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Propagation bounds for quantum dynamics and applications[1]

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based on joint work with

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Outline

- Dynamics of multi-component quantum systems
- Lieb-Robinson Bounds
- Applications

Quantum Heisenberg dynamics of multi-component systems

The structure of the systems under consideration is as follows:

- (finite) collection of quantum systems: spins, qudits, (an)harmonic oscillators, atoms, quantum dots, ... labeled by $x \in \Lambda$.
- Each system has a Hilbert space $\mathcal{H}_x$. The Hilbert space describing the total system is the tensor product

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x.$$

- Each system has a dynamics described by a self-adjoint Hamiltonian H_x (densely) defined on $\mathcal{H}_x$.

- The algebra of observables of the composite system is

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x) = \mathcal{B}(\mathcal{H}_\Lambda).$$

If $X \subset \Lambda$, we have $\mathcal{A}_X \subset \mathcal{A}_\Lambda$, by identifying $A \in \mathcal{A}_X$ with $A \otimes \mathbb{1}_{\Lambda\setminus X} \in \mathcal{A}_\Lambda$.
E.g., for a two-component system, the observable $A \otimes \mathbb{1}$ measures the quantity represented by A for the first component of the system.

Λ will be equipped with a metric. E.g., For $\Lambda \subset \mathbb{Z}^\nu$, we will use the graph distance on the $\mathbb{Z}^\nu$.

Propagation: what and where?

The Hilbert space for a particle in a domain $\Lambda \subset \mathbb{R}^\nu$ is $L^2(\Lambda)$. $\psi(t)$ is then called the wave function, and one can study its propagation in space. This is not what we are interested in here.

Instead, we are interested in bounding the speed with which information (or disturbances) spread through a collection of interacting quantum systems distributed in space. The metric on Λ measures the distance between the different components of the system.

Interactions

We will consider bounded interactions modeled by map Φ from the set of subsets of Λ to $\mathcal{A}_\Lambda$ such that $\Phi(X) \in \mathcal{A}_X$, and $\Phi(X) = \Phi(X)^*$, for all $X \subset \Lambda$. The full Hamiltonian is

$$H = \sum_{x \in \Lambda} H_x \; + \sum_{X \subset \Lambda} \Phi(X).$$

E.g., $\Lambda \subset \mathbb{Z}^\nu$, $\mathcal{H}_x = \mathbb{C}^2$; the spin-1/2 Heisenberg Hamiltonian:

$$H = \sum_{x \in \Lambda} B\sigma_x^3 \; + \sum_{|x-y|=1} J_{xy}(\sigma_x^1\sigma_y^1 + \sigma_x^2\sigma_y^2 + \sigma_x^3\sigma_y^3)$$

The Heisenberg dynamics, $\{\tau_t\}_{t\in\mathbb{R}}$, defined by

$$\tau_t(A) = e^{itH} A e^{-itH}, \quad A \in \mathcal{A}_\Lambda.$$

For a lattice system of oscillators we consider the standard (unbounded) harmonic interaction and anharmonic perturbations of the following form:

$\Lambda \subset \mathbb{Z}^\nu$, $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} L^2(\mathbb{R})$, $p_x = -i\frac{d}{dq_x}$, and

$$H_\Lambda = \sum_{x\in\Lambda} p_x^2 + \omega^2 q_x^2 + V(q_x) + \sum_{|x-y|=1} \lambda(q_x - q_y)^2 + \Phi(q_x - q_y)$$

with V and Φ satisfying suitable assumptions.

Examples of systems that are modeled by such Hamiltonians:

- lattice of magnetic moments associated with the atoms in a magnetic material
- a lattice of coupled oscillators
- a collection of quantum dots interacting through tunneling junctions
- discretizations of field theories (lattice QFT)
- an array of qubits in which quantum information is stored or on which a quantum algorithm is performed.

