mu)(b, a) = \frac{1}{a}\int_{-\infty}^{\infty} h((b-y)/a)d\mu(y). \tag{2.11}$$

The connection between the approximate identity and this transform is

$$-a\frac{\partial}{\partial a}(\tilde{\psi}_a * \mu)(b) = W_h(\mu)(b, a). \tag{2.12}$$

This result follows from

$$-a\frac{\partial}{\partial a}(\frac{1}{a}\psi(\frac{x}{a})) = \frac{1}{a}\Big(\psi(\frac{x}{a}) + \frac{x}{a}\psi'(\frac{x}{a})\Big),$$

and the definitions.

We have the following analogue of Theorem 2.2:

Theorem 2.6. *Let μ be a probability measure. Then we have the following results:*

1. *We have*

$$\lim_{\varepsilon\downarrow 0} \varepsilon\int_\varepsilon^\infty W_h(\mu)(b, a)\frac{da}{a} = \mu(\{b\}). \tag{2.13}$$

2. Let $0 < \alpha \leq 1$. Assume that $(d_\alpha \mu)(b)$ exists. Then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{1-\alpha} \int_\varepsilon^\infty W_h(\mu)(b,a) \frac{da}{a} = c_\alpha (d_\alpha \mu)(b), \tag{2.14}$$

where c_α was defined in Theorem 2.2.

Remark 2.7. We note that for $0 < \alpha < 1$ we can replace $\int_\varepsilon^\infty$ by $\int_\varepsilon^M$ for any $M > 0$. See the proof of the Theorem.

We also have the following analogue of Theorem 2.4(1).

Theorem 2.8. *Let μ be a probability measure. Then for any bounded interval (c,d) we have the following result. Let*

$$C_h = \int_{\mathbf{R}} |h(x)|^2 dx,$$

Then we have

$$\lim_{a \downarrow 0} \int_c^d |W_h(\mu)(b,a)|^2 db = C_h \Big(\sum_{x \in (c,d)} \mu(\{x\})^2 + \tfrac{1}{2} \big(\mu(\{c\})^2 + \mu(\{d\})^2 \big) \Big). \tag{2.15}$$

Even when the quantity $(d_\alpha \mu)(x)$ does not exist, it is possible to say something on the wavelet transforms, to cover the cases of measures which are not supported on the sets where such limits exist. Set

$$C^\alpha_{\mu,\psi}(x) = \limsup_{a \to 0} \frac{\psi_a * \mu}{a^\alpha}(x) \quad \text{and} \quad D^\alpha_\mu(x) = \limsup_{\varepsilon \to 0} \frac{\mu((x-\varepsilon, x+\varepsilon))}{(2\varepsilon)^\alpha}.$$

Then we have

Theorem 2.9. *Let μ be a probability measure, and let ψ satisfy Assumption 2.1. Then $C^\alpha_{\mu,\psi}(x)$ is finite for any x, whenever $D^\alpha_\mu(x)$ is finite for the same x, and, if ψ is non-negative, they are both finite or both infinite.*

Remark 2.10. The above theorem implies that if $\limsup_{a \to 0} |(\tilde{\psi}_a * \mu)(x)| < \infty$ for all $x \in (c,d)$, then there is no singular part of μ supported in (c,d).

Finally as an application of the above theorems we consider $\mathcal{H}$ to be a separable Hilbert space and A a selfadjoint operator. Then

Theorem 2.11. *Suppose A is a selfadjoint operator on $\mathcal{H}$. Consider a function ψ satisfying Assumption 2.1. Then*

1. *λ is in the point spectrum of A, if for some $f \in \mathcal{H}$, $\|f\| = 1$,*

$$\lim_{a\to 0}\langle f, \psi_a(A-\lambda)f\rangle = 0.$$

2. *Let $B \subset \mathbf{R}$ be a Borel set of positive Lebesgue measure. Then $B \cap \sigma_{ac}(A) \neq \emptyset$, if for some $f \in \mathcal{H}$, $\|f\| = 1$,*

$$\lim_{a\to 0}\langle f, \tilde{\psi}_a(A-\lambda)f\rangle \neq 0, \quad \textit{for a.e. } \lambda \in B.$$

3. *The point spectrum of A in (c,d) is empty, if and only if for some orthonormal basis $\{f_n\}$, of $\mathcal{H}$, one has for every n,*

$$\liminf_{a\to 0}\frac{1}{a}\int_c^d |\langle f_n, \psi_a(A-\lambda)f_n\rangle|^2 d\lambda = 0.$$

4. *The absolutely continuous spectrum of A in (c,d) is empty, if and only if for some orthonormal basis $\{f_n\}$ of $\mathcal{H}$, one has for every n and some $0 < p < 1$,*

$$\liminf_{a\to 0}\int_c^d \big|\frac{1}{a}\langle f_n, \psi_a(A-\lambda)f_n\rangle\big|^p d\lambda = 0.$$

3 Proofs

Throughout the computations below the letter C denotes a constant, whose value may vary from line to line.

