Perturbation of near threshold eigenvalues

Arne Jensen

Department of Mathematical Sciences
Aalborg University

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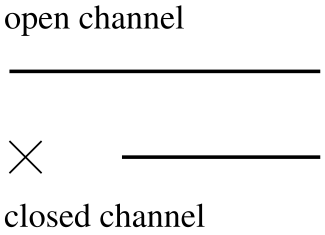
The results presented here are joint work with Gheorghe Nenciu, Bucharest.
Some of the results are also joint work with Victor Dinu, Bucharest.

The problem

Consider a simple two channel Hamiltonian with one open and one closed channel. The closed channel has a bound state close to a threshold of the open channel.



What happens to the bound state, if the two channels are weakly coupled?

The problem

Simplified model. In the closed channel take only the bound state. Assume it is non-degenerate. Model problem ($\varepsilon > 0$):

$$H(\varepsilon) = H + \varepsilon W = \begin{bmatrix} H_{\mathrm{op}} & 0 \\ 0 & E_0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & W_{12} \\ W_{21} & 0 \end{bmatrix} \quad \text{on} \quad \mathcal{H} = \mathcal{H}_{\mathrm{op}} \oplus \mathbf{C}.$$

The operator H_{op} is assumed to have the properties of a Schrödinger operator in odd dimensions, with a threshold at zero. The eigenfunction for eigenvalue E_0 for $\varepsilon = 0$ is denoted by Ψ_0.

Problem: Estimate the survival probability

$$|\langle \Psi_0, e^{-itH(\varepsilon)} \Psi_0 \rangle|^2$$

for ε small, as E_0 is tuned past the threshold.

The problem

This set-up can be realized experimentally, by applying a magnetic field to a system. Physicists use the term Feshbach resonance for the associated phenomenon. I do not know whether the term “resonance” is appropriate in this context.

But the results I present here would lead me to say that in many cases this term is inappropriate.

We are not aware of other rigorous treatments of the problem presented here.

Preliminaries

Define $A_\varepsilon(t) = \langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0\rangle$. Stone's formula:

$$A_\varepsilon(t) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-itx} \,\mathrm{Im}\langle \Psi_0, (H(\varepsilon) - x - i\eta)^{-1}\Psi_0\rangle \, dx$$

Feshbach formula yields:

$$\mathrm{Im}\langle \Psi_0, (H(\varepsilon) - x - i\eta)^{-1}\Psi_0\rangle = \mathrm{Im}\, \frac{1}{F(x + i\eta, \varepsilon)},$$

with

$$F(z, \varepsilon) = E_0 - z - \varepsilon^2 \langle \Psi_0, WQ^*(H_{\mathrm{op}} - z)^{-1} QW\Psi_0\rangle.$$

$Q\colon \mathcal{H}_{\mathrm{op}} \oplus \mathbf{C} \to \mathcal{H}_{\mathrm{op}}$ orthogonal projection.

Case $E_0 < 0$

Assume $E_0 < 0$. Analytic perturbation theory tells us that $H(\varepsilon)$ has an eigenvalue $E(\varepsilon)$ close to E_0 for ε small. We have $E(\varepsilon) \to E_0$ as $\varepsilon \to 0$, and

$$|A_\varepsilon(t) - e^{-itE(\varepsilon)}| \lesssim \varepsilon^2.$$

So we expect the survival probability to be close to one in this case, for E_0 close to 0 and ε small.

Case $E_0 \geq 0$

For $E_0 > 0$ and not too close to 0 we expect the embedded eigenvalue E_0 to become a metastable state with an exponential decay law $|A_\varepsilon(t)|^2 \cong e^{-2\Gamma(\varepsilon)t}$, at least for a considerable time interval.

