On the Distribution of the Eigenvalues for Non-selfadjoint Operators

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1. Possible spectra for selfadjoint operators

Let A be a selfadjoint operator in a Hilbert space $\mathfrak{H}$.

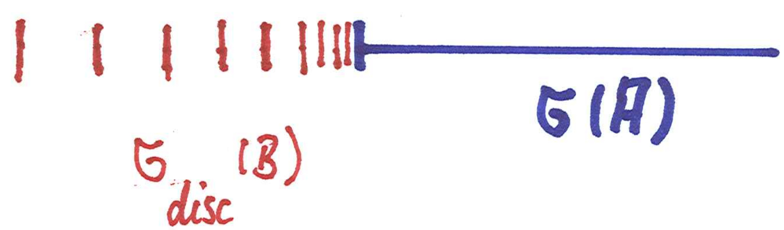
Let B be a perturbation of A given by $B = A + M$.

We are interested in the discrete spectrum of B, i. e. in the isolated eigenvalues of finite multiplicity.

If B is selfadjoint there is only a restricted number of possibilities:

Assume $\sigma(A) = [c\ ,\ \infty)\ ,\quad c \in \mathbb{R}$ (very often $c = 0$) .

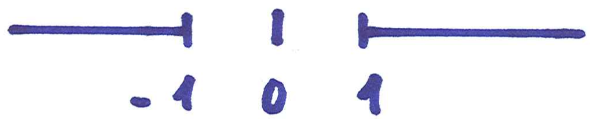
Then $\sigma_{disc}(B)$ may have the form



(atoms, molecules)

The only accumulation point is c .

Another possibility: $\sigma(A)$ has gaps

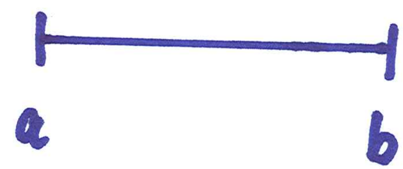


Dirac operators

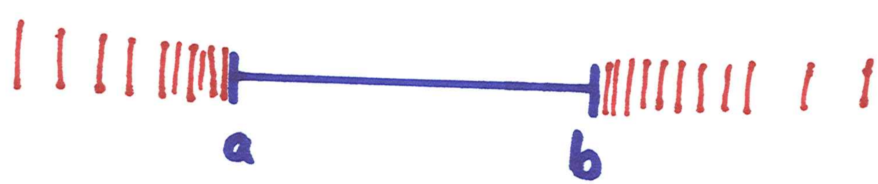
solid state physics

then $\sigma_{disc}(B)$ is

For bounded A , $\sigma(A) = [a, b]$



Then $\sigma_{disc}(B)$ (discrete Schrödinger operators)



2. Possible spectra for non-selfadjoint operators

The variety is much larger.

The spectral theory is much less developed in comparison with the selfadjoint case.

Example 1: For selfadjoint operators it is known that

$$\sigma(B) = \sigma_{\text{ess}}(B) \,\dot{\cup}\, \sigma_{disc}(B) \qquad \text{(disjoint union)}$$

For non-selfadjoint B this must not be true. One prominent example is the shift operator in $l^2(\mathbb{N})$.

Let $(Bf)(n) = f(n+1)\ , n \in \mathbb{N}\ . \quad f \in l^2(\mathbb{N})\ .$

Then

$$\begin{aligned} \sigma_{\mathrm{ess}}(B) &= \{z \in \mathbb{C}\ ;\ |z| = 1\} \\ \sigma(B) &= \{z \in \mathbb{C}\ ; |z| \leqq 1\} \\ \sigma_{disc}(B) &= \emptyset\ . \end{aligned}$$

(see e. g. Kato book, p. 237)

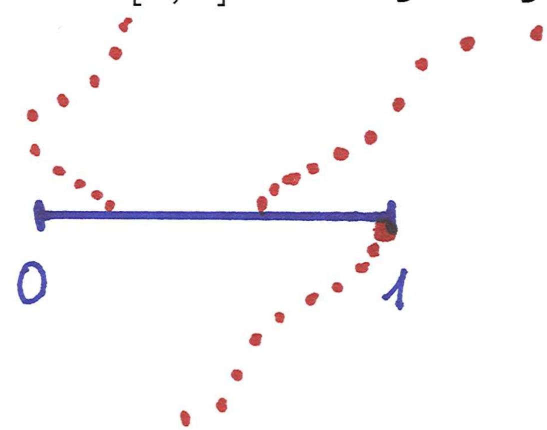
Example 2

Let A be a bounded selfadjoint operator, $\sigma(A) = [0,1]$.

