TU Clausthal

Absolutely continuous spectrum for periodic magnetic fields

Michael J. Gruber, TU Clausthal
Quantum Systems, IMSc Chennai, 2010-08-18

Contents

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The main player

- $H = (\imath\nabla + A)^2 + V$ on $L^2(\mathbb{R}^d)$ (usually $d = 2$)
- V smooth function, A smooth vector field
- $B = \operatorname{curl} A$ magnetic field
- B, V are assumed Γ-periodic (usually $\Gamma = \mathbb{Z}^d$)
- $M := \mathbb{R}^d/\Gamma$

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The main dichotomy

- $V \equiv 0, A \equiv 0$:

  $$\operatorname{spec} \Delta = [0, \infty)$$

  continuous spectrum
- $d = 2$, $B \not\equiv 0$ constant:

  $$\operatorname{spec} H = \{B(1 + 2n) \mid n \in \mathbb{N}_0\}$$

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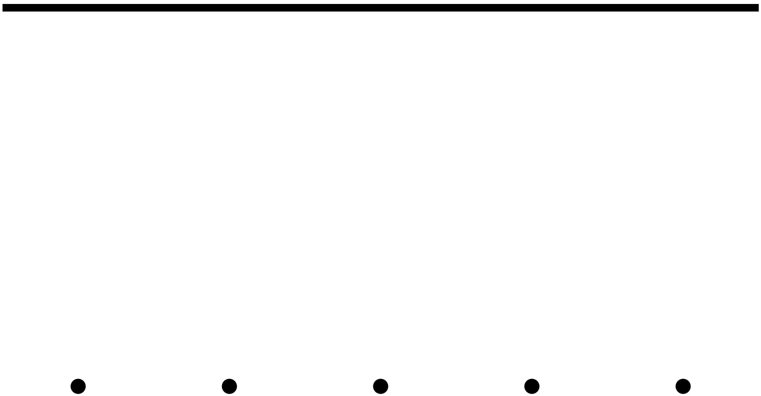

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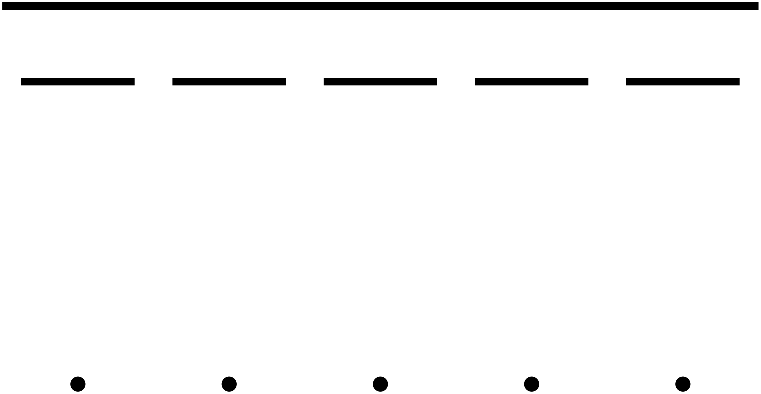

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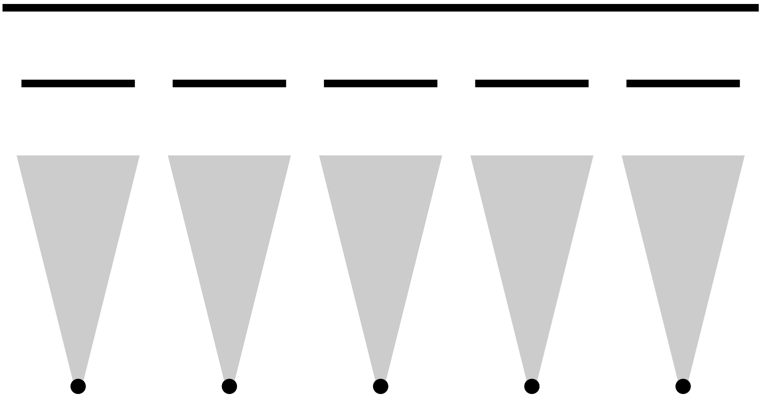