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Outline of the argument: Assume $F(z, \varepsilon)$ sufficiently smooth with boundary values for $\operatorname{Re} z$ close to E_0. Write

$$F(x + i0, \varepsilon) = R(x) + iI(x)$$

Then the equation $R(x) = 0$ has a solution $x_0(\varepsilon)$ nearby E_0. Main contribution to the integral in $A_\varepsilon(t)$ will then come from x close to $x_0(\varepsilon)$ and in this neighborhood

$$\operatorname{Im} \frac{1}{F(x + i\eta, \varepsilon)} \cong \frac{-I(x_0(\varepsilon))}{(x - x_0(\varepsilon))^2 + I(x_0(\varepsilon))^2}$$

Case $E_0 \geq 0$

This type of result leads to exponential decay with $\Gamma(\varepsilon) = -I(x_0(\varepsilon))$. The optimal error estimate is

$$\langle \Psi_0, e^{-itH(\varepsilon)} \Psi_0 \rangle = e^{-it(x_0(\varepsilon) - i\Gamma(\varepsilon))} + O(\varepsilon^2)$$

if we are away from the threshold.

For $x_0(\varepsilon) \sim \varepsilon$ we may get a different error estimate, depending on the nature of the threshold. We may get an error $O(\varepsilon^\nu)$, $\nu = \frac{1}{2}$ or $\nu = \frac{3}{2}$.

We will now look at the case when E_0 can be tuned through zero.

Preliminaries

Define

$$g(z) = \langle \Psi_0, WQ^*(H_{\text{op}} - z)^{-1} QW\Psi_0 \rangle.$$

Assume that there is an interval $(-a, 0)$, $a > 0$, such that H_{op} has no spectrum here.

Assumption

Let $D_a = \{z \in \mathbf{C} \setminus [0, \infty) \mid |z| < a, \}$.
Assume for $z \in D_a$

$$g(z) = \frac{i}{\sqrt{z}} g_{-1} + g_0 - i\sqrt{z} g_1 - z g_2 + O(z^{3/2})$$

and the expansion is once differentiable.

Preliminaries

The Assumption can be verified for Schrödinger operators in odd dimensions.
Examples:

- $H_{\mathrm{op}} = -\frac{d^2}{dr^2} + V(r) \quad \text{on} \quad \mathcal{H}_{\mathrm{op}} = L^2([0,\infty))$
  Dirichlet boundary condition.
  Note: $g_{-1} \neq 0$ can occur for certain V.

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Technique: Asymptotic expansion of the resolvent near the threshold zero. Perturbation argument based on known kernel of free resolvent. The coefficients g_{-1} and g_1 are explicitly computable.

Classification

There are three different types of results, depending on how regular or singular $g(z)$ is at zero.

- Singular case $g_{-1} \neq 0$
- Regular case $g_{-1} = 0$ and $g_1 \neq 0$
- Smooth case $g_{-1} = 0$ and $g_1 = 0$

Smooth case $g_{-1} = 0$ and $g_1 = 0$

In the smooth case one gets the following result:

Theorem

Assume $E_0 \in (-a/2, a/2)$. Write $F(x + i0, \varepsilon) = R(x, \varepsilon) + iI(x, \varepsilon)$. Then for ε sufficiently small there exists a unique solution $x_0(\varepsilon) \in (-a, a)$ to $R(x, \varepsilon) = 0$. Let $\Gamma(\varepsilon) = -I(x_0(\varepsilon), \varepsilon)$ For $\varepsilon > 0$ sufficiently small and all $t > 0$ we have

$$|A_\varepsilon(t) - e^{-it(x_0(\varepsilon) - i\Gamma(\varepsilon))}| \lesssim \varepsilon^2 |\ln \varepsilon|.$$

Singular case $g_{-1} \neq 0$

Main idea is to replace $g(z)$ in $F(z,\varepsilon) = E_0 - z - \varepsilon^2 g(z)$ by the leading terms in the asymptotic expansion. We take

$$H_s(z,\varepsilon) = E_0 - z - \varepsilon^2 g_{-1} i \frac{1}{\sqrt{z}} - \varepsilon^2 g_0 = E - z - \varepsilon^2 g_{-1} i \frac{1}{\sqrt{z}}$$

with $E = E_0 - \varepsilon^2 g_0$.
For $E \geq -a/2$ and ε sufficiently small the equations

$$F(x,\varepsilon) = 0 \quad \text{and} \quad H_s(x,\varepsilon) = 0$$

on $(-a, 0)$ and $(-\infty, 0)$ have unique solutions x_b and $\tilde{x}_b$, respectively.

