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Representations of a Class of Partial Algebras*

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Fundamental notions

Partial algebra

1. A *partial algebra* is a triplet $(\mathfrak{A}, \Gamma, \circ)$ comprising:
   - a *linear space* $\mathfrak{A}$ over $\mathbb{K}$;
   - a *relation* $\Gamma \subseteq \mathfrak{A} \times \mathfrak{A}$ on $\mathfrak{A}$; and
   - a *partial multiplication* $\circ$ such that
     i. $(x, y) \in \Gamma \Leftrightarrow x \circ y \in \mathfrak{A}$;
     ii. $(x, z) \in \Gamma$ and $(y, z) \in \Gamma$ imply $(\alpha x + \beta y, z) \in \Gamma$, and then $(\alpha x + \beta y) \circ z = \alpha(x \circ z) + \beta(y \circ z), \quad \forall\, \alpha, \beta \in \mathbb{C}$.

Partial *-algebra

2. A *partial *-algebra* is a partial algebra $(\mathfrak{A}, \Gamma, \circ)$ equipped with an *involution* * such that
   i. $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;
   ii. $(x \circ y)^* = y^* \circ x^*$, whenever $(x, y) \in \Gamma$.

Studying partial algebras

In general, partial algebras are not algebras in the usual sense. Partial algebras are studied by means of their sets of multipliers.

Multipliers

Let $(\mathfrak{A}, \Gamma, \circ)$ be a partial algebra, $\mathfrak{C} \subset \mathfrak{A}$ and $x \in \mathfrak{A}$. Introduce the sets of multipliers

$$\begin{aligned} M_L(x) &= \{y \in \mathfrak{A} : (y, x) \in \Gamma\} \\ &= \text{left multipliers of } x. \end{aligned}$$

$$\begin{aligned} M_R(x) &= \{y \in \mathfrak{A} : (x, y) \in \Gamma\} \\ &= \text{right multipliers of } x. \end{aligned}$$

$$\begin{aligned} M_L(\mathfrak{C}) &= \bigcap_{\mathfrak{C}} M_L(x) \\ &= \text{left multipliers of } \mathfrak{C}. \end{aligned}$$

$$\begin{aligned} M_R(\mathfrak{C}) &= \bigcap_{\mathfrak{C}} M_R(x) \\ &= \text{right multipliers of } \mathfrak{C}. \end{aligned}$$

Remark

Partial algebras are severely handicapped structures which are often nonassociative. Notions of associativity are introduced as follows.

Associativity

A partial algebra $(\mathfrak{A}, \Gamma, \circ)$ is associative if $x \in M_L(y)$, $y \in M_L(z)$ and $x \circ y \in M_L(z)$, then $y \circ z \in M_R(x)$ and $(x \circ y) \circ z = x \circ (y \circ z)$.

Remark

- This notion of associativity is extremely restrictive. Many partial algebras do not satisfy it.
- A less restrictive notion of associativity is the following.

Semi-associativity

A partial algebra $(\mathfrak{A}, \Gamma, \circ)$ is semi-associative if
$y \in M_R(x) \Rightarrow y \circ z \in M_R(x)$ for every $z \in M_R(\mathfrak{A})$ and
$(x \circ y) \circ z = x \circ (y \circ z)$.

Some examples of partial algebras

Algebras are partial algebras

Every algebra $(\mathfrak{A}, \cdot)$, with $\cdot$ as multiplication, is a partial algebra. Here $\Gamma = \mathfrak{A} \times \mathfrak{A}$.

Partial O-algebras

Let

$$\begin{aligned} X &= \text{a locally convex space over } \mathbb{K}, \\ \mathcal{D} &= \text{a subspace of } X, \text{ and} \\ L(\mathcal{D}, X) &= \text{the set of all linear } X\text{-valued maps, each with domain } \mathcal{D}. \end{aligned}$$

Furnish $L(\mathcal{D}, X)$ with pointwise addition and scalar multiplication, $\circ$ as composition of maps and the relation $\Gamma \equiv \{(x, y) \in L(\mathcal{D}, X) \times L(\mathcal{D}, X) : y\mathcal{D} \subseteq \mathrm{dom}(x)\}$, where $\mathrm{dom}(a)$ denotes the domain of a. Then the triplet $(L(\mathcal{D}, X), \Gamma, \circ)$ is a partial algebra.

