The Quantum Hall Conductance: A rigorous proof of quantization

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Joint work with M. Hastings - Microsoft Research Station Q

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The challenge...

J. Avron and R. Seiler:

``Formulate the theory of the Integer Quantum Hall effect, which explains the quantization of the Hall conductance, so that it applies also for interacting electrons in the thermodynamic limit.''

Rigorous theory still in need of development...

The Hall Effect: Setup

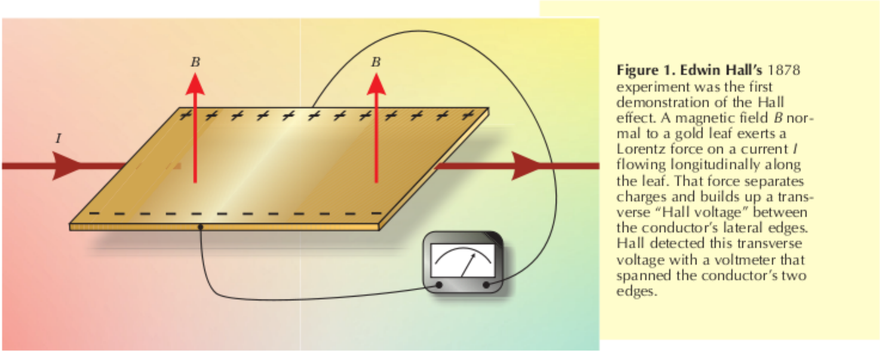

Figure: From J. Avron, *et al.* article in Physics Today, August 2003.

The Quantum Hall Effect: Resistance Plot

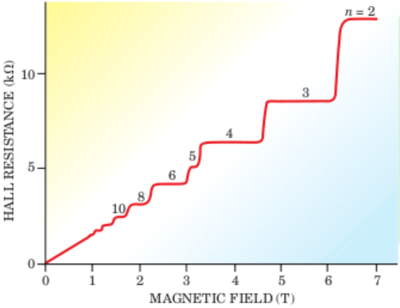

Figure 2. The integer quantum Hall effect. Plotting the Hall resistance (essentially the reciprocal of the Hall conductance) of a low-temperature two-dimensional electron gas against the strength of the imposed magnetic field normal to the gas plane, one finds a stairlike quantized sequence of Hall conductances very precisely equal to ne^2/h, where n is the integer that characterizes each plateau. The natural unit of resistance defined by this effect is about 26 kΩ. (Adapted from M. Paalanen, D. Tsui, A. Gossard, *Phys. Rev. B.* **25**, 5566 [1982].)

Figure: Clear plateaus imply quantization of Hall conductance.

The Quantum Hall Effect: The torus geometry

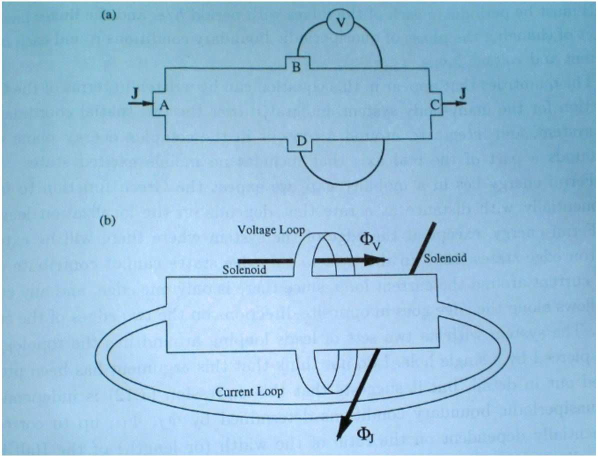

Figure: Magnetic field B deflects Hall current, generated by varying flux Φ_V, moving around voltage loop, while Φ_J monitors changes in J moving around current loop. (Image courtesy of D. Thouless)