Proof of Theorem 2.2: Part (1): Since f is a continuous function of compact support and ψ_a is bounded for each $a > 0$, $f(x)\psi_a(x-y)$ is absolutely integrable and the integral is uniformly bounded in $y \in \mathbf{R}$. Therefore, by an application of Fubini, a change of variable $x \to ax + y$ and dominated convergence theorem, in that order, it follows that

$$\begin{aligned}
\lim_{a\to 0}\int dx\, f(x)\big(\tilde{\psi}_a * \mu\big)(x) &= \lim_{a\to 0}\int dx\, f(x)\int \tilde{\psi}_a(x-y)d\mu(y) \\
&= \lim_{a\to 0}\int d\mu(y)\int f(x)\tilde{\psi}_a(x-y)dx \\
&= \lim_{a\to 0}\int d\mu(y)\int f(ax+y)\psi(x)dx \\
&= \int d\mu(y)\int (\lim_{a\to 0} f(ax+y))\psi(x)dx \\
&= \int f(y)d\mu(y)\cdot\int \psi(x)dx.
\end{aligned}$$

Part (2): This is a direct consequence of the definition of the integral noting that pointwise we have

$$\lim_{a\to 0} \psi_a(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

We also need to use the dominated convergence theorem to interchange the limit and the integral.
Part (3): Let Φ_μ denote the distribution function of μ. Then we have

$$\begin{aligned} \frac{1}{a^\alpha} \int_{\mathbf{R}} & \psi_a(x-y) d\mu(y) \\ &= -\frac{1}{a^\alpha} \int_{\mathbf{R}} \frac{d}{dy} \psi((x-y)/a) \Phi_\mu(y) dy \\ &= \frac{1}{a^\alpha} \int_{\mathbf{R}} \psi'(y) \Phi_\mu(x-ay) dy \\ &= -\int_0^\infty \psi'(y)(2y)^\alpha \frac{\Phi_\mu(x+ay) - \Phi_\mu(x-ay)}{(2ay)^\alpha} dy, \end{aligned} \tag{3.1}$$

where in the first step we used integration by parts, the next step changed variables and in the last step used the oddness of ψ' to split the integral into the positive and negative half lines and multiplied by appropriate powers.

We observe that

$$(d_\alpha \mu)(x) = \lim_{a\to 0} \frac{\Phi_\mu(x+ay) - \Phi_\mu(x-ay)}{(2ay)^\alpha}$$

for each $y \in \mathbf{R}$, and is finite by assumption. Furthermore, the function $(\Phi_\mu(x+ay) - \Phi_\mu(x-ay))(2ay)^{-\alpha}$ is a bounded measurable function, such that we due to (2.1) we can take the limits inside the integral sign in (3.1) and use the dominated convergence theorem.

Now doing an integration by parts gives the value of the integral as stated in the theorem.

Proof of Theorem 2.4: Part (1): We have

$$\frac{1}{a} \int_c^d |\psi_a * \mu(x)|^2 dx = \iint d\mu(y_1)\, d\mu(y_2) \int_c^d dx \frac{1}{a} \overline{\psi_a(x-y_1)} \psi_a(x-y_2).$$

Since the function ψ_a is bounded, the interval (c,d) is bounded, and μ is a probability measure, the right hand side integral converges absolutely, so we used Fubini to interchange integrals to get the equality above. Let

$$h_a(y_1, y_2) = \int_c^d dx\ \frac{1}{a} \overline{\psi_a(x-y_1)} \psi_a(x-y_2).$$

Suppose $y_1 \neq y_2$, then using the bound $|\psi(x)| \leq C\langle x\rangle^{-\delta}$, we see that the bound

$$
\begin{aligned}
|h_a(y_1,y_2)| &\leq \frac{C}{a}\int_{-\infty}^{\infty}\langle (x+y_2-y_1)/a\rangle^{-\delta}\langle x/a\rangle^{-\delta}dx \\
&= \frac{C}{a}\Big(\int_{|x|\leq|y_1-y_2|/2} + \int_{|x|\geq|y_1-y_2|/2}\Big)(\cdots)dx \\
&\leq \frac{Ca^{\delta}}{|y_1-y_2|^{\delta}}\int_{-\infty}^{\infty}\langle x/a\rangle^{-\delta}d(x/a) \\
&\leq \frac{Ca^{\delta}}{|y_1-y_2|^{\delta}}
\end{aligned}
$$