Goal for this lecture

In this lecture I will explain how propagation bounds are an useful tool to investigate complex quantum states of systems with many components, e.g., the ones that describe condensed matter at low temperatures.

Such states are also potentially relevant for quantum computation.

Lieb-Robinson bounds provide an *a priori* upper bound for the rate with which correlations (entanglement, complexity) can spread through the system.

Topics to be discussed

1. **Lieb-Robinson bounds** for the growth rate of the support of time-evolved observables (speed of propagation).
2. The **Exponential Clustering Theorem**: gapped ground states have a finite correlation length.
3. **Existence of the dynamics** for ∞ systems.
4. **Stability of gapped ground states**, Local perturbations perturb locally.
5. **Approximate product structure of gapped ground states**. Area Law for the local entropy of a gapped ground state.

Other applications

- Lieb-Schultz-Mattis Theorem in d dimensions (Hastings, N-Sims)
- Quantized Hall Effect (Hastings-Michalakis)
- Stability of Topologically Ordered Phases (Bravyi-Hastings-Michalakis)
- Complexity of computational problems in quantum many-body theory (Aharonov-Gottesman-Irani-Kempe,...).
- Practical calculation of ground states, equilibrium states, form factors etc. Ideas for new, better algorithms to do such calculations (Verstraete, Osborne, ...).

The support of observables and small commutators

Recall that we can identify $A \in \mathcal{A}_X$ with $A \in \mathcal{A}_Y$, for all Y that contain X ($A = A \otimes \mathbb{1}_{Y \setminus X}$).

- The smallest set X such that $A \in \mathcal{A}_X$, is called the support of A, denoted by $\operatorname{supp} A$. I.e., $A \in \mathcal{A}_X$ iff $\operatorname{supp} A \subset X$.
- Even if interactions are between nearest neighbors only, for generic $A \in \mathcal{A}_\Lambda$, $\operatorname{supp} \tau_t(A) = \Lambda$ for all $t \neq 0$,
- This, however, does *not* mean that quantum dynamics essentially non-local.

For $X, Y \subset \Lambda$, s.t., $X \cap Y = \emptyset$, $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, $AB - BA = [A, B] = 0$: observables with disjoint supports commute. Conversely, if $A \in \mathcal{A}_\Lambda$ satisfies

$$[A, B] = 0, \quad \text{for all } B \in \mathcal{A}_Y \tag{1}$$

then $Y \cap \operatorname{supp} A = \emptyset$.

A more general statement is true: if the commutators in (1) are uniformly small, than A is close to $\mathcal{A}_{\Lambda \setminus Y}$.

Lemma

Let $A \in \mathcal{A}_\Lambda$, $\epsilon \geq 0$, *and* $Y \subset \Lambda$ *be such that*

$$\|[A, B]\| \leq \epsilon \|B\|, \quad \textit{for all } B \in \mathcal{A}_Y \tag{2}$$

then there exists $A' \in A_{\Lambda \setminus Y}$ *such that*

$$\|A' \otimes \mathbb{1} - A\| \leq c\epsilon$$

with $c = 1$ *if* $\dim \mathcal{H}_Y < \infty$ *and one can take* $c = 2$ *in general.*

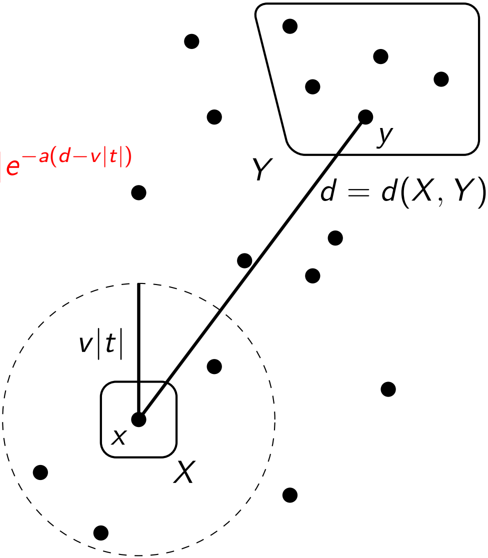
$\Rightarrow$ we can investigate $\operatorname{supp} \tau_t(A)$ by estimating $[\tau_t(A), B]$ for $B \in \mathcal{A}_Y$. This is what Lieb-Robinson bounds are about.