A partial algebra of differential operators

Let $\mathfrak{A}_n$ denote the linear span of all linear ordinary differential operators of order n with continuous coefficients and $\mathfrak{A}$ be the linear span of $\bigcup_n \mathfrak{A}_n$. With $\circ$ as composition of maps and $\Gamma \equiv \{(x, y) \in \mathfrak{A} \times \mathfrak{A} : x \circ y \in \mathfrak{A}\}$, then the triplet $(\mathfrak{A}, \Gamma, \circ)$ is a partial algebra.

Locally convex partial algebras

Let

$$\begin{aligned}
(\mathfrak{A}, \circ, \tau) &= \text{a locally convex algebra with a locally convex topology } \tau \\
\circ &= \text{a separately continuous multiplication on } \mathfrak{A}. \\
\mathfrak{A}[\tau] &= \text{the completion of the locally convex space } (\mathfrak{A}, \tau) \\
\Gamma &\equiv \{(x, y) \in \mathfrak{A}[\tau] \times \mathfrak{A}[\tau] : x \in \mathfrak{A} \text{ or } y \in \mathfrak{A}\}.
\end{aligned}$$

Then the triplet $(\mathfrak{A}[\tau], \Gamma, \circ)$ is a locally convex partial algebra.

Lebesgue spaces as partial algebras

Let $1 \leq d \leq \infty$,

$$(\Omega, \mathcal{B}, \mu) = \text{a finite measure space.}$$

$$L^d(\Omega, \mathcal{B}, \mu) = \text{Lebesgue spaces equipped with pointwise addition and scalar multiplication.}$$

For each $d \in (1, \infty)$, there always exist $r, s \in (d, \infty]$ such that

$$1/r + 1/s = 1/d.$$

Let $d \geq 1$ and

$$\Gamma_d = \{(f, g) \in (L^r(\Omega, \mathcal{B}, \mu) \times L^s(\Omega, \mathcal{B}, \mu) : \min\{r, s\} > d \text{ and } 1/r + 1/s = 1/d\}.$$

Then $(L^d(\Omega, \mathcal{B}, \mu), \Gamma_d, \cdot)$, where $\cdot$ denotes pointwise multiplication in $L^d(\Omega, \mathcal{B}, \mu)$, is a partial algebra.

Lebesgue spaces as partial algebras, contd

In analogy to last example, let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. Equip $L^d(\Omega, \mathcal{B}, \mu)$, $1 \leq d \leq \infty$, with the relation

$$\widetilde{\Gamma}_d = \{(f, g) \in L^d(\Omega, \mathcal{B}, \mu) \cap L^r(\Omega, \mathcal{B}, \mu) \times L^d(\Omega, \mathcal{B}, \mu) \cap L^s(\Omega, \mathcal{B}, \mu) : \min\{r, s\} > d \text{ and } 1/r + 1/s = 1/d\}.$$

Then $(L^d(\Omega, \mathcal{B}, \mu), \widetilde{\Gamma}_d, \cdot)$, where $\cdot$ denotes pointwise multiplication in $L^d(\Omega, \mathcal{B}, \mu)$, is a partial algebra.

Notice that if a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$ is actually *finite*, i.e. $\mu(\Omega) < \infty$, then the relation $\widetilde{\Gamma}_d$ reduces to Γ_d defined above.

