Fractional quantum hall effect in rapidly rotating bose gases

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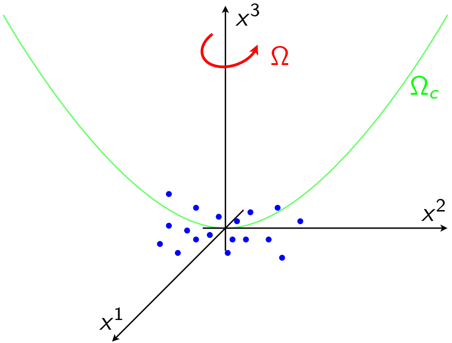
Introduction



N trapped bosons, rotating along the 3rd axis

$x = (x^1, x^2, x^3) \in \mathbb{R}^3$

$z = x^1 + ix^2 \in \mathbb{C}$

$\Omega \geq 0$

Hamiltonian in rotating frame, acting on $\bigvee_1^N L^2(\mathbb{R}^3)$ (notation $L = x \times p$, $p = -i\hbar\nabla$):

$$\sum_{j=1}^{N} \left(\frac{|p_j|^2}{2m} + \frac{m\Omega_c^2}{2}|x_j|^2 - \Omega e_3 \cdot L_j \right) + \sum_{1 \leq i < j \leq N} W(x_i - x_j)$$

The one-body operator

For $A(x) := (-x^2, x^1, 0) = e_3 \times x$, $p \cdot A(x) = A(x) \cdot p = e_3 \cdot L$. Hence

$$\frac{|p|^2}{2m} + \frac{m\Omega_c^2|x|^2}{2} - \Omega e_3 \cdot L = \frac{|p - m\Omega_c A(x)|^2}{2m} + \frac{m\Omega_c^2|x^3|^2}{2} + (\Omega_c - \Omega)e_3 \cdot L$$

unbounded from below for $\Omega > \Omega_c$.

$$\text{Fast rotation} \iff 0 < \omega = \frac{\Omega_c - \Omega}{\Omega_c} \ll 1.$$

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$$\text{FAST ROTATION} \iff 0 < \omega = \frac{\Omega_c - \Omega}{\Omega_c} \ll 1.$$

Choice of units: $\Omega_c = m = \hbar = 1$. Note:

$$\sigma_{L^2(\mathbb{R}^3)}\left(\frac{|p - A(x)|^2 + |x^3|^2}{2}\right) = \left\{3\left(k + \frac{1}{2}\right),\ k = 0, 1, 2, \ldots\right\},$$

$$\ker\left(|p - A(x)|^2 + |x^3|^2 - 3\right) = \left\{F(x^1 + ix^2)e^{-|x|^2/2},\ F \text{ holom.}\right\} := \mathfrak{H}_1.$$

Interaction

We vary the typical length of the interaction W.

• **Scattering length:** assume $W \geq 0$ is radial and decays fast enough.
Let f = sol. of the zero-scattering eq.

$$\begin{cases} -\Delta f + Wf = 0 \\ f(|x|) \to 1 \text{ when } |x| \to \infty \end{cases}$$

Then $f(x) \simeq 1 - \frac{a}{|x|}$ for $|x| \gg 1$. a = scattering length of W.
Scaling: $a\left(\lambda^{-2} W(\cdot/\lambda)\right) = \lambda a(W)$.

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• **Hamiltonian to be studied:**

$$H^N_{\omega,a} = \sum_{j=1}^N \left(\frac{|p_j - A(x_j)|^2 + |x_j^3|^2 - 3}{2} + \omega e_3 \cdot L_j \right) + \sum_{1 \leq i < j \leq N} W_a(x_i - x_j).$$

with $W_a = a^{-2}W(\cdot/a)$ with W fixed having scat. length 1.

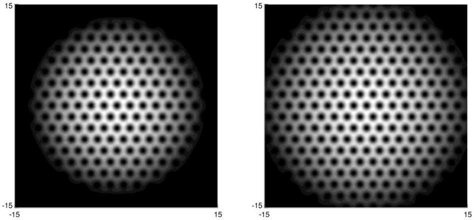
$$E^N(\omega, a) = \inf \sigma_{\bigvee_1^N L^2(\mathbb{R}^3)} H^N_{\omega,a}.$$

The Gross-Pitaevskii regime

For $0 < \omega \leq 1$ **fixed**, Gross-Pitaevskii (GP) regime [LSY00,LS06]:

$$\lim_{\substack{a\to 0,\ Na\to g,\\ N^{1/3}a\to 0}} \frac{E^N(\omega,a)}{E^N_{\rm GP}(\omega,g)} = 1,$$

$$E^N_{\rm GP}(\omega,g) = \inf_{\int|\varphi|^2=N}\left(\left\langle\left(\frac{|p-A|^2+|x^3|^2-3}{2}+\omega e_3\cdot L\right)\varphi,\varphi\right\rangle + 4\pi g\int_{\mathbb{R}^3}|\varphi|^4\right).$$



Condensation [LS02, LS06].
When $\omega \to 0 \Rightarrow \exists$ vortices [AB+].