Lieb-Robinson bound :

$\exists C, v, a > 0$ such that

$$\| [\tau_t(A), B] \| \leq C\|A\| \|B\| e^{-a(d - v|t|)}$$

where $t \in \mathbb{R}$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, for finite $X, Y \subset V$, and $d = d(X, Y)$. The first such estimates were proved by Lieb & Robinson (1972).



Lieb-Robinson bounds $\Longrightarrow$ local approximation of $\tau_t(A)$

Up to small corrections, the diameter of the support of $\tau_t(A)$ does not grow faster than linearly in t. More precisely, by the Lemma, one has the following result:

There exist $C > 0$, such that for all $\delta > 0$, there exists A_t^δ, supported in a ball of radius $v|t| + \delta$ such that $\|A_t^\delta - \tau_t(A)\| \leq C\|A\|e^{-a\delta}$.

The usefulness of this property for a wide variety of applications was first realized by Matthew Hastings (2004).

Lieb-Robinson bounds

Theorem (Lieb-Robinson 1972, N-Sims 06, Hastings-Koma 06, N-Sims-Ogata 06, Eisert-Osborne 06, N-Sims 2007)

Let $a, \epsilon > 0$ and assume that the interactions $\Phi(X)$ satisfy

$$\|\Phi\|_a := \sup_{x \neq y} (1 + d(x,y))^{\nu+\epsilon} e^{a d(x,y)} \sum_{X \ni x,y} \|\Phi(X)\| < \infty$$

Then, there exists constants C and v (depending only on $a, \epsilon, \|\Phi\|_a, \nu$), such that for all local observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, one has the bound

$$\|[\tau_t(A), B]\| \leq C\|A\| \, \|B\| \, \min(|X|, |Y|) e^{-a(d(X,Y) - v|t|)}$$

Note: $|X|$ can be replaced by $|\partial X|$, etc, with a suitable definition of the boundary ∂X.

1. Existence of the Thermodynamic limit

In bulk matter or in large systems, we expect to be able to model the behavior at a given location up to some finite time t, based on what we know about the local environment. We don't need to know the state of the entire system or even its size. In particular, we would like to have a well-defined dynamics for local observables of the infinite system.

Basic question: does

$$\tau_t^\Lambda(A) \longrightarrow \tau_t(A), \text{ as } \Lambda \uparrow \mathbb{Z}^\nu \text{ ?}$$

Is there a well-defined dynamics for the ∞ system?

One can use Lieb-Robinson bounds to establish the existence of the thermodynamic limit:

Let Λ_n be an increasing exhausting sequence of finite subsets of an infinite system with local Hamiltonians of the form

$$H_{\Lambda_n} = \sum_{X \subset \Lambda_n} \Phi(X)$$

The essential observation is the following bound: for $n > m$

$$\|\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A)\| \leq \sum_{X \subset \Lambda_n, X \cap \Lambda_n \setminus \Lambda_m \neq \emptyset} \int_0^{|t|} \|[\Phi(X), \tau_t^{\Lambda_m}(A)]\| \, ds.$$

Define $f(t) = \tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A)$. Then

$$\begin{aligned} f'(t) &= i[H_{\Lambda_n}, \tau_t^{\Lambda_n}(A)] - i[H_{\Lambda_n}, \tau_t^{\Lambda_m}(A)] \\ &\quad - i[H_{\Lambda_m}, \tau_t^{\Lambda_m}(A)] + i[H_{\Lambda_n}, \tau_t^{\Lambda_m}(A)] \\ &= i[H_{\Lambda_n}, f(t)] + i[(H_{\Lambda_n} - H_{\Lambda_m}), \tau_t^{\Lambda_m}(A)] \end{aligned}$$