The partial algebraic structure of an algebra

Observation

Let $\mathfrak{A}$ be an algebra, with multiplication $\cdot$, and $\mathcal{M}$ a subspace of $\mathfrak{A}$. Introduce

$$\Gamma = \{(a,b) \in \mathcal{M} \times \mathcal{M} : a \cdot b \in \mathcal{M}\}.$$

Then, $(\mathcal{M}, \Gamma, \cdot)$ is a partial algebra.
Thus, every algebra possesses two types of subalgebras, namely: subalgebras, in the usual sense, and partial subalgebras.

Remark

The following is a connection between algebras and partial algebras.

Theorem [Ekhaguere (2008)]

Every algebra is an inductive limit of partial algebras.

Concrete Partial *-Algebras

Central to the *structural analysis* and *representation theory* of abstract partial *-algebras are the *concrete partial *-algebras* which we introduce as follows.

Notation

$(\mathcal{D}, \langle\cdot,\cdot\rangle)$ = a pre-Hilbert space with completion $(\mathcal{H}, \langle\cdot,\cdot\rangle)$

$L(\mathcal{D}, \mathcal{H})$ = the set of all closable $\mathcal{H}$-valued linear operators x with $\mathrm{dom}(x) = \mathcal{D}$

$$L(\mathcal{D}) = \{x \in L(\mathcal{D}, \mathcal{H}) : x\mathcal{D} \subset \mathcal{D}\}$$

$$L^+(\mathcal{D}, \mathcal{H}) = \{x \in L(\mathcal{D}, \mathcal{H}) : \mathrm{dom}(x^*) \supset \mathcal{D}\}$$

$$L^+(\mathcal{D}) = \{x \in L^+(\mathcal{D}, \mathcal{H}) \cap L(\mathcal{D}) : x^*\mathcal{D} \subset \mathcal{D}\}.$$

Proposition

The sets introduced above have the following properties.

- $L(\mathcal{D}, \mathcal{H})$ is a linear space with pointwise addition and scalar multiplication.
- $L(\mathcal{D})$ is a subspace of $L(\mathcal{D}, \mathcal{H})$.
- $L(\mathcal{D})$ is an algebra when equipped with the usual notion of composition of maps.
- $L^{+}(\mathcal{D})$ is a *-algebra, with the same operations as on $L(\mathcal{D})$ and involution $x \mapsto x^{+} = x^{*} \lceil_{\mathcal{D}}$.

Remark

The sets $L^{+}(\mathcal{D}, \mathcal{H})$, $L^{+}(\mathcal{D})$ and $B(\mathcal{H})$ may coincide. Indeed, we have the following.

Theorem

If there exists some $x \in L^+(\mathcal{D},\mathcal{H})$ such that $\overline{x} = x$ on $\mathcal{D}$ and $x^+ \in M_L(x)$, then automatically $\mathcal{D} = \mathcal{H}$, whence $L^+(\mathcal{D},\mathcal{H}) = L^+(\mathcal{H}) = B(\mathcal{H})$. In particular, if $x \in L^+(\mathcal{D})$ is closed on $\mathcal{D}$, then $\mathcal{D} = \mathcal{H}$ and $L^+(\mathcal{D},\mathcal{H}) = L^+(\mathcal{H}) = B(\mathcal{H})$.

The relation Γ

Introduce the *relation* Γ and *partial multiplication* $\Box$ as follows.

$$\begin{aligned} \Gamma &= \{(x,y) \in L^+(\mathcal{D},\mathcal{H}) \times L^+(\mathcal{D},\mathcal{H}) : y\mathcal{D} \subset \operatorname{dom}(x^{+*}) \\ &\qquad \text{and} \quad x^+\mathcal{D} \subset \operatorname{dom}(y^*)\} \\ x \Box y &= x^{+*}y \text{ whenever } (x,y) \in \Gamma. \end{aligned}$$

Theorem

$(L^+(\mathcal{D},\mathcal{H}), \Gamma, \Box, {}^+)$ is a partial *-algebra.