Calculations of the particle density of an atomic BEC in the mean-field LLL regime. Cooper, Komineas & Read. *Phys. Rev. A* **70**, (2004).

[LSY00] Lieb, Seiringer & Yngvason. *Phys. Rev. A* **61**, (2000). ($\omega = 1$). [LS06] Lieb & Seiringer. *Commun. Math. Phys.* **264**, (2006). ($0 < \omega < 1$). [LS02] Lieb & Seiringer. *Phys. Rev. Lett.* **88**, (2002).

[AB+] Aftalion, Blanc & Dalibard. *Phys. Rev. A* **71**, (2005). Aftalion & Blanc. *SIAM J. Math. Anal.* **38**, (2006). Aftalion, Blanc & Nier. *J. Funct. Anal.* **241**, (2006).

FQHE Regime

In GP theory, the number of vortices is

$$N_v \sim \sqrt{\frac{g}{\omega}} \sim \sqrt{\frac{Na}{\omega}}.$$

In physics literature:

$$\text{phase transition } \boxed{\text{GP}} \rightsquigarrow \boxed{\text{FQHE}} \text{ when } \frac{N_v}{N} \sim 1$$

We introduce

$$\kappa = \frac{a}{N\omega}$$

and look at the regime $a \to 0$, $\omega \to 0$ $(N \to \infty)$ and $\kappa = O(1)$.

Rmk. The bosonic FQHE was not yet observed experimentaly.

Lowest Landau Level (LLL) effective model I

$$H^N_{\omega,a} = \sum_{j=1}^{N}\left(\frac{|p_j - A(x_j)|^2 + |x_j^3|^2 - 3}{2} + \omega e_3\cdot L_j\right) + \sum_{1\leq i<j\leq N} W_a(x_i - x_j)$$

For $a = 0$ and $\omega = 0$, $E^N(0,0) = 0$ with ground states:

$$\mathfrak{H}_N = \bigvee_1^N \mathfrak{H}_1 = \left\{F(x_1^1 + ix_1^2, ..., x_N^1 + ix_N^2)e^{-\frac{\sum_{i=1}^N |x_i|^2}{2}} \ :\ F \text{ symm. and hol.}\right\}.$$

$$E^N_{\rm LLL}(\omega, a) = \inf_{\substack{\Psi\in\mathfrak{H}_N,\\ \|\Psi\|=1}} \left\langle\left(\omega\sum_{j=1}^N e_3\cdot L_j + 4\pi a \sum_{1\leq i<j\leq N}\delta(x_i - x_j)\right)\Psi, \Psi\right\rangle$$

Rmk 1. makes sense since all $\Psi\in\mathfrak{H}_N$ are smooth.

Rmk 2. $E^N_{\rm LLL}(\omega, a)$ is **not** obtained by restricting $H^N_{\omega,a}$ to the LLL.
Would give: $a\int W > 4\pi a$.

Validity of LLL model

Theorem (Validity of LLL model [LS09])

Let $W \geq 0$ decaying fast enough and $\kappa = a/N\omega$. One has for a constant $c = c(\kappa, W)$:

$$\left(1 - cNa^{1/3} - ca^{1/9}\right) E^N_{LLL}(\omega, a) \leq E^N(\omega, a) \leq E^N_{LLL}(\omega, a)\frac{1 + cN^{1/2}a}{1 - ca}.$$

Upper bd: $a \ll N^{-1/2}$. **Lower bd:** $a \ll N^{-3}$. **Expected:** $a \ll N^{-1/3}$.

- **FQHE/LLL regime attained even for N finite.**

$$E^N(\omega, a) \underset{\substack{\omega\to 0,\\ a/N\omega=\kappa}}{\sim} E^N_{\rm LLL}(\omega, a) \quad \text{for } N \text{ fixed.}$$

[LS09] M.L. & R. Seiringer. Strongly correlated phases in rapidly rotating Bose gases. *J. Stat. Phys*, **137** (2009), no. 5-6, p. 1040–1062.