Therefore, since $f(0) = 0$,

$$\begin{aligned} \|f(t)\| &\leq \int_0^t \|\text{non-norm-conserving part of } f'(t)\| \, dt \\ &\leq \int_0^t \|[(H_{\Lambda_n} - H_{\Lambda_m}), \tau_t^{\Lambda_m}(A)]\| \end{aligned}$$

which for Hamiltonians of the form under consideration implies the formula on the previous slide.

Theorem

Let $a \geq 0$, and Φ such that $\|\Phi\|_a < \infty$. Then, the dynamics $\{\tau_t\}_{t\in\mathbb{R}}$ corresponding to Φ exists as a strongly continuous, one-parameter group of automorphisms on $\mathcal{A}$. In particular,

$$\lim_{n\to\infty} \|\tau_t^{\Lambda_n}(A) - \tau_t(A)\| = 0$$

for all $A \in \mathcal{A} = \overline{\bigcup_n \mathcal{A}_{\Lambda_n}}$. The convergence is uniform for t in compact sets and independent of the choice of exhausting sequence $\{\Lambda_n\}$.

Anharmonic oscillator lattice

$$H = \sum_{x \in \mathbb{Z}^\nu} p_x^2 + \omega^2 q_x^2 + V(q_x) \quad + \sum_{|x-y|=1} \lambda (q_x - q_y)^2$$

with a sufficiently nice bounded V. Since the canonical operators (p's, q's , a's, a^*'s,...) are all unbounded, we will instead work with the Weyl operators:

$$W(f) = \exp\left[\frac{i}{\sqrt{2}}\left(a(f) + a^*(f)\right)\right] ,$$

where the creation and annihilation operators for each oscillator are defined by

$$a_x = \frac{1}{\sqrt{2}}(q_x + ip_x) \quad \text{and} \quad a_x^* = \frac{1}{\sqrt{2}}(q_x - ip_x) .$$

Theorem (N-Schlein-Sims-Starr-Zagrebnov, RMP 2010)

Assume $\omega > 0$, $\|k\hat{V}(k)\|_1 < \infty$, $\|k^2\hat{V}(k)\|_1 < \infty$. *For all* $f \in \ell^1(\mathbb{Z}^\nu)$, *and all* $t \in \mathbb{R}$, *the limit*

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu} \tau_t^\Lambda(W(f)) = \tau_t^\infty(W(f))$$

converges in the operator norm topology and the resulting the dynamics is continuous in t *in the weak operator topology.*

The convergence has to be considered on a suitable Hilbert space (a regular representation), e.g., Fock space.

The essential argument for the proof of this theorem is again based on a Lieb-Ronbinson bound.

Theorem (N-Raz-Schlein-Sims, CMP 2009)

Let $\lambda \geq 0, \omega > 0$, *and* V *such that* $\|k^2 \hat{V}(k)\|_1 < \infty$. *Then, for all* $f, g \in \ell^1(\mathbb{Z}^\nu)$, *we have*

$$\left\| \left[\tau_t^\Lambda(W(f)), W(g)\right] \right\| \leq C \sum_{x,y} |f(x)|\,|g(y)| e^{-2(d(x,y)-v|t|)}$$

with

$$v = 6\sqrt{\omega^2 + 4\nu\lambda} + c\|k^2 \hat{V}(k)\|_1.$$

2. The Exponential Clustering Theorem

In a relativistic quantum field theory, the speed of light plays the role of an automatic bound for the Lieb-Robinson velocity. This implies decay of correlations in a QFT with a gap and a unique vacuum (Ruelle, others). Fredenhagen proved an exponential bound for the decay: $\sim e^{-\gamma c^{-1}|x|}$, i.e., $\xi \leq c/\gamma$. In a QFT the gap γ is interpreted as the mass of the lightest particle.

