QUANTUM GAUSSIAN CHANNELS

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Objectives:

1. Motivate Gaussian Channels
   in Quantum Information Theory

2. Describe some recent results by
   Solomon Ivan & Krishna Kumar
   of this Institute

System $S \longleftrightarrow$ Hilbert space $\mathcal{H}_S$, or $\mathcal{H}$

States are $\rho \in \mathcal{B}(\mathcal{H})$ with $\rho \geq 0, \quad \mathrm{tr}\,\rho = 1$

A state ρ assigns value to observable B through the pairing $(\rho, B) = \mathrm{tr}(\rho B)$

Observables are self-adjoint $\Longleftrightarrow$ values are real

Unit vectors $|\psi\rangle \in \mathcal{H} \longleftrightarrow$ pure states $\rho = |\psi\rangle\langle\psi|$.

Pure states are one-dimensional projections; others are **mixed states**.

State space Ω_S is the collection of all states of S.

Ω is **convex**: $\quad \rho_1,\, \rho_2 \in \Omega \Rightarrow p\rho_1 + (1-p)\rho_2 \in \Omega_S$, for all $0 < p < 1$

Pure states, and only pure states, are the **extremals** of Ω.

Composite systems:

Composite system $A+S \;\longleftrightarrow\; \mathcal{H}_{AS}=\mathcal{H}_A\otimes\mathcal{H}_S$

Reduced states of subsystems are partial traces:
$\rho^A=\mathrm{tr}_S(\rho^{AS}),\ \rho^B=\mathrm{tr}_A(\rho^{AS})$

ρ^{AS} pure state $\not\Rightarrow$ $\rho^A,\ \rho^S$ pure.
This is one consequence of entanglement

Given a bipartite mixed state ρ^{AS} of $A+S$,
it can be written as an **ensemble**, or convex sum,
of pure states

$$\rho^{AS}=\sum_j p_j|\psi_j^{AS}\rangle\langle\psi_j^{AS}|,\ p_j>0,\ \sum_j p_j=1$$

in **innumerable** number of ways.

Too many extremals. No Caratheodory!

$|\psi_j^{AS}\rangle$'s are not required to be orthogonal or linearly independent.

ρ^{AS} is **separable** iff we can find an ensemble

$$\rho^{AS} = \sum_j p_j \, \xi_j^A \otimes \zeta_j^S, \;\; p_j > 0, \;\; \sum_j p_j = 1$$

where ξ_j^A and ζ_j^S are respectively states of A, S.

If ρ^{AS} is not separable, it is **entangled**.

Separable states form a convex subset $\Omega^{(\mathrm{sep})} \subset \Omega$.

Checking separability of a given mixed state remains an open problem for $\dim \mathcal{H}_A = \dim \mathcal{H}_S \geq 3$.

Maps and Physical Processes or Channels:

Quantum evolution is basically unitary:

$$\rho \to \rho(t) = U(t)\rho U(t)^\dagger$$

Unitary evolution on a composite system may not appear unitary on subsytems. It is linear, though.

$$\rho_S \to \rho_S(t) = \mathrm{tr}_A \left(U_{AS}(|0\rangle_A {}_A\langle 0| \otimes \rho_S) U_{AS}^\dagger \right)$$

$|0\rangle_A {}_A\langle 0|$ is a reference state of the ancilla A.
We ask: What is the most general linear map Φ on state space Ω, permitted within the quantum theory.

A map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is positive iff $\Phi(\rho) \geq 0$ for all $\rho \geq 0$.

Positive maps on system S form a convex cone.
But the extremals are not known for $\dim \mathcal{H}_S \geq 3$.

Clearly, a physical map Φ_S should be positive. But positivity of Φ_S does not imply that the 'trivial' extension $\mathrm{Id}_A \otimes \Phi_S$ is positive.

A positive map Φ_S is **completely positive** if the extension $\mathrm{Id}_A \otimes \Phi_S$ is positive, for every n where n is the dimension of $\mathcal{H}_A$, the ancilla Hilbert space.

For any collection of operators $W_k \in \mathcal{B}(\mathcal{H}_S)$ the map

$$\Phi_S : \rho_S \to \rho_S' = \sum_k W_k \rho_S W_K^\dagger$$

is manifestly CP. The good news is: All CP maps are of this form. Thus the extremals of the convex cone of CP maps are of the form

$$\rho_S \to \rho_S' = W \rho_S W^\dagger, \quad W \in \mathcal{B}(\mathcal{H}_S).$$

Since physical processes have to preserve trace of ρ, we conclude trace-preserving CP maps (CP-T) correspond to physical processes.

Λ^{CP-T} — trace-preserving CP maps

Λ^{CP-U} — unital or identity-preserving CP maps

Λ^{CP-TU} — trace-preserving unital CP maps. Also called doubly stochastic maps.