Partial O^*-Algebras

Definition

A linear subspace $\mathfrak{M}$ of $L^+(\mathcal{D}, \mathcal{H})$ is called a *partial O^*-algebra* on $\mathcal{D}$ provided that

- $\mathfrak{M}$ is $+$-invariant, i.e. $x \in \mathfrak{M} \Rightarrow x^+ \in \mathfrak{M}$;
- $x, y \in \mathfrak{M}$, with $(x, y) \in \Gamma$, $\Rightarrow x \Box y \in \mathfrak{M}$.

Remark

- $(L^+(\mathcal{D}, \mathcal{H}), \Gamma, \Box, {}^+)$ will be denoted simply by $L_w^+(\mathcal{D}, \mathcal{H})$. The latter is the maximal weak partial *-algebra of closable $\mathcal{H}$-valued operators on $\mathcal{D}$.
- $L^+(\mathcal{D}, \mathcal{H})$ admits several locally convex topologies, including the following: τ_w, τ_s, τ_s^+, $\tau_{\sigma w}$, $\tau_{\sigma s}$, and $\tau_{\sigma s}^+$.

Notation

Let $\mathfrak{M}$ be a partial O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$. Introduce

$$\mathcal{D}^*(\mathfrak{M}) \equiv \cap_{x \in \mathfrak{M}} \mathcal{D}(x^*)$$
$$\widehat{\mathcal{D}}(\mathfrak{M}) \equiv \cap_{x \in \mathfrak{M}} \mathcal{D}(\overline{x})$$

Definition

A partial O^*-algebra $\mathfrak{M}$ on $\mathcal{D}$ in $\mathcal{H}$ is called

- *self-adjoint* if $\mathcal{D} = \mathcal{D}^*(\mathfrak{M})$;
- *essentially self-adjoint* if $\mathcal{D}^*(\mathfrak{M}) = \widehat{\mathcal{D}}(\mathfrak{M})$

Remark

The following statements are easily checked.

Proposition

Let $\mathfrak{M} \subset L_w^+(\mathcal{D}, \mathcal{H})$ be a self-adjoint partial O^*-algebra. Then, $\mathfrak{M}$ is a locally convex

(i) partial O-algebra, with a separately continuous partial multiplication, when furnished with the topologies τ_s and $\tau_{\sigma s}$;

(ii) partial O^*-algebra, with a separately continuous partial multiplication, when furnished with any of the topologies: τ_w, τ_s^+, $\tau_{\sigma w}$, and $\tau_{\sigma s}^+$.

Commutants and bicommutants

A partial O^*-algebra possesses, in general, a multiplicity of commutants and bicommutants. We limit ourselves to the following notions.

Commutants

Let $\mathcal{C}$ be a $^+$-invariant subset of $L_w^+(\mathcal{D},\mathcal{H})$. Then the commutants $\mathcal{C}'_\sigma$ and $\mathcal{C}'_c$ are defined as follows:

$$\begin{aligned}\mathcal{C}'_\sigma &= \{X \in L_w^+(\mathcal{D},\mathcal{H}) : \langle X\xi, A^+\eta\rangle = \langle A\xi, X^+\eta\rangle, \\ &\qquad \forall\, A \in \mathcal{C}, \text{ and } \xi,\eta \in \mathcal{D}\}\end{aligned}$$

$$\begin{aligned}\mathcal{C}'_c &= \{X \in L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H}) : \langle A \cdot X\xi, \eta\rangle = \\ &\qquad \langle \xi, A^+ \cdot X^+\eta\rangle, \ \forall\, A \in \mathcal{C}, \text{ and } \xi,\eta \in \mathcal{D}\}\end{aligned}$$

These commutants are related, since $\mathcal{C}'_c = \mathcal{C}'_\sigma \cap L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H})$.