In condensed matter physics, a gap also implies exponential decay under general conditions (Hastings 2004, N-Sims 2006, Hastings-Koma 2006).

Theorem (N-Sims 2006, Hastings-Koma 2006)

Suppose that $\|\Phi\|_a < \infty$ *for some* $a > 0$ *and that* H *has a spectral gap* $\gamma > 0$ *above a unique ground state* $\langle\cdot\rangle$*. Then, for all* $A \in \mathcal{A}_X$*,* $B \in \mathcal{A}_Y$*,*

$$|\langle AB\rangle - \langle A\rangle\langle B\rangle| \leq c\|A\|\,\|B\|\min(|X|,|Y|)e^{-\mu d(X,Y)}.$$

with $\mu = (a\gamma)/(\gamma + 4\|\Phi\|_a)$*. If interaction is of finite range:*

$$|\langle AB\rangle - \langle A\rangle\langle B\rangle| \leq c\|A\|\,\|B\|\min(|\partial_\Phi X|,|\partial_\Phi Y|)e^{-\mu d(X,Y)}.$$

with

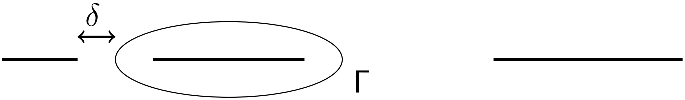
$$\partial_\Phi X = \{x \in X \mid \exists Z \text{ with } x \in Z, Z \cap X^c \neq \emptyset, \text{ and } \Phi(Z) \neq 0\}.$$

3. Local perturbations perturb locally

We begin with an instructive result which does not by itself imply the later results to be discussed but which makes them at least plausible. Consider a system with a Hamiltonian of the form

$$H_s = H_0 + s\Phi, s \in [0, 1]$$

and suppose that the spectrum of H_s can be decomposed into two parts $\Sigma_1(s)$ and $\Sigma_2(s)$ such that, for all s, $\Sigma_1(s) \subset I$, a bounded interval, and $\Sigma_2(s) \subset \mathbb{R} \setminus (I - \delta) \cup (I + \delta)$, for some fixed $\delta > 0$.



Let P_s denote the spectral projection of H_s corresponding to $\Sigma_1(s)$. One can show that

$$P_s = U_s P_0 U_s^*$$

with U_s, the unitary solution U_s of

$$\frac{d}{ds} U_s = iD_s U_s, U_0 = \mathbb{1},$$

and

$$D_s = \int_{-\infty}^{\infty} dt\, F_\delta(t) e^{itH_s} \Phi e^{-itH_s}$$

The function $F_\delta \in L^1(\mathbb{R})$ can be chosen to decay faster than any power.

E.g., consider a system with Hamiltonian H_s of the form $H_0 + s\Phi$, with the following additional assumptions:

1. The system is defined on a metric graph and there are constants $C(A, B)$, $\mu > 0$ and a Lieb-Robinson velocity $v \geq 0$ such that for all $s \in [0, 1]$

$$\|[\tau_t^{(s)}(A), B]\| \leq C(A, B)e^{-\mu(d - v|t|)}$$

   Here, τ_t is the Heisenberg dynamics generated by H_s, $d = \mathrm{dist}(\mathrm{supp}\, A, \mathrm{supp}\, B)$, and $C(A, B)$ is of a suitable form such as $C\|A\| \, \|B\| \min(|\mathrm{supp}\, A|, |\mathrm{supp}\, B|)$.
2. $\Phi = \Phi^*$ is a bounded perturbation of finite support.