Refering to the operator-sum representation

$$\Phi_S : \rho_S \rightarrow \rho_S' = \sum_k W_k \, \rho_S \, W_K^\dagger$$

,

$\Sigma_k \, W_k^\dagger W_K = \mathrm{Id}$, for Λ^{CP-T}

$\Sigma_k \, W_k W_K^\dagger = \mathrm{Id}$, for Λ^{CP-U}

$\Sigma_k \, W_k^\dagger W_K = \mathrm{Id}$ & $\Sigma_k \, W_k W_K^\dagger = \mathrm{Id}$, for Λ^{CP-TU}

Λ^{CP-TU} is the intersection of Λ^{CP-T} and Λ^{CP-U}

The extremals of the convex sets Λ^{CP-T}, Λ^{CP-U}, and Λ^{CP-TU} are not known for $n \geq 3$

Convex sum of unitary maps is doubly stochastic.

The converse, Birkhoff theorem, true only for $n = 2$.

Gaussian States

Oscillator, hermitian position-momentum operators q, p, annihilation creation operators a, $a^\dagger$:

$$a = \frac{1}{\sqrt{2}}(q+ip), \quad [q,\, p] = qp - pq = i, \quad [a,\, a^\dagger] = 1.$$

$H = (aa^\dagger + a^\dagger a)/2$, vacuum $|0\rangle$: $\quad a|0\rangle = 0$.

Displacement operator (in phase space):

$\alpha = (\alpha_1 + \alpha_2)/\sqrt{2}$

$$\begin{aligned} D(\alpha) &= \exp(\alpha a^\dagger - \alpha^* a) = \exp\{-i(\alpha_1 p - \alpha_2 q)\} \\ &= \exp(i\alpha_1\alpha_2/2)\exp(-i\alpha_1 p)\exp(i\alpha_2 q)\} \\ &= \exp(-i\alpha_1\alpha_2/2)\exp(i\alpha_2 q)\exp(-i\alpha_1 p)\} \end{aligned}$$

Coherent state: $|\alpha\rangle = D(\alpha)|0\rangle$, one coherent state for each complex number α. $\; a|\alpha\rangle = \alpha|\alpha\rangle$.

Squeeze operator: $S(\xi) = \exp(\xi a^2 - \xi^* a^{\dagger\,2})$

Coherent states are Gaussian states.
The unitary operators $S(\xi)$, $\exp(-i\,tH)$, and $D(\alpha)$ take Gaussian states to Gaussians. All Gaussian pure states are obtained by the action of these — the semidirect product of the symplectic and Weyl groups — on the vacuum state.

$$D(\alpha)D(\beta) = \exp[-(\alpha\beta^* - \beta\alpha^*)/2]D(\alpha+\beta)$$

.

$$\mathrm{tr}(\,D^\dagger(\alpha)D(\beta)\,) = \pi\delta(\alpha)$$

.

Given a state, density operator ρ, compute the **characteristic function** $\chi_W(\alpha) = \mathrm{tr}(\,\rho D(\alpha)\,)$.

Construct the Wigner phase space distribution through Fourier transformation:

$$W(\alpha) = \pi^{-1} \int d^2\beta \exp(\alpha\beta^* - \alpha^*\beta)\chi_W(\beta)$$
$$= (2\pi)^{-1} \int d\beta_1 d\beta_2 \exp[i(\alpha_2\beta_1 - \alpha_1\beta_2)]\,\chi_W(\beta)$$

Completeness of the displacement operators, and invertibility of F.T. operation imply that ρ, $\chi_W(\alpha)$ and $W(\alpha)$ all have identical information.

A state ρ is Gaussian iff its $\chi_W(\alpha)$ [equivalently, its $W(\alpha)$] is gaussian.

Thermal states and their unitary transforms $U\rho U^\dagger$ under the symplectic and Weyl groups are Gaussian states.

Gaussian states are almost the only states readily available to experimenters. And hence their importance.

Gaussian Channels

A CP-T map Φ is a Gaussian channel iff the output $\Phi(\rho)$ is Gaussian for every Gaussian input ρ.

Action of a Gaussian channel takes the form:

$$\chi(\alpha) \to \chi'(\alpha) = \chi(X\alpha)\exp(-\frac{1}{2}\alpha^T Y\alpha),$$

where X, Y are to obey some constraints.

Addition of classical noise is an easy to construct Gaussian channel:

$$\chi(\alpha) \to \chi(\alpha)\exp(-\frac{1}{2}a\,|\alpha|^2),\ a \geq 0$$

So any given Gaussian channel can be followed by such a channel almost 'free'.