Bicommutants

Associated with the commutants above are the following four bicommutants.

$$
\begin{aligned}
\mathcal{C}''_{\sigma\sigma} &= \{X \in L^+_w(\mathcal{D},\mathcal{H}) : \langle X\xi, A^+\eta\rangle = \\
&\qquad \langle A\xi, X^+\eta\rangle,\ \forall\ A \in \mathcal{C}'_\sigma,\ \text{and}\ \xi,\eta \in \mathcal{D}\} \\
\mathcal{C}''_{cc} &= \{X \in L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H}) : \langle A \cdot X\xi, \eta\rangle = \\
&\qquad \langle \xi, A^+ \cdot X^+\eta\rangle,\ \forall\ A \in \mathcal{C}'_c,\ \text{and}\ \xi,\eta \in \mathcal{D}\} \\
\mathcal{C}''_{c\sigma} &= \{X \in L^+_w(\mathcal{D},\mathcal{H}) : \langle X\xi, A^+\eta\rangle = \\
&\qquad \langle A\xi, X^+\eta\rangle,\ \forall\ A \in \mathcal{C}'_c,\ \text{and}\ \xi,\eta \in \mathcal{D}\} \\
\mathcal{C}''_{\sigma c} &= \{X \in L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H}) : \langle A \cdot X\xi, \eta\rangle = \\
&\qquad \langle \xi, A^+ \cdot X^+\eta\rangle,\ \forall\ A \in \mathcal{C}'_\sigma,\ \text{and}\ \xi,\eta \in \mathcal{D}\}
\end{aligned}
$$

About these bicommutants, there are the following facts.

Proposition

Let $\mathcal{C}$ be a $+$-invariant subset of $L_w^+(\mathcal{D},\mathcal{H})$. Then

(i) $\mathcal{C}''_{\sigma c} = \mathcal{C}''_{\sigma\sigma} \cap L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H})$;

(ii) $\mathcal{C}''_{\sigma\sigma} \subseteq \mathcal{C}''_{c\sigma} \subset L_w^+(\mathcal{D},\mathcal{H})$; and $\mathcal{C}''_{\sigma c} \subseteq \mathcal{C}''_{cc} \subseteq L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H})$;

(iii) $\mathcal{C}''_{\sigma c} \subseteq \mathcal{C}''_{\sigma\sigma}$; and $\mathcal{C}''_{cc} \subseteq \mathcal{C}''_{c\sigma}$.

(iv) The sets $\mathcal{C}'_{\sigma}$, $\mathcal{C}''_{\sigma\sigma}$ and $\mathcal{C}''_{c\sigma}$ are t_s^*-closed in $L_w^+(\mathcal{D},\mathcal{H})$. Similarly, the sets $\mathcal{C}'_{c}$, $\mathcal{C}''_{\sigma c}$ and $\mathcal{C}''_{cc}$ are t_s^*-closed in $L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H})$. Moreover, each of the sets $\mathcal{C}''_{\sigma\sigma}$ and $\mathcal{C}''_{c\sigma}$ contains $\mathcal{C}$.

Bitraces

A pair $(\tau, \mathcal{N}_\tau)$, with $\mathcal{N}_\tau \subset \mathcal{M}$, is a *bitrace* on $\mathcal{M}$ provided that

(i) $\tau \in \mathrm{wgt}(\mathcal{M})$;

(ii) $\tau(x, y) = \tau(y^+, x^+), \quad x, \ y \in \mathcal{M}$;

(iii) $\mathcal{N}_\tau$ is an ideal of $\mathcal{M}$; and

(iv) the restriction of τ to $\mathcal{N}_\tau \times \mathcal{N}_\tau$ (denoted in the sequel again by τ) is a positive sesquilinear form on $\mathcal{N}_\tau$.