Let B_R denote the following 'fattening' of $\mathrm{supp}\,\Phi$:

$$B_R = \{x \text{ such that there exists } y \in \mathrm{supp}\,\Phi \text{ with } d(x,y) \leq R\}$$

Theorem

For any $m > 0$, there is a constant C_m such that, for every R large enough there exists a unitary V_R with $\mathrm{supp}\, V_R \subset B_R$ and

$$\|P_1 - V_R^* P_0 V_R\| \leq C_m R^{-m}$$

Idea of the proof:

The LR bounds imply that there exists a constant $C(\Phi)$ and $\Phi(R,s,t) = \Phi(R,s,t)^*$ with $\mathrm{supp}\,\Phi(R,s,t) \subset B_R$ and such that

$$\|\tau_t^{(s)}(\Phi) - \Phi(R,s,t)\| \leq C(\Phi) e^{-\mu(R - v|t|)}$$

Pick a choice of F_δ which satisfies

$$|F_\delta(t)| \leq F_n |t|^{-n}$$

to define $D_s(R)$ with $\mathrm{supp}(D_s(R)) \subset B_R$ as follows:

$$D_s(R) = \int_{-T}^{T} dt\, F_\delta(t) \Phi(R, s, t).$$

With $T = (\mu R - m \log R)/(\mu v)$ and $n = m + 1$, we have

$$\|D_s(R) - D_s\| \leq \frac{F_n \|\Phi\|}{n-1} |T|^{-m} + C(\Phi) e^{-\mu(R - vT)} \leq C'_m R^{-m}$$

One now just has to define V_R to be solution at $s = 1$ of

$$-i \frac{\partial}{\partial s} V_s = D_s(R) V_s,\, V_0 = \mathbb{1}.$$

Meaning of this result for an isolated eigenvector:
If $P_0 = |\psi_0\rangle \langle\psi_0|$, we get that we see that, for any n, there is an operator V_R with $\mathrm{diam}\,\mathrm{supp}(\Phi) \leq r + R$ such that

$$\|\psi_1 - A_R\psi_0\| \leq C_n\|\Phi\|R^{-n}.$$

E.g., if the ground state ψ_s of H_s is isolated in the sense of above, ψ_s is well-approximated by a local perturbation of ψ_0.The effect of a local operator can of course be felt far away if there are long-range correlations in ψ_0. Those correlations can be controlled with the Exponential Clustering Theorem.

4. Structure of gapped ground states

The Exponential Clustering Theorem says that a non-vanishing gap γ implies a finite correlation length ξ. Can one say more? E.g., with the goal of devising better algorithms to compute ground states? Or to do quantum computation?

It turns out that gapped ground states have an approximate product structure.

An approximation factorization theorem

We will consider a system of the following type: Let Λ be a finite subset of $\mathbb{Z}^\nu$. At each $x \in \Lambda$, we have a finite-dimensional Hilbert space of dimension n_x. Let

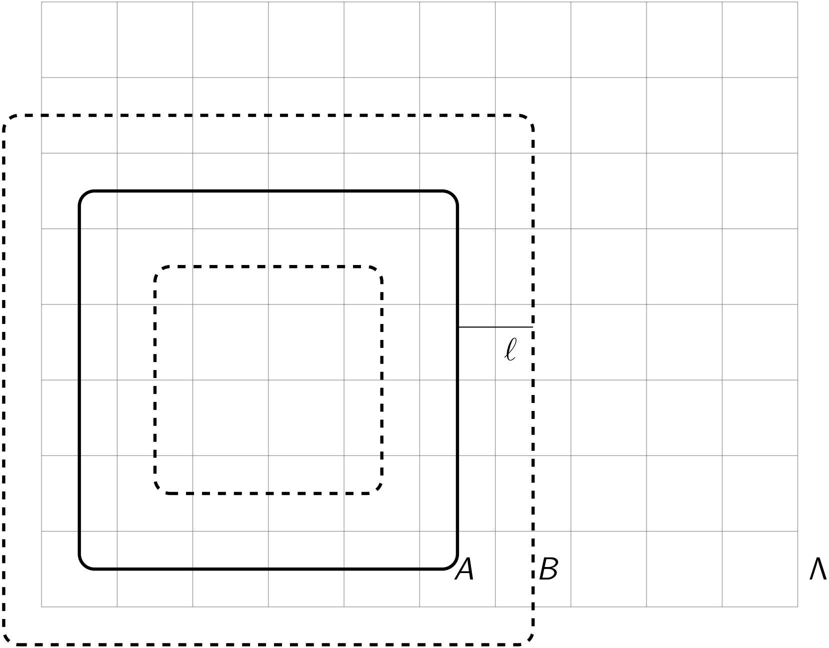
$$H_V = \sum_{\{x,y\} \subset \Lambda, |x-y|=1} \Phi(x,y),$$