The Hall conductance formula

From Kubo's formula to Hall conductance

- Let $S_H(\Phi_J, \Phi_V)$ be the Hall conductance as the two fluxes change.
- Assume $|\Psi_0(\Phi_J, \Phi_V)\rangle$ ground state of Hamiltonian $H = H(\Phi_J, \Phi_V)$
- Current operators: $J_x = \partial H / \partial \Phi_J$, $J_y = \partial H / \partial \Phi_V$
- Kubo's formula for $S_H(\Phi_J, \Phi_V)$:

$$i\hbar \langle \Psi_0(\Phi_J, \Phi_V)| \left(J_y \frac{\mathcal{P}}{(H - E_0)^2} J_x - J_x \frac{\mathcal{P}}{(H - E_0)^2} J_y \right) |\Psi_0(\Phi_J, \Phi_V)\rangle$$

- Perturbation theory: $\left| \frac{\partial \Psi_0}{\partial \Phi_V} \right\rangle = -\frac{\mathcal{P}}{H - E_0} \frac{\partial \mathcal{H}}{\partial \Phi_V} |\Psi_0\rangle$ (*parallel transport* assumed)
- Hall conductance:

$$S_H(\Phi_J, \Phi_V) = i\hbar \left(\left\langle \frac{\partial \Psi_0}{\partial \Phi_V}, \frac{\partial \Psi_0}{\partial \Phi_J} \right\rangle - \left\langle \frac{\partial \Psi_0}{\partial \Phi_J}, \frac{\partial \Psi_0}{\partial \Phi_V} \right\rangle \right)$$

The Berry phase argument

Winding numbers...

- Assume *parallel transport* in Φ_J and Φ_V during evolution of the transported ground state $\Psi_0(\Phi_J, \Phi_V)$ around a square of size h/e (in flux space.) Then, we have the following relation between the geometric phase $e^{i\eta(\Phi_J,\Phi_V)}$ and the Hall conductance from Kubo's formula:

$$\nabla \times (\nabla\eta(\Phi_J, \Phi_V)) \cdot \hat{n}(\vec{J}, \vec{V}) = S_H(\Phi_J, \Phi_V)/\hbar$$

- For "flux-averaged" conductance $\overline{S_H}$ and $\Sigma = [0, h/e] \times [0, h/e]$, the above relation implies:

$$\overline{S_H} = \frac{\hbar}{(h/e)^2} \int_\Sigma \nabla \times (\nabla\eta) \cdot d\hat{n} = \frac{e^2}{2\pi h} \oint \nabla\eta(\vec{\Phi}) \cdot d\vec{\Phi} = k\frac{e^2}{h}$$

Relaxing assumptions

Removing the averaging...

The previous argument is a “rigorous” version of Laughlin’s argument, but it relies on a set of assumptions:

1. Strict spectral gap for all Φ_J and Φ_V.
2. Averaged Hall conductance is close to actual Hall conductance at $\Phi_J = \Phi_V = 0$.
3. It assumes the torus geometry.

We relax assumption 1 to strict gap only at $\Phi_J = \Phi_V = 0$ and we remove assumption 2. We keep assumption 3.

Note: Powerful techniques from **non-commutative geometry** have been employed to remove assumption 3, yet these techniques apply to non-interacting electron models.

Our Model

- We consider the discrete, tight-binding model of interacting electrons on a lattice.
- Electrons have orbitals centered at sites in $T = L \times L$, a finite subset of $\mathbb{Z}^2$.
- At each site $s \in T$, we introduce the charge operator q_s with eigenvalues 0, 1 and 2, representing the number of electrons occupying the site.
- The Hamiltonian H_0 has a unique ground state $|\Psi_0\rangle$ and a spectral gap $\gamma > 0$ to the first excited state.
- The total charge $Q = \sum_{s \in T} q_s$ is conserved, so $[Q, H_0] = 0$.
- We introduce magnetic fluxes through twists θ_x, θ_y at the boundary, following Niu-Thouless and Avron-Seiler. The twist operators are given by:

$$R_X(\theta_x, A) = e^{i\theta_x Q_X} A e^{-i\theta_x Q_X} = R_X(\theta_x + 2\pi, A), \quad Q_X = \sum_{1 \leq x(s) \leq L/2} q_s$$

$$R_Y(\theta_y, A) = e^{i\theta_y Q_Y} A e^{-i\theta_y Q_Y} = R_Y(\theta_y + 2\pi, A), \quad Q_Y = \sum_{1 \leq y(s) \leq L/2} q_s.$$