Regular bitraces

1. A bitrace $(\tau, \mathcal{N}_\tau)$ on $\mathcal{M}$ is *regular* if

(i) $\mathcal{N}_\tau \neq \{0\}$; and

(ii) there is a subspace $\mathcal{G}_\tau \subset M(\mathcal{M}) \cap \mathcal{N}_\tau$ such that

(a) the linear span $[\lambda_\tau(\mathcal{G}_\tau)]$ of $\lambda_\tau(\mathcal{G}_\tau)$ is dense in $\mathcal{H}_\tau$; and

(b)(α) $\tau(x_1 \cdot b_1, x_2 \cdot b_2) = \tau(b_1, (x_1^+ \cdot x_2) \cdot b_2)$

(β) $\tau(b_1 \cdot z_1, b_2 \cdot z_2) = \tau(b_1, b_2 \cdot (z_2 \cdot z_1^+))$

for all $b_1, b_2 \in \mathcal{G}_\tau$, $x_1, x_2 \in \mathcal{M}$ with $x_1^+ \in L(x_2)$, and $z_1, z_2 \in \mathcal{M}$ with $z_2 \in L(z_1^+)$.

Core of a bitrace

If $(\tau, \mathcal{N}_\tau) \in \mathrm{btr}(\mathcal{M})$ is *regular*, then the subspace $\mathcal{G}_\tau$ is a *core*.

Classification of bitraces

Let $\mathcal{M}$ be a unital partial O^*-algebra on $\mathcal{D}$, with unit e. A bitrace $\tau \in \mathrm{btr}(\mathcal{M})$ is
(i) *finite* if $e \in \mathcal{N}_\tau$;
(ii) *semifinite* if there is a net $\{t_\alpha\} \subset \mathcal{G}_\tau \cap \mathcal{M}_+$, satisfying $\|\pi_\tau(t_\alpha)\| \leq 1$ for each α and $\{\pi_\tau(t_\alpha)\} \subset L^+(\mathcal{D}_\tau)$, such that $\{\pi_\tau(t_\alpha)\}$ converges strongly to the identity element of $\mathrm{B}(\mathcal{H}_\tau)$.
(iii) *normal* if for each t_s^*-convergent increasing net $\{t_\alpha\} \subset \mathcal{G}_\tau \cap \mathcal{M}_+$, satisfying $\|\pi_\tau(t_\alpha)\| \leq 1$ for each α, with limit $t \in \mathcal{G}_\tau \cap \mathcal{M}_+$, the net $\{\tau(t_\alpha, x)\}$ converges to $\tau(t, x)$ for every $x \in \mathcal{G}$.

Call a partial O^*-algebra $\mathfrak{M} \subset L_w^+(\mathcal{D}, \mathcal{H})$ a partial W^*-algebra if $\mathfrak{M}$ is t_s^*-closed and $\mathfrak{M}''_{c\sigma} = \mathfrak{M}$.

Classification of partial W^*-algebras

Partial W^*-algebras may be classified as follows by means of the type of bitraces that are defined on them.

Definition

A partial W^*-algebra $\mathcal{M}$ on $\mathcal{D}$ is

(i) *finite* if there is a faithful, normal, regular finite bitrace on $\mathcal{M}$;

(ii) *semifinite* if there is a faithful, normal, regular semifinte bitrace on $\mathcal{M}$;

(iii) *properly infinite* if there is no nonzero normal regular finite bitrace on $\mathcal{M}$; and

(iv) *purely infinite* if there is no nonzero normal regular semifinite bitrace on $\mathcal{M}$.

What next?

- Study the *structure* of partial W^*-algebras.
- Use the *classification scheme* introduced above to study *types* of partial W^*-algebras.
- Study *representations* of abstract partial W^*-algebras by concrete partial W^*-algebras.
- Study nets of partial W^*-algebras of, in general, unbounded observables.
- Study partial W^*-dynamical systems (Ekhaguere (1991)).

Representations

Many aspects of partial algebras are studied by means of homomorphisms, especially representations.