LLL effective model II

Bargmann space: $\mathcal{B}_N = \left\{ F \text{ symm. and hol. } : \int |F|^2 e^{-\sum_{i=1}^N |x_i|^2} < \infty \right\}.$

$$E_{\rm LLL}^N(\omega, a) = \inf \sigma_{\mathcal{B}_N} \left(\omega \mathcal{L}_N + 4\pi a \mathcal{I}_N \right),$$

$$\mathcal{L}_N = \sum_{j=1}^N z_j \partial_{z_j}, \qquad \mathcal{I}_N = \sum_{i<j} \delta_{ij}.$$

$$\left(\sum_{i<j} \delta_{ij} \right) F = (2\pi)^{-3/2} \sum_{i<j} F\left(z_1, \dots, \frac{z_i + z_j}{2}, \dots, \frac{z_i + z_j}{2}, \dots, z_N \right)$$

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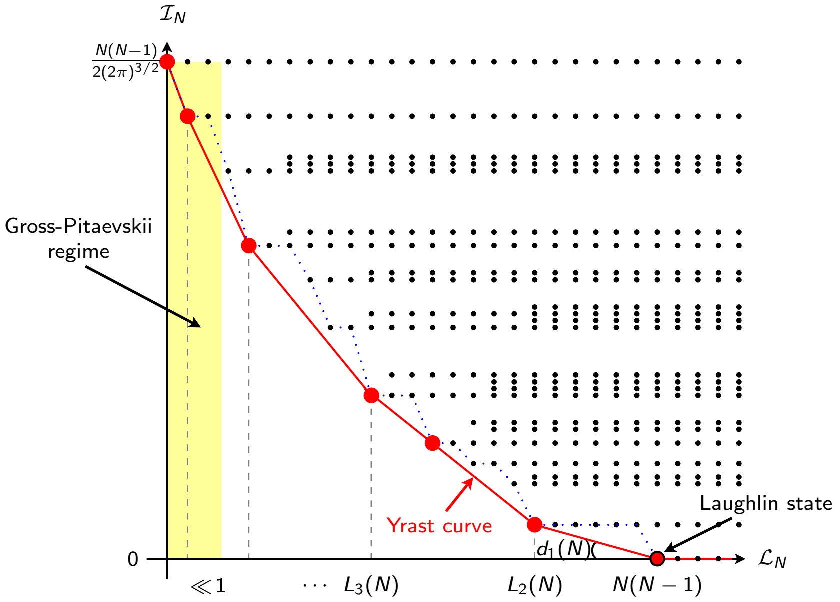
$$\left(\sum_{i<j} \delta_{ij}\right) F = (2\pi)^{-3/2} \sum_{i<j} F\left(z_1, ..., \frac{z_i+z_j}{2}, ..., \frac{z_i+z_j}{2}, ..., z_N\right)$$

- $\mathcal{L}_N$ and $\mathcal{I}_N$ commute! $\Rightarrow$ pick common eigenstates depending on a/ω.
- **Laughlin wavefunction [Lau83]:**
  $\ker(\mathcal{I}_N) = \{F_{\rm Laugh}(z_1, ..., z_N) G(z_1, ..., z_N) \ : \ G \text{ symm. and hol.}\}.$

$$F_{\rm Laugh}(z_1, ..., z_N) = c \prod_{i<j} (z_i - z_j)^2.$$

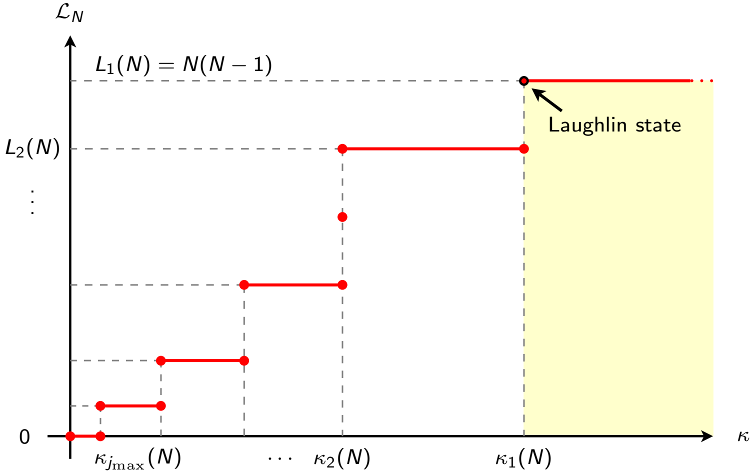
[Lau83] Laughlin, *Phys. Rev. Lett.* **50**, (1983).

Joint spectrum of $\mathcal{L}_N$ and $\mathcal{I}_N$



Plateaux for angular momentum

$\exists \kappa_1(N) > \kappa_2(N) > \cdots \mid$ GS is constant for $\kappa \in (\kappa_j(N), \kappa_{j+1}(N))$,
$\kappa_j(N) = -1/4\pi d_j(N)$.



CV of states for fixed N

$\Pi_N(\kappa)$= orth. projector on GS of the reduced LLL model (in $L^2(\mathbb{R}^{3N})$).