Singular case $g_{-1} \neq 0$

We have $g_{-1} > 0$. Choose parameters

$$s = (\varepsilon^2 g_{-1})^{2/3} t \quad \text{and} \quad f = (\varepsilon^2 g_{-1})^{-2/3} E.$$

Let $v(f)$ be the (unique) solution of $f + v - v^{-1/2} = 0$ on $(0, \infty)$.

Theorem

Assume $E \in [-a/2, (a/2)\varepsilon^{4/5}]$. Then for $t \geq 0$ and ε small we have

$$\left| A_\varepsilon(t) - \frac{2v(f)^{3/2}}{2v(f)^{3/2}+1} e^{-itx_b} - \frac{1}{\pi} \int_0^\infty \frac{y^{1/2}}{y(f-y)^2+1} e^{-isy} dy \right| \lesssim \varepsilon^{4/5}.$$

Singular case $g_{-1} \neq 0$

Comments:

- We always get a bound state close to zero.
- The decay law is definitely non-exponential.

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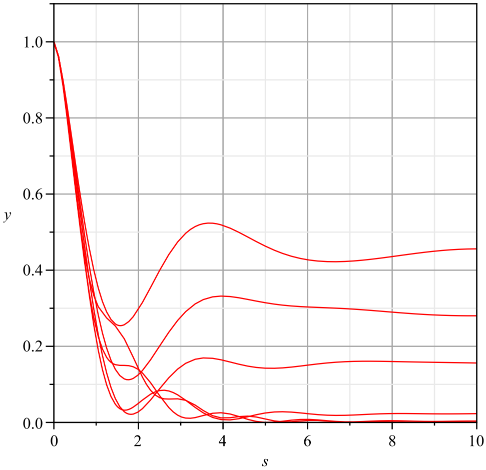
Only quantity not computable from the model $H_s(z,\varepsilon)$ is x_b. Thus we replace x_b by $\tilde{x}_b$. We have that for $-\varepsilon^{4/3} \lesssim E \lesssim \varepsilon^{4/5}$

$$\left| A_\varepsilon(t) - \frac{2v(f)^{3/2}}{2v(f)^{3/2}+1} e^{isv(f)} - \frac{1}{\pi}\int_0^\infty \frac{y^{1/2}}{y(f-y)^2+1} e^{-isy}\,dy \right| \lesssim \varepsilon^{4/5} + s\varepsilon^{4/3}.$$

Note that $t\tilde{x}_b = -sv(f)$.

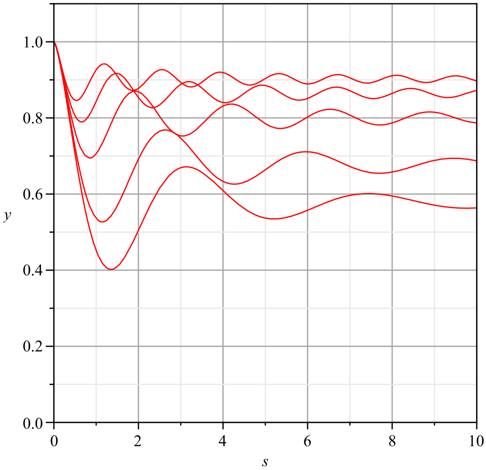
Decay laws, singular case

$f = 0, 0.5, 1, 2, 3, 4$ from top to bottom.



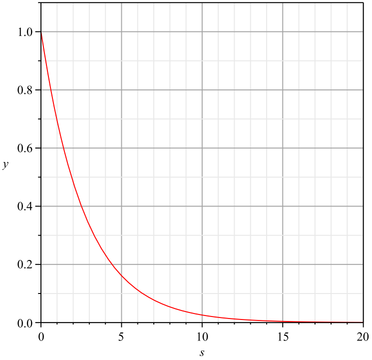
Decay laws, singular case

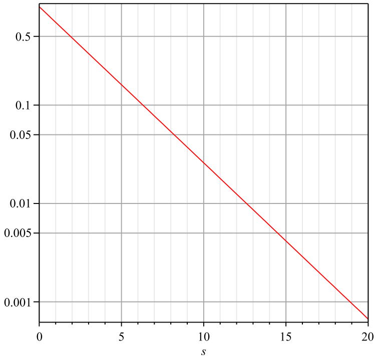
$f = -4, -3, -2, -1, -0.5$, from top to bottom.