Similarly given a Gaussian channel, it can be preceded and followed by independent unitary Gaussian channels corresponding to elements of the semidirect product of the symplectic and Weyl groups.

Classification of Gaussian Channels:

So, Gaussian Channels

$$\chi(\alpha) \rightarrow \chi'(\alpha) = \chi(X\alpha)\exp(-\frac{1}{2}\alpha^T Y \alpha),$$

have to be classified modulo these 'inexpensive' maps. And we have the following four families of **minimal noise** channels, determined by $\det X$ alone. Y is then a multiple of the identity in every case.

$A_2: \quad X = \mathrm{diag}\,[1,\, 0], \quad Y = 1;$

$C_1: \quad X = \mathrm{diag}\,[\kappa,\, \kappa], \quad Y = 1 - \kappa^2, \quad 0 \leq \kappa \leq 1$
Attenuator or beam-splitter;

$C_2: \quad X = \mathrm{diag}\,[\kappa,\, \kappa], \quad Y = \kappa^2 - 1, \quad 1 \leq \kappa \leq \infty$
Amplifier

$D: \quad X = \mathrm{diag}\,[\kappa,\, -\kappa], \quad Y = 1 + \kappa^2, \quad \kappa > 0$
transpose or phase conjugator.

For each case Solomon and Krishna Kumar have developed the operator-sum representation

$$\rho \longrightarrow \sum_k W_k \rho W_k$$

Phase Conjugator:

Define $a = \frac{\kappa}{\sqrt{1+\kappa^2}}, \quad b = \frac{1}{\sqrt{1+\kappa^2}}$. Then

$$W_k = b \sum_{n=0}^{k} b^n (-a)^{k-n} \sqrt{{}^k C_n} |k-n\rangle\langle n|, \quad k = 0, 1, 2, \cdots$$

$$\sum_k W_k^\dagger W_k = \mathbb{1}, \quad \text{but} \sum_k W_k W_k^\dagger = \kappa^{-2} \mathbb{1}$$

Not unital, but 'almost' unital.

Genuine unital for $\kappa = 1$.

Attenuator:

$$W_k = \sum_{m=0}^{\infty} \sqrt{{}^{m+k} C_k} (\sqrt{1-\kappa^2})^k \kappa^m |m\rangle\langle m+k|, \; k = 0, 1,$$

$$\sum_k W_k^\dagger W_k = \mathbb{1}, \quad \text{but} \sum_k W_k W_k^\dagger = \kappa^{-2} \mathbb{1}$$

Not unital, but 'almost' unital.

Genuine unital for $\kappa = 1$, but this corresponds to the trivial unit map.

Amplifier:

Define $a = \frac{\sqrt{\kappa^2-1}}{\kappa}$, $\;b = \frac{1}{\kappa}$. Then

$$W_k = b \sum_{m=0}^{\infty} \sqrt{{}^{m+k}C_k}\, a^k b^m |m+k\rangle\langle m|, \quad k = 0, 1, 2, \cdots$$

.

$$\sum_k W_k^\dagger W_k = \mathbb{1}, \quad \text{but} \sum_k W_k W_k^\dagger = \kappa^{-2}\mathbb{1}$$

Not unital, but 'almost' unital.

Genuine unital for $\kappa = 1$, but this corresponds to the trivial unit map.

The Singular case $X = \mathrm{diag}\,[1,\, 0]$

This is an interesting case. The Kraus operators can be written as a continuous family of rank one operators:

$$W_x = |x/\sqrt{2}\rangle_{\mathrm{coh}}\ {}_{\mathrm{pos}}\langle x|$$

.

Conjecture: All these minimal noise Gaussian channels are extremals of $\Lambda^{\mathrm{CP-T}}$.

Other Phase Space Distributions: P & Q

$$\chi_P(\alpha) = \chi_W(\alpha)\exp(+|\alpha|^2/2)$$

$$\chi_Q(\alpha) = \chi_W(\alpha)\exp(-|\alpha|^2/2)$$

$$Q(\alpha) = \pi^{-1}\langle\alpha|\rho|\alpha\rangle$$

Positive and $\leq 1/\pi$ throughout the phase space.

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$$

State **classical** if $P(\alpha)$ is pointwise nonnegative.

$$\rho \rightarrow \rho' = \int d^2\alpha Q(\alpha)|\alpha\rangle\langle\alpha|$$
$$\rho \rightarrow \rho' = \int d^2\alpha Q(\alpha^*)|\alpha\rangle\langle\alpha|$$
$$\rho \rightarrow \rho' = \int dq\langle q|\rho|q\rangle|q\rangle\langle q|_{\mathrm{coh}}$$

All these three are Gaussian channels

They all break nonclassicality.

Do they break entanglement?

Yes, and these are the only minimum noise EB channels.