Definition

1. A *homomorphism* σ from a partial algebra $(\mathfrak{A}_1, \Gamma_1, \circ)$ to another partial algebra $(\mathfrak{A}_2, \Gamma_2, \cdot)$ is a linear map such that
   - (a) $(x, y) \in \Gamma_1 \Rightarrow (\sigma(x), \sigma(y)) \in \Gamma_2$ and $\sigma(x \circ y) = \sigma(x) \cdot \sigma(y)$.
   - (b) $\sigma(e_1) = \sigma(e_2)$, if the partial algebras are unital with units e_1 and e_2
   - (c) $\sigma(x^*) = (\sigma(x))^\#$ if the partial algebras are involutive.
2. An *isomorphism* σ of a partial algebra $(\mathfrak{A}_1, \Gamma_1, \circ)$ to another partial algebra $(\mathfrak{A}_2, \Gamma_2, \cdot)$ is a bijective homomorphism σ such that σ^{-1} is also a homomorphism of $(\mathfrak{A}_2, \Gamma_2, \cdot)$ onto $(\mathfrak{A}_1, \Gamma_1, \circ)$.

Partial Algebras of Class E

States

A state on a partial algebra $(\mathfrak{A}, \Gamma, \cdot)$ is a sesquilinear form $\varphi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{C}$ satisfying $\varphi(x, x) \geq 0$, for all $x \in \mathfrak{A}$.

Notation

Write $\text{sesq}(\mathfrak{A})$ for the set of all states on $(\mathfrak{A}, \Gamma, \cdot)$.

Proposition

Every state φ on a partial algebra $(\mathfrak{A}, \Gamma, \cdot)$ gives rise to a triplet $(\mathcal{H}_\varphi, \mathcal{D}_\varphi, \pi_\varphi)$ comprising:

- a Hilbert space $\mathcal{H}_\varphi$;
- a dense subspace $\mathcal{D}_\varphi$; and
- a map π_φ from $\mathfrak{A}$ to $L(\mathcal{D}_\varphi, \mathcal{H}_\varphi)$.

Invariant states

Call $\varphi \in \text{sesq}(\mathfrak{A})$ invariant if there is a nonzero subspace $\mathfrak{A}_\varphi$ of $M_R(\mathfrak{A})$ such that

- $\lambda(\mathfrak{A}_\varphi) \equiv \mathfrak{A}_\varphi/\ker(\varphi)$ is dense in $\mathcal{H}_\varphi$;
- $\langle \pi_\varphi(a)^+ \lambda(x), \lambda(b \circ y)\rangle = \langle \lambda(x), \lambda((a \circ b) \circ y)\rangle$, $\forall\ x, y \in \mathfrak{A}_\varphi$, with $(a, b) \in \Gamma$.

Remark

A partial algebra may not possess any nonzero invariant states. For example, the partial algebras $L^d(\mathbb{R}, dx)$, $1 \leq d < 2$, equipped with pointwise multiplication, possess no bounded nonzero invariant states. In view of this, we introduce a class of partial algebras which I call *partial algebras of class* E.

Definition

Say that a partial algebra $(\mathfrak{A}, \mathbf{\Gamma}, \cdot)$ is of class E if it possesses the following properties:

(i) $\mathfrak{A}$ has a nonempty set $\mathrm{sesq}(\mathfrak{A})_{\mathrm{inv}}$ of nonzero invariant states;

(ii) there is a subset of $\mathrm{sesq}(\mathfrak{A})_{\mathrm{inv}}$ which separates the points of $\mathfrak{A}$.

Remark

The following is a generalization of the usual characterization of abstract C^*-algebras.

Theorem [Ekhaguere (2008)]

Every partial algebra $(\mathfrak{A}, \Gamma, \cdot)$ of class E is isomorphic to some partial subalgebra of $(L(\mathcal{D}, \mathcal{H}), \mathbf{\Gamma}, \circ)$, where $\mathcal{D}$ is a pre-Hilbert space whose completion is $\mathcal{H}$, $\mathbf{\Gamma} = \{(x, y) \in L(\mathcal{D}, \mathcal{H}) \times L(\mathcal{D}, \mathcal{H}) : y\mathcal{D} \subset \mathrm{dom}(x)\}$, and with composition $\circ$ of maps as the partial multiplication on $L(\mathcal{D}, \mathcal{H})$.