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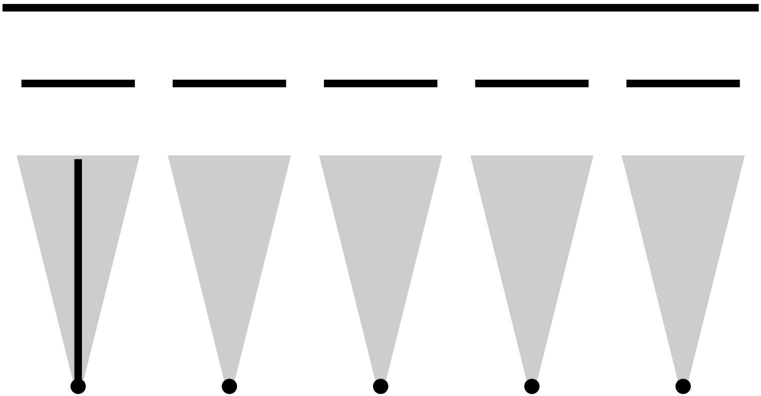

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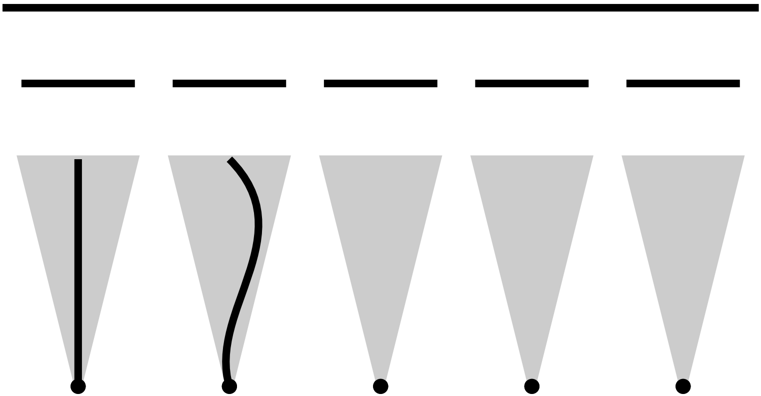

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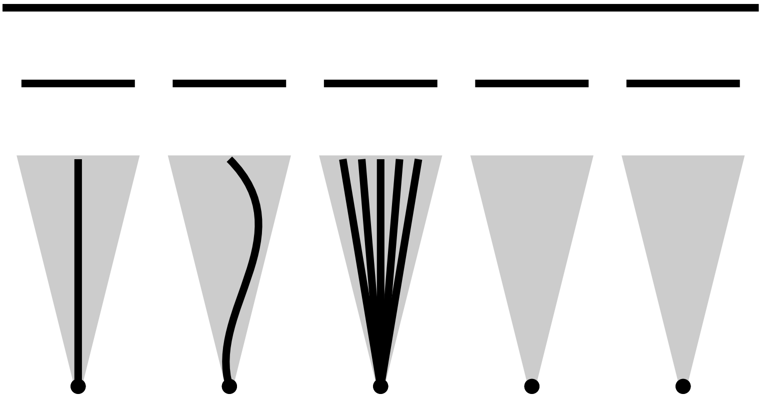