Let $B = A + M$ and M a rank one operator.

If B is selfadjoint, $\sigma_{disc}(B)$ consists of at most one eigenvalue in $\mathbb{R} \setminus [0,1]$.

If B is not selfadjoint, G. Katriel (2008) has constructed a rank one operator M such that $\sigma_{disc}(B)$ has infinitely many eigenvalues in $\mathbb{C} \setminus [0,1]$. They may accumulate at any point in $[0,1]$.



Example 3: $L^2(\mathbb{R}_+)$

Let $Bf = -\frac{d^2 f}{dx^2} + Vf$ with boundary conditions of the form: $f(0) = hf'(0)$ on $L^2(0, \infty)$.

If V and h are real (selfadjoint case) $\sigma_{disc}(B)$ is finite

if $$\int_0^\infty x\, |V(x)|\, dx < \infty .$$

If V and h are complex $\sigma_{disc}(B)$ is finite

if

$$\sup_{0 \leq x < \infty} |V(x)| e^{\epsilon x} < \infty \ , \quad \epsilon > 0 \ ,$$

strong decrease!

and if

$$\int_0^\infty x \, |V(x)| \, e^{\epsilon x} \, dx < \infty \ , \quad \epsilon > 0 \ .$$

Pavlov (1966) showed that $\sigma_{disc}(B)$ is also finite if

$$\sup_{0 \le x < \infty} |V(x)| e^{\epsilon\sqrt{x}} < \infty \, .$$

This seems to be the borderline.

Because Pavlov (1967) also showed that $\sigma_{disc}(B)$ is not finite if only

$$\sup_{0 \le x < \infty} |V(x)| e^{\epsilon |x|^{\alpha}} < \infty \quad \text{for} \quad \alpha < \tfrac{1}{2} \, .$$

And the accumulation points can be somewhere in $(0, \infty)$.

Example 4: Localization of the discrete spectrum

Consider $B = -\dfrac{d^2}{dx^2} + V$ in $L^2(\mathbb{R})$.

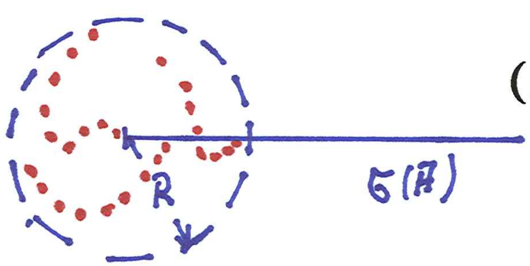
V complex valued and $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Then V is relatively compact w.r.t. $-\dfrac{d^2}{dx^2}$.

$$\sigma(B) = [0,\infty) \cup \{\text{eigenvalues accumulating on } [0,\infty)\}.$$

But the spectrum is localized, i.e.

$$\sigma(B) \subseteq [0,\infty) \cup \{z \in \mathbb{C}\ ;\ |z| \leq \overbrace{\frac{1}{4}\|V\|^2_{L^1}}^{R}\}.$$



(Abranov, Aslanyan, Davies, 2001)

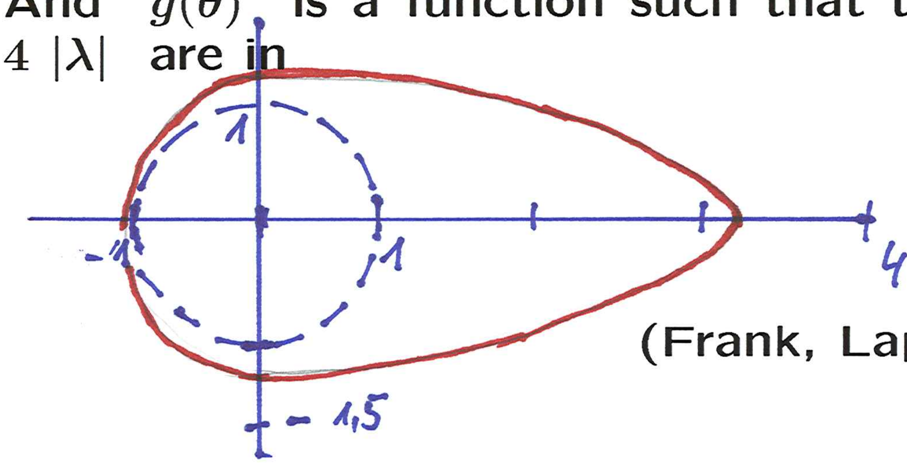
A similar result holds for

$B = -\frac{d^2}{dx^2} + V$ in $L^2(0,\infty)$ with Dirichlet boundary conditions.