Remark

As indicated earlier, every algebra possesses two types of subalgebras: *subalgebras* in the usual sense and *partial subalgebras*. The following result indicates that partial algebras of class E arise as such partial subalgebras.

Theorem [Ekhaguere (2008)]

Every partial algebra of class E may be identified with a partial subalgebra of some algebra.

Representations determined by bitraces

Let $\tau \in \mathrm{Btr}(\mathcal{M})$ and $\mathcal{G}_\tau$ be a core for τ. On the dense subspace $\mathcal{D}_\tau \subset \mathcal{H}_\tau$, define the linear maps $\pi_\tau(x)$ and $\rho_\tau(x)$, $x \in \mathcal{M}$, by

$$\pi_\tau(x)\lambda_\tau(y) = \lambda_\tau(x \cdot y),\ x \in \mathcal{M},\ y \in \mathcal{G}_\tau;$$

$$\rho_\tau(x)\lambda_\tau(y) = \lambda_\tau(y \cdot x),\ x \in \mathcal{M},\ y \in \mathcal{G}_\tau,$$

and denote $\overline{\pi_\tau(x)}$ (resp. $\overline{\rho_\tau(x)}$) again by $\pi_\tau(x)$) (resp. $\rho_\tau(x)$), $x \in \mathcal{M}$. Then, π_τ (resp. ρ_τ) is a representation (resp. antirepresentation) of $\mathcal{M}$ in $L^+(\mathcal{D}_\tau, \mathcal{H}_\tau)$.

Since $\tau(x^+, x^+) = \tau(x, x),\ x \in \mathcal{N}_\tau$, it follows that the involution in $\mathcal{N}_\tau$ is $\|\cdot\|_\tau$-isometric and hence extends to an antilinear isometry $J : \mathcal{H}_\tau \longrightarrow \mathcal{H}_\tau,\ x \mapsto x^+$, satisfying $J^2 = I$, the identity map on $\mathcal{H}_\tau$.

Theorem [Ekhaguere (2007)]

Let $\mathcal{M}$ be a standard, unital, partial O^*-algebra and $(\tau, \mathcal{N}_\tau) \in \mathrm{Btr}(\mathcal{M})$ a semifinite normal bitrace on $\mathcal{M}$. Then, π_τ (resp. ρ_τ) is a normal *representation* (resp. *antirepresentation*) of $\mathcal{M}$ into $L^+(\mathcal{D}_\tau, \mathcal{H}_\tau)$ satisfying the following properties:

(i) $J\pi_\tau(x)J = \rho_\tau(x^+)$, and $J\rho_\tau(x)J = \pi_\tau(x^+)$, $x \in \mathcal{M}$;

(ii) $\pi_\tau(\mathcal{M})''_{\sigma c} \subset \rho_\tau(\mathcal{M})'_c \subset \pi_\tau(\mathcal{M})''_{c\sigma}$,

(iii) $\pi_\tau(\mathcal{M})'''_{cc\sigma} \subset \rho_\tau(\mathcal{M})''_{c\sigma} \subset \pi_\tau(\mathcal{M})'''_{\sigma c\sigma}$.

(iv) $\rho_\tau(\mathcal{M})''_{\sigma c} \subset \pi_\tau(\mathcal{M})'_c \subset \rho_\tau(\mathcal{M})''_{c\sigma}$,

(v) $\rho_\tau(\mathcal{M})'''_{cc\sigma} \subset \pi_\tau(\mathcal{M})''_{c\sigma} \subset \rho_\tau(\mathcal{M})'''_{\sigma c\sigma}$.

Remark

If $\mathcal{A}$ is a W^*-algebra of operators, and π and ρ are the *-representation and *-antirepresentation, respectively, determined by some unbounded trace on $\mathcal{A}$, then we simply have the following relationships: $\pi(\mathcal{A})' = \rho(\mathcal{A})$ and $\rho(\mathcal{A})' = \pi(\mathcal{A})$. Parts (ii) to (v) of the last Theorem reduce to these results if $\mathcal{A}$ is a W^*-algebra.

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