is valid. It follows that $\lim_{a\to 0} h_a(y_1,y_2) = 0$ for $y_1 \neq y_2$. It remains to consider $y_1 = y_2$. This is done by noting that

$$
h_a(y_1,y_1) = \int_c^d \frac{1}{a}|\psi_a(x-y_1)|^2 dx = \int_{(c-y_1)/a}^{(d-y_1)/a} |\psi(x)|^2 dx,
$$

from which taking limits, we obtain the stated value for the coefficient, either C or $C/2$, based on whether $c < y_1 < d$ or $y_1 = c, d$, using the evenness of ψ. Now to complete the proof, we note the estimate

$$
|h_a(y_1,y_2)| \leq C\int_{\mathbf{R}}\langle x/a\rangle^{-\delta}d(x/a) \leq C_0,
$$

where the constant C_0 is independent of a, y_1, and y_2. Thus the proof is completed used the dominated convergence theorem.

Part (2): We adapt the arguments in [3] to the case at hand. We split the measure in three components: $\mu = \mu_1+\mu_2+\mu_3$. Here $d\mu_1 = (1-\chi_{[c-1,d+1]})d\mu$, $d\mu_2 = g dx$ with $g \in L^1([c-1,d+1])$, and μ_3 is purely singular, and supported on $[c-1,d+1]$. We have for $x \in [c,d]$ the estimate

$$
|(\tilde{\psi}_a * \mu_1)(x)| \leq C\int_{\mathbf{R}\setminus[c-1,d+1]} a^{-1}\langle (x-y)/a\rangle^{-\delta}d\mu_1(y) \leq Ca^{\delta-1}.
$$

We now look at the μ_2 part. We have, for $0 < p < 1$, by the reverse Hölder inequality

$$
\int_c^d \big|(\tilde{\psi}_a * g)(x) - A_\psi g(x)\big|^p dx \leq \Big(\int_c^d \big|(\tilde{\psi}_a * g)(x) - A_\psi g(x)\big|dx\Big)^p (d-c)^{1-p},
$$

which implies that $\tilde{\psi}_a * g \to A_\psi g$ in $L^p((c,d))$, $0 < p \leq 1$.

Now we will show that the singular part μ_3 does not contribute to the limit. So assume that μ_3 is purely singular and that its support S is contained in $[c-1, d+1]$. Since μ_3 is singular, by the definition of support, S satisfies $\mu_3(\mathbf{R} \setminus S) = 0$ and $|S| = 0$, with $|\cdot|$ denoting the Lebesgue measure. By the regularity of the Lebesgue measure, given an $\varepsilon > 0$, there is an open set $O \subset (c-2, d+2)$, such that $S \subset O$, with $|O \setminus S| < \varepsilon$. We also have $|O| \leq |O \setminus S| + |S| < \varepsilon$. For the same ε, since the measure μ_3 is regular, we also have a compact $K \subset S$, such that $\mu_3(S \setminus K) < \varepsilon$. In addition, since $K \subset S$, and S has Lebesgue measure zero, K also has Lebesgue measure zero.

The above reverse Hölder inequality gives

$$\begin{aligned}\int_c^d \big|(\tilde{\psi}_a * \mu_3)(x)\big|^p dx &= \int_O \big|(\tilde{\psi}_a * \mu_3)(x)\big|^p dx + \int_{(c,d)\setminus O} \big|(\tilde{\psi}_a * \mu_3)(x)\big|^p dx \\ &\leq |O|^{1-p} \mu_3((c,d))^p \|\psi\|_1^p \\ &\quad + |d-c|^{1-p} \Big(\int_{(c,d)\setminus O} \big|(\tilde{\psi}_a * \mu_3)(x)\big| dx \Big)^p \\ &\leq C\varepsilon^{1-p} + |d-c|^{1-p} \Big(\int_{(c,d)\setminus O} \big|(\tilde{\psi}_a * \mu_3)(x)\big| dx \Big)^p.\end{aligned}$$

Now consider a bounded continuous function h which is 1 on $(c,d) \setminus O$, and 0 on K.