Decay laws, singular case

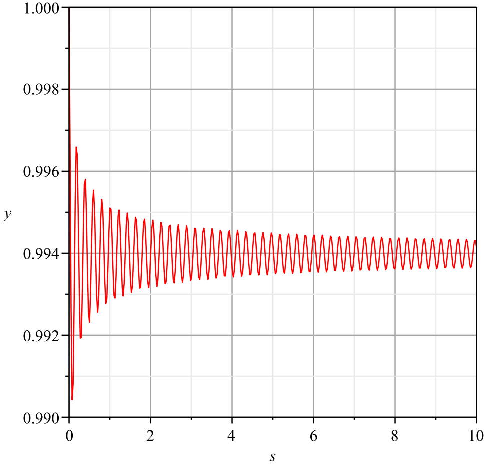
$f = 30$, left hand figure linear vertical scale, right hand figure logarithmic vertical scale.





Decay laws, singular case

$f = -30$. Note vertical scale.



Regular case, $g_{-1} = 0$, $g_1 \neq 0$

Note that $g_1 < 0$.
Model function

$$H_r(z,\varepsilon) = E_0 - z - \varepsilon^2(g_0 - ig_1\sqrt{z} - g_2 z) = b(\tilde{E} - z + i\tilde{g}_1\sqrt{z}),$$

where $b = 1 - \varepsilon^2 g_2$, $\tilde{E} = (E_0 - \varepsilon^2 g_0)/b$, and $\tilde{g}_1 = g_1/b$.
For $\tilde{E} \geq 0$ and ε sufficiently small we have that $F(x,\varepsilon) > 0$ on $(-a, 0)$, and $H_r(x,\varepsilon) > 0$ on $(-\infty, 0)$.
For $-a/2 \leq \tilde{E} < 0$ and ε sufficiently small the equations $F(x,\varepsilon) = 0$ and $H_r(x,\varepsilon) = 0$ on $(-a, 0)$ and $(-\infty, 0)$ have unique solutions x_b and $\tilde{x}_b$, respectively.

Regular case, $g_{-1} = 0$, $g_1 \neq 0$

Choose parameters $\tilde{s} = (\varepsilon^2 \tilde{g}_1)^2 t$ and $\tilde{f} = (\varepsilon^2 \tilde{g}_1)^{-2} \tilde{E}$.

Theorem

For $\tilde{E} \geq 0$ we have

$$\left| A_\varepsilon(t) - \frac{1}{\pi} \int_0^\infty \frac{y^{1/2}}{(\tilde{f} - y)^2 + y} e^{-i\tilde{s}y} dy \right| \lesssim \varepsilon^{4/3},$$

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For $\tilde{E} \leq 0$ we have

$$\left| A_\varepsilon(t) - \frac{\sqrt{1 + 4|\tilde{f}|} - 1}{\sqrt{1 + 4|\tilde{f}|}} e^{-itx_b} - \frac{1}{\pi} \int_0^\infty \frac{y^{1/2}}{(\tilde{f} - y)^2 + y} e^{-i\tilde{s}y} dy \right| \lesssim \varepsilon^{4/3}.$$

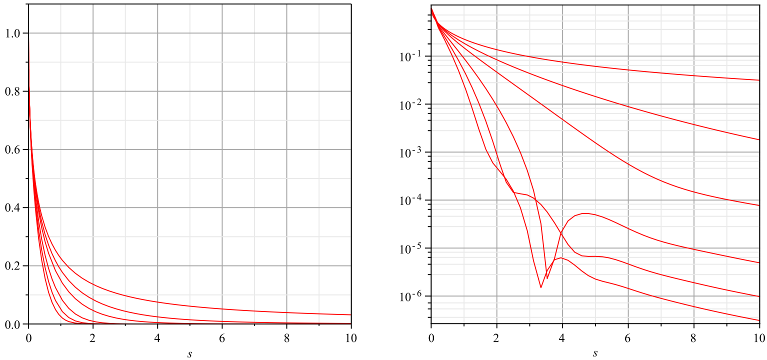
Regular case, $g_{-1} = 0$, $g_1 \neq 0$