The Hamiltonian interactions

- Let $\Phi(X) = \Phi^\dagger(X)$, be an interaction associated with a set of sites $X \subset T$.
- The Hamiltonian is given by $H_0 = \sum_{X \subset T} \Phi(X)$.
- Interactions $\Phi(X)$ are uniformly bounded ($\sup_{s \in T} \sum_{X \ni s} \|\Phi(X)\| \leq J$) and have finite range R ($\mathrm{diam}(X) > R \implies \|\Phi(X)\| = 0$;exponential decay is also fine.)

Example

Let $a_j^\dagger$ and a_j be the creation and annihilation operators for a particle at site j. The following Hamiltonian in 2-D has interactions satisfying the above assumptions for $J = 4\max\{J_1, 2J_2\}$ and $R = 2$:

$$H_0 = J_1 \sum_{|i-j|=1} e^{i\phi} a_i^\dagger a_j + J_2 \sum_{|i-j|=2} a_i^\dagger a_i a_j^\dagger a_j.$$

The Hamiltonian in flux-space

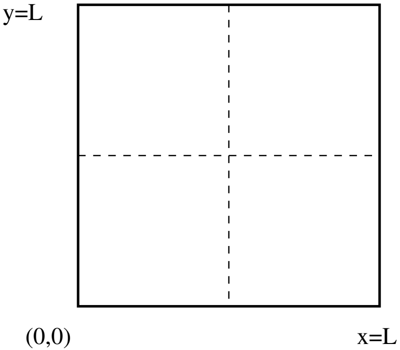

Figure: Lines illustrating how the twists are defined on the torus. The twists ϕ_x, ϕ_y affect interactions close to the vertical and horizontal dashed lines, respectively, while the twists θ_x, θ_y affect interactions close to the vertical and horizontal solid lines.

Quantization of the Hall conductance

The Hall conductance σ_H for a system of *interacting* particles described by the Hamiltonian H_0 on $T = L \times L$, a finite subset of the two dimensional lattice, with periodic boundary conditions, finite range R, finite strength J interactions and a non-vanishing spectral gap $\gamma > 0$ as L increases, satisfies the quantization condition:

Theorem

For all sufficiently large L:

$$\left|\sigma_H - n \cdot \frac{e^2}{h}\right| \leq C_R L^3 \, e^{-\frac{L}{\xi \ln^2 L}}, \tag{1}$$

for some $n \in \mathbb{N}$, C_R a polynomial in R and ξ a correlation length depending only on $R, \frac{J}{\gamma}$. The quantity e^2/h denotes the square of the electron charge divided by Planck's constant.

Outline of proof - I

Step by step...

- Introduce *quasi-adiabatically* loop-evolved states:

$$|\Psi_{\circlearrowleft}(r)\rangle = V_{\circlearrowleft}(0,0,r)\,|\Psi_0(0,0)\rangle\,, \tag{2}$$

  where the unitary $V_{\circlearrowleft}(0,0,r)$ describes the quasi-adiabatic evolution of the initial ground state $|\Psi_0(0,0)\rangle$ around a square of size r, starting at the origin in flux-space and moving counter-clockwise.