Example: with $\Psi_{\rm Laugh} = \pi^{-N/4} F_{\rm Laugh} e^{-|x|^2/2}$,

$$\Pi_N(\kappa) = |\Psi_{\rm Laugh}\rangle \langle \Psi_{\rm Laugh}| \quad \text{for } \kappa > \kappa_1(N).$$

Theorem (CV of states [LS09])

Let $\kappa > 0$ and $N \geq 2$ fixed. Denote $\Psi^N_{\omega_n,a_n}$ any sequence of GS of $H^N_{\omega_n,a_n}$, with $a_n \to 0$ and $a_n/N\omega_n \to \kappa$. Then

$$\lim_{n\to\infty} \left\| \Psi^N_{\omega_n,a_n} - \Pi_N(\kappa)\Psi^N_{\omega_n,a_n} \right\| = 0.$$

In particular if $\kappa > \kappa_1(N)$, $\Psi^N_{\omega_n,a_n} \to \Psi_{\rm Laugh}$, up to a phase factor.

Idea of proof: upper bound

• **Potential in the LLL:**

$$\left(\int_{\mathbb{R}^3} g(x)e^{-a|x|^2}\,dx\right)\delta_0 \leq g(\cdot)_{|\mathcal{B}_1} \leq \left(\int_{\mathbb{R}^3} g\right)\delta_0 + C\int_{\mathbb{R}^3} g(x)\frac{|x|^4}{1+|x|^4}\,dx.$$

• **Dyson test function:** $\Psi = \prod_{i<j} f_a(|x_i - x_j|)\Psi_{\rm LLL}(x_1,\dots,x_N)$,

where $f_a \simeq$ zero-scattering solution. Then

$$\left\langle \Psi, H^N_{\omega,a}\Psi\right\rangle = E^N_{\rm LLL}(\omega,a)\,\|\Psi\|^2 + \left\langle \sum_{i<j}\left((f_a')^2 + W_a f_a^2\right)_{ij}\Psi_{\rm LLL},\Psi_{\rm LLL}\right\rangle + \text{errors}.$$

$$\int (f_a')^2 + W_a f_a^2 \simeq 4\pi a.$$

[Dys57] Dyson, *Phys. Rev.*, **106** (1957).
[LSSY] Lieb, Seiringer, Solovej, Yngvason. The mathematics of the Bose gas and its condensation, Oberwolfach Seminars, Birkhäuser (2005).

Idea of proof: lower bound

Idea: bound from below interaction by $a\times$ fixed (N-body) potential, using (part of) kinetic energy.

Lemma (Dyson-type inequality)

Let W a potential supported in $B(0,R_0)$ and $y=(s,y^3)\in\mathbb{R}^3$. We have for all $R>R_0$ and all ψ

$$\int_{|x-y|\leq R} e^{-|x|^2}\left(|\partial_{x^3}\psi(x)|^2+|\partial_{\bar z}\psi(x)|^2+W_a(x-y)|\psi(x)|^2\right)dx$$

$$\geq 4\pi a\, e^{-(|y^3|+R)^2+|s|^2}\left|\frac{1}{4\pi R^2}\int_{|x-y|=R} e^{-\bar s z}\psi(x)\,dx\right|^2 .$$

Trick: if $\psi\in\mathcal{B}_1$, $\displaystyle\frac{1}{4\pi R^2}\int_{|x-y|=R} e^{-\bar s z}\psi(x)\,dx = e^{-|s|^2}\psi(y)$.

$$\Longrightarrow\quad H^N_{\omega,a}\geq\sum_{j=1}^N\left(\theta h_j+\omega(e_3\cdot L_j)\right)+4\pi a(1-\theta)\sum_{i<j}U_{ij}$$

with U_{ij} complicated N-body term. Then: perturbation theory.

Conclusion

Summary:

- System understood for $a \ll 1$ and $\omega \ll 1$ such that $\kappa = a/N\omega = O(1)$, at least for N fixed;
- highly correlated 'universal' regime with plateaux / FQHE.

Perspectives:

- Estimates in N not good enough yet;
- need to understand better effective Hamiltonian in the limit $N \to \infty$;
- *conjecture 1:*

$$\omega \mathcal{L}_N + 4\pi a \mathcal{I}_N = 4\pi N a \left(\frac{1}{4\pi\kappa} \underbrace{\frac{\mathcal{L}_N}{N^2}}_{O(1)} + \underbrace{\frac{\mathcal{I}_N}{N}}_{O(1)} \right);$$

- *conjecture 2:*

$$\kappa_1 \to \kappa > 0 \quad \text{in the limit } N \to \infty.$$

Reference:

M.L. & R. Seiringer. Strongly correlated phases in rapidly rotating Bose gases. *J. Stat. Phys*, **137** (2009), no. 5-6, p. 1040–1062.