with $\|\Phi(x,y)\| \leq J$. Suppose H_V has a unique ground state and denote by P_0 the corresponding projection, and let $\gamma > 0$ be the gap above the ground state energy.

For a set $A \subset \Lambda$, the boundary of A, denoted by ∂A, is

$$\partial A = \{x \in A \mid \text{there exists } y \in \Lambda \setminus A, \text{with } |x-y| = 1\}.$$

For $\ell \geq 1$, define $B(\ell) = \{x \in \Lambda \mid d(x, \partial A) < \ell\}$.


ℓ
A
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Λ

The following generalizes a result by Hastings (2007):

Theorem (Hamza-Michalakis-N-Sims JMP2009)

There exists $\xi > 0$ (given explicitly in terms of d, J, and γ), such that for any sufficiently large $\ell > 0$, and any $A \subset \Lambda$, there exist two orthogonal projections $P_A \in \mathcal{A}_A$, and $P_{\Lambda \setminus A} \in \mathcal{A}_{\Lambda \setminus A}$, and an operator $P_B \in \mathcal{A}_{B(\ell)}$ with $\|P_B\| \leq 1$, such that

$$\|P_B(P_A \otimes P_{\Lambda \setminus A}) - P_0\| \leq C(\xi)|\partial A|^2 e^{-\ell/\xi}$$

where $C(\xi)$ is an explicit polynomial in ξ.

Area Law for Gapped 1-Dim'l Systems

The Area Law Conjecture states that the entropy of local restrictions of a gapped ground state grows no faster than the surface area for the local region. In general, this means that for $X \subset \Lambda$ and $\rho_X \in \mathcal{A}_X$ the density matrix describing the restriction of the state to $\mathcal{A}_X$, then

$$S(\rho_X) = -\text{Tr}\rho_X \log \rho_X \leq C|\partial X|$$

So far proved only for one-dimensional systems (in general).

Theorem (Hastings 07)

Consider a quantum spin chain with $\dim \mathcal{H}_x \leq n$, *and with Hamiltonian*

$$H = \sum_x \Phi(\{x, x+1\})$$

with $\|\Phi(\{x, x+1\})\| \leq J$ *and assume* H *had a unique ground state with a gap* $\gamma > 0$ *in the spectrum above the ground state. There exist constants* C *and* $\xi < \infty$ *depending only on* J *and* γ, *such that*

$$S(\rho_{[a,b]}) \leq C\xi \log(\xi) \log(n) 2^{\xi \log(n)}$$

The proof of this theorem uses the factorization property of gapped ground states which in turn relies on Lieb-Robinson bounds in an essential way.

Concluding remarks

- The local structure of interactions in physical systems implies a finite speed of propagation.
- Bounds on the speed of propagation of the Lieb-Robinson type allow for a (quasi-) local analysis of the dynamics and existence of the dynamics for infinite systems.
- The locality of the dynamics in turn implies useful properties of important states of the system, such as the ground state (and equilibrium states).
- Propagation bounds can also be derived for irreversible dynamics (Markov semigroups), Poulin 2010.