Let $\lambda = |\lambda| e^{i\Theta}$ be an eigenvalue of B . Then

$$|\lambda|^{\frac{1}{2}} \leqq \frac{1}{2}\, g(\theta) \int_0^\infty |V(x)|\, dx$$

And $g(\theta)$ is a function such that the maximal values of $4\,|\lambda|$ are in



(Frank, Laptev, Seiringer, 2009)

For general $L^2(\mathbb{R}^d)$, $d \geq 2$, $B = -\Delta + V$,

Safronov (2008) showed that for

$$|V(x)| \leq \; c \, \frac{1}{(1+|x|)^q} \; , \;\; q > 1 \; ,$$

$\sigma_{disc}(B)$ is contained in a disc.

Frank (2010) proved for $d \geqq 2$ and $0 < \gamma \leq \frac{d}{2}$,

that any eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ of B satisfies

$$|\lambda|^\gamma \leqq D_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma + \frac{d}{2}} \, dx \; ,$$

where $D_{\gamma,d}$ is a constant independent of V .

Concluding these remarks there are several problems:

1) In which region of $\mathbb{C}$ are the non-real eigenvalues of B located?

2) Can they accumulate at infinity?

3) Are they contained in a disc or in a bounded region?

4) A rough estimate can be given by the numerical range:

$$\text{Num}(B) = \{< Bf, f >,\ \ f \in \mathrm{dom\,B};\ \ \|\mathrm{f}\| = 1\}$$

If B is closed, if $\mathbb{C} \setminus \overline{\text{Num}(B)}$ is connected and if $\mathbb{C} \setminus \overline{\text{Num}(B)}$ contains at least one point not in $\sigma(B)$, then $\sigma(B) \subseteq \overline{\text{Num}(B)}$.

5) What is the essential spectrum of a non-selfadjoint operator?

There are several ($\geqq 5$) definitions of the essential spectrum. We follow

E. B. Davies: Linear operators and their spectra Cambridge 2007, Chapters: 4.3. and 11.2.

Definition:

A bounded operator P in a Banachspace is Fredholm iff ran(P) is closed and if both

$$\dim \text{ker}(P) < \infty \quad , \quad (\text{null}(P) < \infty)$$

and $\dim \text{coker}(P) < \infty \quad , \quad (\text{def}(P) < \infty)$.

$\lambda \in \sigma_{\text{ess}}(P)$ iff $(\lambda - P)$ is not a Fredholm operator

(Davies allows semi-Fredholm points).

Let P be a closed unbounded operator acting in a Banachspace, let $\lambda \notin \sigma(P)$. Then

$$z \in \sigma_{\text{ess}}(P) \quad \text{iff} \quad z \neq \lambda \quad \text{and} \quad (\lambda - z)^{-1} \in \sigma_{\text{ess}}\{(\lambda I - P)^{-1}\}.$$

$\mathbb{C} \setminus \sigma_{\text{ess}}(P)$ may consist of a bounded and an unbounded set.

In the unbounded set $\sigma(P)$ consists only of a countable set of isolated eigenvalues with finite algebraic and geometric multiplicities.

If A and B are closed unbounded operators and if

$s \in \text{res}(A) \cap \text{res}(B)$ exists such that $(s-B)^{-1} - (s-A)^{-1}$ is compact,

then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$. (Weyl's Theorem, Davies p. 331).

In the following we consider a selfadjoint A with

$\sigma(A) = [0, \infty)$, and closed (nonselfadjoint) operator B in $\mathfrak{H}$.

For some $s \in \text{res}(A) \cap \text{res}(B)$ we assume

$(s-B)^{-1} - (s-A)^{-1} \in \mathcal{S}_p$.

Then $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A) = [0, \infty)$.

Hence $\mathbb{C} \setminus [0, \infty)$ is the unbounded set in the complement of $\sigma_{\text{ess}}(B)$.

$\Rightarrow$ In $\mathbb{C} \setminus [0, \infty)$ there are only eigenvalues, isolated, with finite multiplicities.