The only parameter not computable is x_b. We replace by $\tilde{x}_b$, where $\tilde{x}_b = -(\varepsilon^2 \tilde{g}_1)^2 \tilde{v}(\tilde{f})$ and $\tilde{v}(\tilde{f}) = \frac{1}{4}(\sqrt{1+4|\tilde{f}|}-1)^2$. Suppose $-\varepsilon^4 \lesssim E < 0$. Then for all $t > 0$ and sufficiently small ε we have

$$\left| A_\varepsilon(t) - \frac{\sqrt{1+4|\tilde{f}|}-1}{\sqrt{1+4|\tilde{f}|}} e^{i\tilde{s}\tilde{v}(\tilde{f})} - \frac{1}{\pi}\int_0^\infty \frac{y^{1/2}}{(\tilde{f}-y)^2+y} e^{-i\tilde{s}y}\, dy \right| \lesssim \varepsilon^{4/3} + \tilde{s}\varepsilon^4.$$

Decay laws, regular case

$\tilde{f} = 0, 0.5, 1, 2, 3, 4$ from top to bottom. Left hand figure linear vertical scale, right hand figure logarithmic vertical scale.



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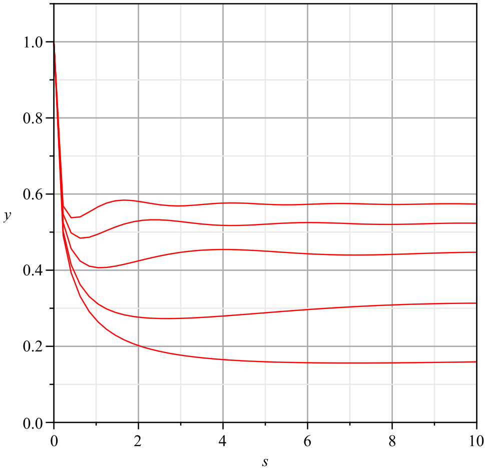

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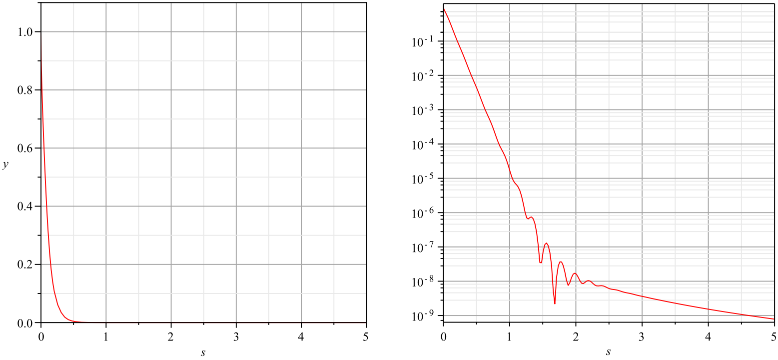

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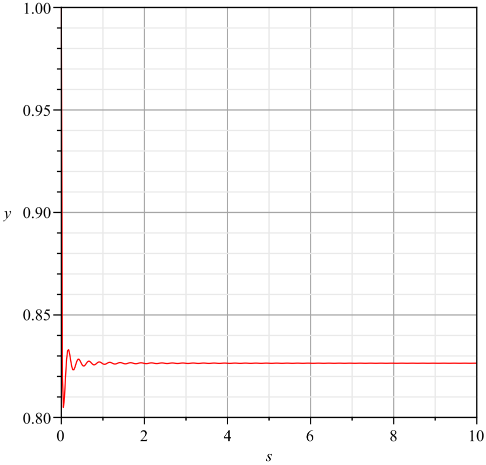
$\tilde{f} = -30$. Note the vertical scale. from top to bottom.



A uniqueness result

We may ask in which sense the behavior found here is unique. We look at the general framework, with a family of Hamiltonians $H(\varepsilon)$ on $\mathcal{H}$, an orthogonal projection P_0, and an effective Hamiltonian $h(\varepsilon)$ on $P_0\mathcal{H}$.