- The unitary $V_{\circlearrowleft}(0,0,r)$ is given by the following product of unitaries:

$$V_{\circlearrowleft}(0,0,r) = U_Y^{\dagger}(0,0,r)\,U_X^{\dagger}(0,r,r)\,U_Y(r,0,r)\,U_X(0,0,r), \tag{3}$$

  where $U_Y(\theta_x,\theta_y,r)$ drives the evolution in flux-space from (θ_x,θ_y) to (θ_x,θ_y+r) and $U_X(0,0,r)$ takes us from (θ_x,θ_y) to (θ_x+r,θ_y).

▸ The generators of the quasi-adiabatic evolution

Outline of proof - II

¡The three amigos!

- Start with a simple inequality involving the Hall conductance σ_H:

$$\left|\sigma_H \frac{2\pi h}{e^2} - 2\pi n\right| \leq \sqrt{2}\left|1 - e^{i\left(\sigma_H \frac{2\pi h}{e^2} - 2\pi n\right)}\right|,$$

which holds whenever the r.h.s. is ≤ 1.

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  which holds whenever the r.h.s. is ≤ 1.
- Upper-bound the term in the r.h.s. by the sum of the norms of following quantities:
  1. Lucky: $1 - \langle \Psi_0, \Psi_{\circlearrowleft}(2\pi)\rangle$
  2. Dusty: $\langle \Psi_0, \Psi_{\circlearrowleft}(r)\rangle^{\left(\frac{2\pi}{r}\right)^2} - e^{i\,\sigma_H \frac{2\pi h}{e^2}}$
  3. Nasty: $\langle \Psi_0, \Psi_{\circlearrowleft}(2\pi)\rangle - \langle \Psi_0, \Psi_{\circlearrowleft}(r)\rangle^{\left(\frac{2\pi}{r}\right)^2}$.

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- We will bound each term by an **exponentially decaying** quantity in the linear size L. The last term is the hardest to bound, hence the name.

Outline of proof - III

Lucky

Using locality (**Lieb-Robinson**) bounds on evolutions generated by singly-twisted Hamiltonians around the loop $\Lambda : (0,0) \to (2\pi, 0) \to (2\pi, 2\pi) \to (0, 2\pi) \to (0,0)$, **energy estimates** at the end of each leg imply the following phase estimate:

Corollary

For a constant $C_R > 0$, the following bound holds:

$$|\langle \Psi_0, \Psi_{\circlearrowleft}(2\pi)\rangle - 1| \leq C_R \left(\frac{J}{\gamma}\right)^2 L^4 e^{-\frac{L}{8R}}. \tag{4}$$

Note that at the end of each leg of the evolution, the Hamiltonians are equal to H_0, due to the periodicity of the flux (Aharonov-Bohm effect). Moreover, the energy estimates follow from the introduction of the (far away) twists $\phi_x = -\theta_x$ and $\phi_y = -\theta_y$, which open up the gap to $\gamma > 0$!

Outline of proof - IV

Dusty

The phase accumulated during perfect adiabatic evolution by parallel transport around a counter-clockwise loop of size r at the origin (in flux space) is given by:

$$\text{Berry phase: } \phi(r) = 2\int_0^r d\theta_x \int_0^r d\theta_y \, \mathrm{Im}\left\{\left\langle \partial_{\theta_y}\Psi_0(\theta_x,\theta_y), \partial_{\theta_x}\Psi_0(\theta_x,\theta_y)\right\rangle\right\}$$

Corollary

For $r \leq C_{\gamma,R,J}/L$, the following phase estimates hold:

$$\langle \Psi_0, \Psi_{\circlearrowleft}(r)\rangle = e^{i\phi(r)} \implies \langle \Psi_0, \Psi_{\circlearrowleft}(r)\rangle^{\left(\frac{2\pi}{r}\right)^2} = e^{i\phi(r)\left(\frac{2\pi}{r}\right)^2}. \quad (5)$$

Outline of proof - IV

Dusty meets Hall...