They may accumulate at any point of

$\sigma(A) = [0, \infty)$.

3. Quantitative estimates

The next theorem gives some answers to these questions

Theorem (Hansmann, Katriel, Demuth, 2008)

Let A be a selfadjoint operator in a Hilbert space $\mathfrak{H}$.

Let $\sigma(A) = [0, \infty)$.

Let B be a closed operator in $\mathfrak{H}$.

Assume that for some $s \in \text{res}(B) \cap \text{res}(A)$

$R := (s - B)^{-1} - (s - A)^{-1} \in \mathcal{S}_p \quad p > 0$. (Schatten class)

Then eigenvalues of B satisfy

$$\sum_{\lambda \in \sigma_{disc}(B)} \frac{\text{dist}(\lambda, [0,\infty))^\gamma}{|\lambda|^{\gamma/2} \; (1+|\lambda|)^\gamma} \leq c \; \|R\|_{S_p}^p$$

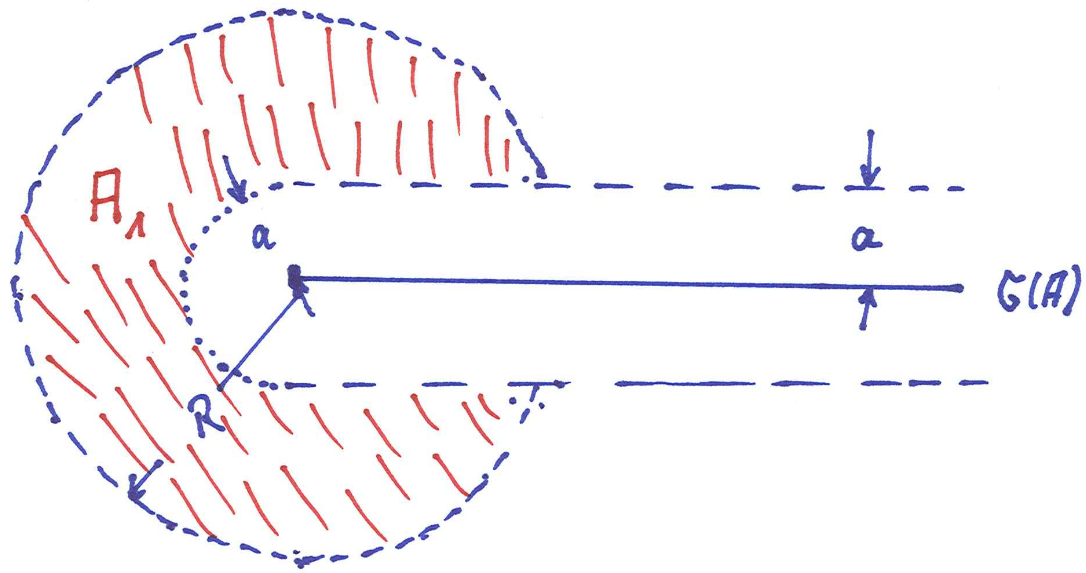
$$\begin{aligned} c &= c_{\gamma,p} \; . \\ \gamma &\geq \max\; (1+p, \;\; 2p) \\ p &> \;\; 0 \; . \end{aligned}$$

This formula gives a possible distribution of the eigenvalues near $[0,\infty) = \sigma(A)$ and also of their number in certain domains of $\mathbb{C}$.

Example 1:

Let $A_1 := \{z : \text{dist}[z, [0, \infty)] \geq a, \ |z| < R\}$

i.e.



Then

$$c_0 \geqq \sum_{\lambda \in \mathsf{disc}(B)} \frac{(\mathsf{dist}[\lambda, [0, \infty)])^\gamma}{|\lambda|^{\gamma/2}\ (1 + |\lambda|)^\gamma}$$