Our results have the structure

$$P_0 e^{-itH(\varepsilon)} P_0 = e^{-ith(\varepsilon)} P_0 + \delta(\varepsilon, t), \quad t > 0, \qquad (*)$$

where

$$\sup_{t>0} \|\delta(\varepsilon, t)\| \leq C\varepsilon^p \quad \text{for some } p > 0. \qquad (**)$$

A uniqueness result

Theorem

Assume Rank $P_0 = 1$. *Assume that* $h^1(\varepsilon)$ *and* $h^2(\varepsilon)$ *both satisfy* $(*)$ *and* $(**)$, *with the same value for* p. *Assume that for some* $c_0 > 0$ *and* $q > 0$ *we have*

$$-c_0\varepsilon^q P_0 \leq Im\, h^1(\varepsilon) \leq 0 \quad \text{for } 0 \leq \varepsilon < \varepsilon_0. \tag{1}$$

Then for ε_0 *sufficiently small we have*

$$\|h^1(\varepsilon) - h^2(\varepsilon)\|_{\mathcal{B}(P_0\mathcal{H})} \leq C\varepsilon^{p+q}, \quad 0 \leq \varepsilon < \varepsilon_0. \tag{2}$$

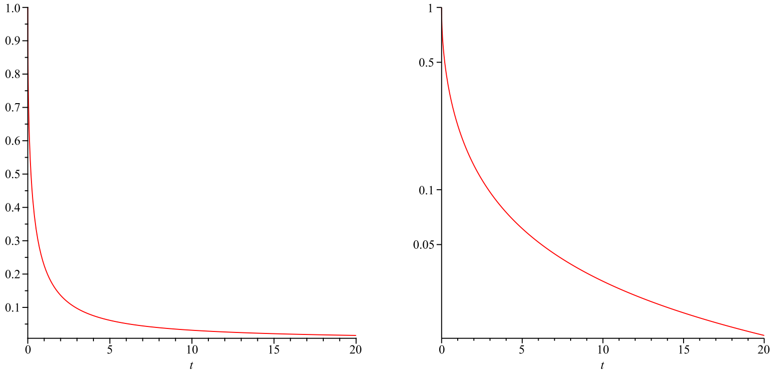
Note that a closely related result has been obtained by Cattaneo-Graf-Hunziker.

Plot, regular case

Regular case: threshold regime, $p > 4$, $s = \beta^2 \varepsilon^4 t$ the parameter

$$A_\varepsilon(t) = e^{is}(1 - \mathrm{erf}(e^{i\pi/4} s^{1/2})) + \mathcal{O}(\varepsilon^{p-4})$$

Plot of $|A_\varepsilon(t)|^2$ and $\log|A_\varepsilon(t)|^2$:



Summary

The message of this talk is summarized as:
Perturbation of a simple eigenvalue for operator of type $H_\varepsilon = H_0 + \varepsilon W$, $H_0 = -\Delta + V$, $V(x) \to 0$ as $|x| to \infty$ sufficiently fast, or ...

- Away from thresholds general results are available, either analytic perturbation theory or Fermi Golden Rule type results for embedded eigenvalues
- Near thresholds *no general results* available. For specific models results can be obtained, but are complicated.

Some ideas

Stone's formula

$$P_0 e^{-itH(\varepsilon)} P_0 = \lim_{\eta \to 0} \frac{1}{\pi} \int_{\sigma(H(\varepsilon))} dx\, e^{-itx} \operatorname{Im} P_0 (H(\varepsilon) - x - i\eta)^{-1} P_0.$$

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Localize near E_0: $I_\varepsilon = (e_0(\varepsilon) - d(\varepsilon), e_0(\varepsilon) + d(\varepsilon))$.

Some ideas

Stone's formula

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Localize near E_0: $I_\varepsilon = (e_0(\varepsilon) - d(\varepsilon), e_0(\varepsilon) + d(\varepsilon))$. Use Schur-Livsic-Feshbach-Grushin formula. Howland's formulation:

$$P_0 (H - z)^{-1} P_0 = F(z, \varepsilon)^{-1},$$

where

$$F(z, \varepsilon) = E_0 P_0 + \varepsilon P_0 W P_0 - \varepsilon^2 P_0 W Q_0 R_{0,\varepsilon}(z) Q_0 W P_0 - z P_0.$$

Here

$$R_{0,\varepsilon}(z) = (Q_0 H(\varepsilon) Q_0 - z Q_0)^{-1} \quad \text{on} \quad Q_0 \mathcal{H}.$$