Let $\phi(r)$ be the phase defined previously. Then, for $r > 0$ we have:

$$\left|\phi(r)/r^2 - 2\,\mathrm{Im}\left\{\left\langle \partial_{\theta_y}\Psi_0(0,\theta_y)_{\theta_y=0}, \partial_{\theta_x}\Psi_0(\theta_x,0)_{\theta_x=0}\right\rangle\right\}\right| \le C\left(q_{\max}R^2\frac{J}{\gamma}\right)^2\cdot r, \quad (6)$$

Recall that the Hall conductance is defined by Kubo's formula:

$$\sigma_H = 2\,\mathrm{Im}\left\{\left\langle \partial_{\theta_y}\Psi_0(0,\theta_y)_{\theta_y=0}, \partial_{\theta_x}\Psi_0(\theta_x,0)_{\theta_x=0}\right\rangle\right\}\cdot\left(2\pi\,\frac{e^2}{h}\right) \quad (7)$$

You do the math... OK, I 'll do the math:

$$\left|\langle\Psi_0,\Psi_{\circlearrowleft}(r)\rangle^{\left(\frac{2\pi}{r}\right)^2} - e^{i\,\sigma_H\frac{2\pi h}{e^2}}\right| \le C'\left(q_{\max}R^2\frac{J}{\gamma}\right)^2\cdot r, \quad (8)$$

for $r \le C_{\gamma,R,J}L^{-1}$. In fact, we will take r to decay (almost) exponentially in L.

Outline of proof - V

Nasty

Using locality estimates (**second order Lieb-Robinson bounds**) for the adiabatic evolution operators U_X and U_Y, we get a *Translation Lemma* which bounds our last amigo...

Corollary

For a numeric constant $C > 0$*, the following bound holds:*

$$\left| \langle \Psi_0, \Psi_{\circlearrowleft}(2\pi) \rangle - \langle \Psi_0, \Psi_{\circlearrowleft}(r) \rangle^{\left(\frac{2\pi}{r}\right)^2} \right| \leq C_{R,q_{\max}} L^3 \left(\frac{e^{-\frac{2L}{\xi \ln^2 L}}}{r} \right).$$

At this point, we choose (optimally):

$$r = \frac{2\pi}{\lfloor e^{L/\xi \ln^2 L} \rfloor}. \tag{9}$$

Reverse engineering Stokes - I

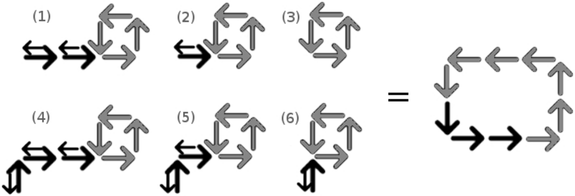

Figure: The evolution of $|\Psi_0\rangle$ into $|\Psi_{\circlearrowleft}(2\pi)\rangle$ can be decomposed into a series of evolutions through the intermediate states $|\Psi_{\circlearrowleft}(\theta_x, \theta_y, r)\rangle$.

Decomposition accomplished...

Using the above decomposition process, we may write $\langle \Psi_0, \Psi_{\circlearrowleft}(2\pi)\rangle$ as a product of overlaps $\langle \Psi_0, \Psi_{\circlearrowleft}(\theta_x, \theta_y, r)\rangle$, with (θ_x, θ_y) points on the induced flux-space lattice, up to exponentially small error.

Reverse engineering Stokes - II

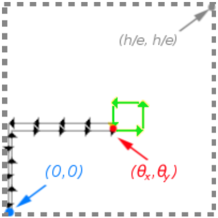

Figure: The state $|\Psi_0\rangle$ evolves quasi-adiabatically along the cyclic path shown to the state $|\Psi_{\circlearrowleft}(\theta_x, \theta_y, r)\rangle$.

Evolution overlap...