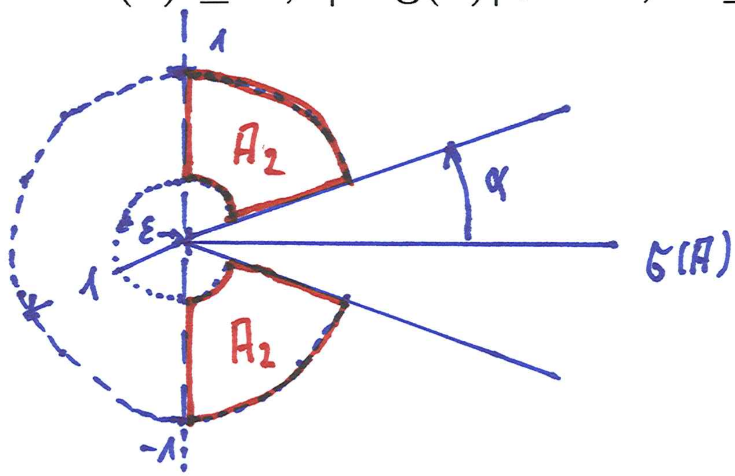
$$\geqq a^\gamma \frac{1}{R^{\gamma/2}} \frac{1}{(1+R)^\gamma} N_B(A_1) ,$$

where $N_B(A_1)$ is the number of eigenvalues of B in A_1 , or

$$N_B(A_1) \leq c_0 \frac{1}{a^\gamma} R^{\gamma/2}\ (1+R)^\gamma .$$

Example 2:

Let $A_2 = \{z : Re(z) \geqq 0\,,\ |\arg(z)| > \ \alpha\,,\ \epsilon \leq |z| \leq 1\}$



Then dist $[\lambda, [0, \infty)] = |\lambda|\ |\sin(\arg(\lambda))| \geq |\lambda|\ |\sin\alpha|$

Such that:

$$N_B(A_2) \leq 2^\gamma\ c_0\ \frac{1}{|\sin\alpha|^\gamma}\ \frac{1}{\epsilon^{\gamma/2}}\ .$$

From

$$\sum_{\lambda \in \sigma_{disc}(B)} \frac{\text{dist}(\lambda, [0,\infty))^\gamma}{|\lambda|^{\gamma/2} \ (1+|\lambda|)^\gamma} \leq c \ \|R\|_{S_p}^p$$

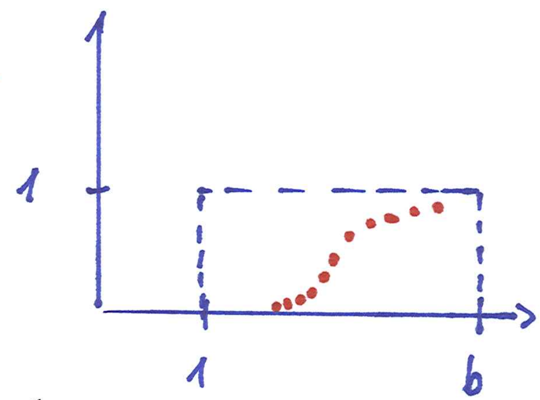
we get for the left hand plane:

$$-\omega_0 \leq Re\lambda \leq 0$$

$$\sum_{\lambda \in \sigma_{\text{disc}(B)}} |\lambda|^{\gamma/2} \leq c \ \|R\|_{S_p}^p \ .$$

Here we used that the numerical range of B is contained in $\{z \in \mathbb{C} : \ Rez \geq -\omega_0\}$, $\omega > 0$.

For $1 \leq Re\ \lambda \leq b$ and $0 < |\mathrm{Im}\ \lambda| \leq 1$



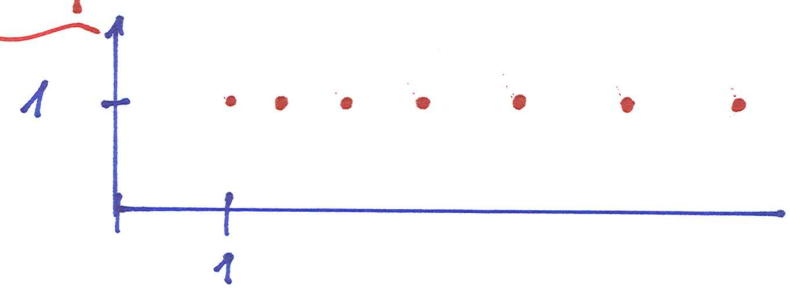
we get

$$\sum_{\lambda \in \sigma_{\mathrm{disc}}(B)} |\mathrm{Im}\ \lambda|^{\gamma} \leq c\ \|R\|_{S_p}^{p} .$$

And if for instance $|\mathrm{Im}\ \lambda| = 1$, $Re\ \lambda > 1$

then

$$\sum_{\lambda \in \sigma_{\mathrm{disc}}(B)} |\lambda|^{-\frac{3}{2}\gamma} \leq c\ \|R\|_{S_p}^{p} .$$



4. Proof method

The original idea is based on Jensen's identity.