Then using Assumption 2.1, that $|\psi(x)| \leq C\langle x\rangle^{-\delta}$, and setting $\phi(x) = \langle x\rangle^{-\delta}$,

$$\begin{aligned}\int_{(c,d)\setminus O} \big|(\tilde{\psi}_a * \mu_3)(x)\big| dx &\leq \int_{(c,d)\setminus O} \frac{1}{a} \int_{\mathbf{R}} |\psi_a(x-y)| d\mu_3(y) dx \\ &\leq C \int_{(c,d)\setminus O} \frac{1}{a} \int_{\mathbf{R}} \langle (x-y)/a\rangle^{-\delta} d\mu_3(y) dx \\ &\leq C \int_{(c,d)\setminus O} h(x) (\tilde{\phi}_a * \mu_3)(x) dx.\end{aligned}$$

The function ϕ satisfies Assumption 2.1, so the Theorem 2.2(1) is applicable with ψ replaced by ϕ there. Therefore the last term, which has positive integrand, converges to $\int_{(c,d)\setminus O} h(x) d\mu(x)$ as a goes to zero, which is bounded by $\int_{(c,d)\setminus K} d\mu(x)$,

$$\int_{(c,d)\setminus O} h(x) d\mu(x) \leq \mu((c,d) \setminus K) \leq \mu((c,d) \setminus S) + \mu(S \setminus K) < \varepsilon,$$

using the facts that $\mu((c,d) \setminus S) = 0$ and $\mu(S \setminus K) < \varepsilon$.

Using the inequality $(a+b+c)^p \leq a^p + b^p + c^p$ for $0 < p < 1$ and non-negative numbers a, b, c, we have

$$\begin{aligned}\int_c^d |(\tilde{\psi}_a * \mu)(x) - A_\psi g(x)|^p dx \leq & \int_c^d |(\tilde{\psi}_a * \mu_1)(x)|^p dx \\ & + \int_c^d |(\tilde{\psi}_a * \mu_2)(x) - A_\psi g(x)|^p dx \\ & + \int_c^d |(\tilde{\psi}_a * \mu_3)(x)|^p dx\end{aligned}$$

Putting the above estimates together and using that ε is arbitrary, one gets

$$\lim_{a\to 0} \int_c^d |(\tilde{\psi}_a * \mu)(x) - A_\psi g(x)|^p dx = 0.$$

Now the spaces $L^p((c,d))$, $0 < p < 1$, are metric spaces with the metric $d(f,g) = \|f-g\|_p^p$. It then follows from the triangle inequality for this metric that

$$\lim_{a\to 0} \int_c^d \left|(\tilde{\psi}_a * \mu)(x)\right|^p dx = |A_\psi|^p \int_c^d |g(x)|^p dx.$$

Since $g = \frac{d\mu_{\mathrm{ac}}}{dx}$, the result follows.

Proof of Theorem 2.6: Let $0 < \varepsilon < M < \infty$. It follows from (2.12) that we have

$$\int_\varepsilon^M W_h(\mu)(b,a)\frac{da}{a} = (\tilde{\psi}_\varepsilon * \mu)(b) - (\tilde{\psi}_M * \mu)(b).$$

The results now follow from Theorem 2.2. □

Proof of Theorem 2.8: The proof is entirely analogous to the proof of Theorem 2.4, replacing ψ by h and adjusting the powers of a. □

Proof of Theorem 2.9: Consider the case when $D^\alpha_\mu(x)$ is finite for some x and for some fixed α. Then for any $0 < y < 1$, $\mu(x-y, x+y) \leq C|y|^\alpha$ for some finite constant C. So, using the last line in equation 3.1 and estimating the right hand side there, one has, by assumption 2.1,

$$\left|\frac{1}{a^\alpha}(\psi_a * \mu)(x)\right| \leq C \int_0^\infty |\psi'(y)|(2y)^\alpha dy \leq C \int_0^\infty \langle y\rangle^{-\delta}|y|^{-1+\alpha} dy < \infty.$$

Now taking the lim sup of the left hand side the finiteness of $C^\alpha_{\mu,\psi}$ follows.

On the other hand, since ψ is positive continuous with $\psi(0) = 1$, there is a $\beta > 0$ such that $\psi(y) > 1/2, \ -\beta < y < \beta$. Using this and the evenness of ψ,

$$\begin{aligned}
\frac{1}{a^\alpha}(\psi_a * \mu)(x) &= \frac{1}{a^\alpha}\int \psi_a(x-y)d\mu(y) = \int \psi(y/a)d\mu(y+x) \\
&\geq \frac{1}{a^\alpha}\int_{-\beta a}^{\beta a} \tfrac{1}{2} d\mu(y+x) \\
&\geq \frac{1}{2a^\alpha}[\mu(x+a\beta) - \mu(x-a\beta)],
\end{aligned}$$

where $\psi \geq 0$ is used to get the first inequality above. The above inequalities immediately imply, since β is fixed, that $D_\mu^\alpha(x) = \infty$ implies the same for $C_{\mu,\psi}^\alpha(x)$. □

Proof of Theorem 2.11: Parts (1) and (2) are a direct application of Theorem 2.2(2) and (3) respectively. Parts (3) and (4) are a direct application of Corollary 2.5 (1) and (3) respectively.

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