Let $|\Psi_{\circlearrowleft}(\theta_x, \theta_y, r)\rangle$ be the state that is quasi-adiabatically evolved from the blue dot to the red dot, around the loop and back. Taylor expanding the evolution around the small green loop, it can be shown that $\langle \Psi_0, \Psi_{\circlearrowleft}(\theta_x, \theta_y, r)\rangle$ is given, up to small corrections, by $\langle \Psi_0, \Psi_{\circlearrowleft}(r)\rangle$

Fractional Hall Effect and Mobility Gaps

Lieb-Robinson Bounds and Topological Order

- Fractional Hall Effect... How do we model this in our context? Degenerate ground state?
  Update: Apparently, yes! The extension to the fractional case is pretty straightforward once we have the framework explaining the integer case. An extra assumption of topological order in the low energy sector is necessary to make sure that all low energy states have the *same* fractional conductance.
- Spectral gap corresponds to Landau level filling + quantum harmonic oscillator energy barrier $\hbar\omega$. In Fractional QHE gap opens due to long-range interactions. However, **disorder** produces low energy **localized** excitations. Can we relax strict gap assumption and assume only a gap between localized and de-localized states? What exactly is the definition of localization for a many-body state?
  Update: Recent work by Hastings has answered this in the affirmative! He defines a notion of localization that can be plugged into the current proof to relax the **spectral gap assumption** into a **mobility gap assumption**.

An "open" question...

Opening the torus

- How important is the torus geometry for this argument? Boundary edge effects in real experiments (FQHE). How do we model those? Add reservoirs at the twists on the boundary and open the geometry? Recall that total charge Q was conserved, so that twists could be concentrated near "boundary" lines anywhere on the torus.
- Did we answer Avron and Seiler's challenge?

Thank you!

The Princeton webpage...

Open Problems in Mathematical Physics

MΠΦ

Quick Access:

Soft phases in 2D O(N) models

Quantum Hall Conductance

Exponents & dimensions SOLVED!

LRO for Quantum Heisengerg Ferr.

Extended states

Impossibility theorems

Fermi gas

Entropy production

Separatrix separation

Navier-Stokes

Spin glass

Optimal flux

Bose-Einstein condensation

This page leads to a collection of significant open problems gathered from colleagues during the academic year 1998/99. They are offered in the belief that good challenges stimulate our work, tempered by the dictum that preformulated questions should not discourage one from seeking new perspectives.

All are invited to send pertinent *comments*, references to *solutions*, and *contributions* for this page to *M. Aizenman (Editor):* aizenman@princeton.edu

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List by Contributors **By order of submission**

General Framework	**Quantum Field Theory**
Statistical Physics	**Quantum Many Body Systems**
Geometry and Physics	**Schroedinger Operators**
Disordered Systems	**Non-equilibrium**
Relativity and Gravitation	**Dynamical Systems**
Fluid Dynamics	

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Return

The Quasi-Adiabatic evolution - I

Rise of the super-operator...

- Introducing the generators of the quasi-adiabatic evolution:

$$\mathcal{S}_\gamma(H,A) = \int_{-\infty}^{\infty} s_\gamma(t) \left(\int_0^t e^{iuH} A e^{-iuH} du \right) dt, \qquad (10)$$

$$s_\gamma(t) = c_\gamma \prod_{n=1}^{\infty} sinc^2(a_n t), \quad \sum_{n\geq 1} a_n = \gamma/3, \quad \|s_\gamma\|_1 = \hat{s}_\gamma(0) = 1. \qquad (11)$$

Choosing $a_n = a_1 \cdot (n \ln^2 n)^{-1}$, yields **near-exponential decay** for $s_\gamma(t)$.

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- The following operator generates the quasi-adiabatic evolution U_X:

$$\mathcal{D}_X(\theta_x, \theta_y) \quad = \quad \mathcal{S}_\gamma \left(H(\theta_x, \theta_y),\ \partial_{\theta_x} H(\theta_x, \theta_y) \right).$$

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- WHY this definition? $\mathcal{D}_X$ generates **exact** adiabatic evolution when $H(\theta_x, \theta_y)$ has gap $\geq \frac{2}{3}\gamma$. For larger flux angles, locality estimates take over!

◂ Return