Let $h : U \to \mathbb{C}$ be a holomorphic function

$U-$ open unit disc.

Let $h(0) = 1$.

Let Ω_r be an open disc centered at zero, radius r and $\overline{\Omega}_r \subset U$.

Then (see e.g. Rudin, Real and Complex Analysis, p. 307)

Jensen's identity

$$\int_0^r \frac{n(u)}{u}\,du = \log\left(\prod_{\substack{z\in\overline{\Omega}_r\\ h(z)=0}} \frac{r}{|z|}\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\Theta})|\,d\Theta$$

$n(u)-$ number of zeros of $h(.)$ in $\overline{\Omega}_u$, $0 < u < r$.

At first G. Katriel has used this idea for studying $\sigma_{disc}(B)$ of selfadjoint B where

$$h(z) = \det(1 - F(z))$$

with a trace class operator $F(z)$ given by

$$F(z) = z\left(1 - z\ e^{-A}\right)^{-1}\left(e^{-B} - e^{-A}\right)$$

assuming $e^{-B} - e^{-A}$ to be a trace class operator.

In this case

$$\sigma_{disc}(B) = \{\log z\ ,\ |z| < 1\ ,\ h(z) = 0\}\ .$$

In this situation we found

$$\sum_{\lambda \in \sigma_{\mathsf{disc}(B)}} |\lambda|^\gamma \leq \Gamma(\gamma+1)\ \zeta(\gamma-1)\ \|e^{-B} - e^{-A}\|_{\mathsf{tr}} \quad \text{with } \gamma > 2.$$

For non-selfadjoint B the use of semigroups becomes problematic.

In this case it is more appropriate to study resolvents:

$$R := (s - B)^{-1} - (s - A)^{-1} \in \mathcal{S}_1 \quad \text{or} \quad \in \mathcal{S}_p \ .$$

Thus we got the result mentioned above

$$\sum_{\lambda \in \sigma_{disc}(B)} \frac{[\text{dist}(\lambda, [0, \infty))]^\gamma}{|\lambda|^{\gamma/2} \ (1 + |\lambda|)^\gamma} \leq c_{\gamma, p} \ \|R\|_{S_p}^p$$

$$\text{for } \gamma \geq \max(1 + p, \ 2p)$$
$$p > 0 \ .$$

Improvements

Borichev, Golinskii, Kupin (2008/09) found bounds on the zeros of a holomorphic function in the unit disc in terms of the growth near the boundary. They are able to take into account isolated values ξ_i on δU .

Assume that

$$\log\ |h(z)|\ \leq\ K_0\ \frac{1}{(1-|z|)^{\alpha}}\ \prod_{j=1}^{N}\ \frac{1}{|z-\xi_j|^{\beta j}}$$

with $\quad |\xi_j| = 1\ , \quad \alpha \geq 0\ , \quad \beta_j \geq 0\ .$

Then the zeros of $h(.)$ satisfy

$$\sum_{h(z)=0} (1-|z|)^{\alpha+1+\tau} \prod_{j=1}^{N} |z-\xi_j|^{(\beta_j-1+\tau)_+}$$

$$\leqq c(\alpha, \beta_j, \xi_j, \tau) \cdot K_0 ,$$

with some $\tau > 0$. Here $(q)_+ = 0$ if $q \leq 0$ and

$(q)_+ = q$ if $q \geq 0$.

What is $h(z)$ in our case?

Let $F(\lambda) = (\lambda + s)(s + B)^{-1} M(\lambda - A)^{-1}$

Then $\lambda \in \sigma_{\text{disc}}(B)$ iff $1 - F(\lambda)$ is not invertible.

$F(\lambda)$ is in S_p (for instance trace class).

Then $\lambda \in \sigma_{\text{disc}}(B)$ iff $\det_p \, (1 - F(\lambda)) =: f(\lambda) = 0$,

$\lambda \in \mathbb{C} \setminus [0, \infty)$.

Now let

$$\phi_s(z) = -s\left(\frac{z+1}{z-1}\right) .$$

$\phi_s(.)$ is a conformal mapping from $\mathbb{U} \to \mathbb{C} \setminus [0, \infty)$.

We take $\quad h(z) = f(\phi_s(z)).$

Then

$$\sigma_{\text{disc}}(B) = \{\phi_s(z),\ z \in \mathbb{U},\ h(z) = 0\}, \qquad s \in \text{res}(B) \cap \text{res}(A).$$

The estimates for $h(z)$ are determined by the estimates of $||M(\lambda - A)^{-1}||_{S_p}$.

For $B = A + M$, $\sigma(A) = [0, \infty)$, $\sigma(B) = \sigma_{disc}(B) \ \dot{\cup} \ [0, \infty)$

we assume that

$$\|M(\lambda - A)^{-1}\|_{S_p}^p \leq K \frac{|\lambda|^\beta}{[\text{dist}(\lambda, [0, \infty))]^\alpha} \tag{1}$$

$\alpha \geq 0$, $\beta \in \mathbb{R}$. For $\tau > 0$ we define

$$\begin{aligned} \eta_1 &= \alpha + 1 + \tau \\ \eta_2 &= ((\alpha - 2\beta)_+ - 1 + \tau)_+ \\ \eta_3 &= -\alpha + \beta - \tau \end{aligned}$$

By assumption (1) $\sigma(B) \subset \{\lambda : Re\lambda \geqq -\omega_0, \ \omega_0 > 0\}$.

Then it holds

$$\sum_{\lambda \in \sigma_{disc}(B) \cap U} \frac{[\text{dist}(\lambda, [0,\infty))]^{\eta_1}}{|\lambda|^{\frac{\eta_1}{2} - \frac{\eta_2}{2}} \, ||\lambda| + \omega_0|^{\eta_3 + \frac{1}{2}\eta_1 + \frac{1}{2}\eta_3}} \leq c \cdot K \ .$$

And

$$\sum_{\lambda \in \sigma_{disc}(B) \cap U^c} \frac{[\text{dist}(\lambda, [0,\infty))]^{\eta_1}}{|\lambda|^{\beta+1+\tau}} \leq c \cdot K \quad \text{with} \quad c = c(\alpha, \beta, p, \tau, \omega_0).$$

5. Schrödinger Operators

Take $A = -\Delta$, $B = A + M_V$, $\mathfrak{H} = L^2(\mathbb{R}^d)$,

with $(M_V f)(x) = V(x) f(x)$, $V(.)$ complex valued.

For $d \geqq 2$ we have

$$\|M_V(\lambda - A)^{-1}\|_{S_p}^p \leq c_{p,d} \, \|V\|_{L^p}^p \, \frac{|\lambda|^{\frac{d}{2}-1}}{[\mathrm{dist}(\lambda, [0,\infty))]^{p-1}}$$

for $p > \frac{d}{2}$.

That means in this case $\alpha = p - 1, \ \beta = \frac{d}{2} - 1$.

That implies for the exponents

$$\eta_1 = p + \tau \ , \quad \tau > 0 \ ,$$

$$\eta_2 = \begin{cases} p - d + \tau & \text{for } p \geq d \\ 0 & \text{for } \frac{d}{2} < p < d \ , \ \tau \text{ small.} \end{cases}$$

$$\eta_3 = \frac{d}{2} - p - \tau \qquad \text{(which is negative).}$$

For $p \geqq d$ and eigenvalues in U :

$$\sum_{\lambda \in \sigma_{disc}(B) \cap \, U} \frac{[\text{dist}(\lambda, [0, \infty))]^{p+\tau}}{|\lambda|^{d/2}} \leq c_{p,\tau} \, \|V\|_{L^p}^p \qquad (2)$$

For $d > p > \frac{d}{2}$, τ small

$$\sum_{\lambda \in \sigma_{disc}(B) \cap \, U} \frac{[\text{dist}(\lambda, [0,\infty))]^{p+\tau}}{|\lambda|^{\frac{p}{2}+\frac{\tau}{2}}} \leq c_{p,\tau} \, \|V\|_{L^p}^p \qquad (3)$$

And the eigenvalues of B in U^c and $p > \frac{d}{2}$:

$$\sum_{\lambda \in \sigma_{disc}(B) \cap \, U^c} \frac{[\text{dist}(\lambda, [0,\infty))]^{p+\tau}}{|\lambda|^{\frac{d}{2}+2\tau}} \leq c_{p,\tau} \, \|V\|_{L^p}^p \qquad (4)$$

This is somewhat better than the results by Frank, Laptev, Lieb, Seiringer (2008) or Laptev, Safronov (2008). They need for (3) $p > \frac{d}{2} + 1$.

The detailed proof is published in
Journal Functional Analysis 257 (2009).