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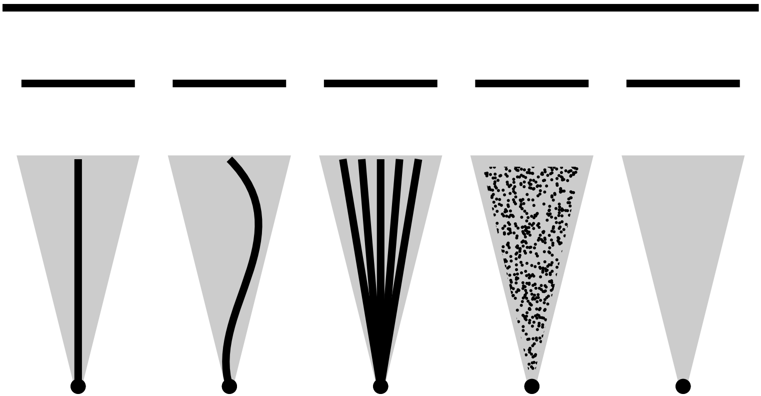

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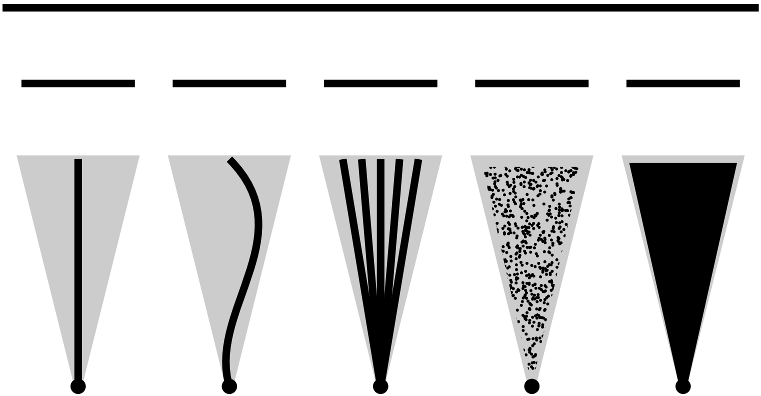

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Magnetic curvature

- Riemannian manifold (X, g), Hilbert space $L^2(X)$
- magnetic field is exact 2-form $b = \mathrm{d}a \in \Omega^2(X)$
- a defines connection $\mathrm{d}_a \coloneqq \mathrm{d} - \imath \frac{ea}{\hbar}$ on the trivial complex line bundle $L \coloneqq X \times \mathbb{C}$
- curvature = magnetic field:

$$\mathrm{curv}(\mathrm{d}_a) = \frac{1}{2\pi} \mathrm{d} \frac{ea}{\hbar} = \frac{e}{h} b = \frac{1}{\Phi_0} b$$

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Flux and Chern number

- b periodic $\Leftrightarrow b \in \Omega^2(X)$ induced by $b_M \in \Omega^2(M)$
- b exact $\Leftrightarrow b = \mathrm{d}a$, but a need not be periodic
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- L is induced by line bundle L_M over $M \Leftrightarrow$ $[b_M/\Phi_0] \in H^2(M, \mathbb{Z})$
- $d = 2$, Euclidean: $\Phi = \int_{[0,1]^2} B(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_M b_M$
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Bloch/Floquet decomposition

Only for abelian Γ, rational flux:

$$\begin{aligned} L^2(X) &\simeq \int_{\hat{\Gamma}}^{\oplus} L^2(M)\,\mathrm{d}\chi, \\ H &\simeq \int_{\hat{\Gamma}}^{\oplus} H_\chi\,\mathrm{d}\chi, \text{ with} \\ \hat{\Gamma} &= \text{character space} \end{aligned}$$

Write $\chi(\gamma) = e^{\imath(k,\gamma)}$ for some $k \in \mathbb{R}^d$. Then

$$H_\chi = H(k) = (\imath\nabla - k + A)^2 + V = (\mathrm{d}_a + \imath k)^*(\mathrm{d}_a + \imath k) + V.$$

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- $\operatorname{spec} H = \bigcup_{\chi\in\hat{\Gamma}} \operatorname{spec} H_\chi = \bigcup_{n\in\mathbb{N}_0} \lambda_n(\hat{\Gamma})$ is a union of countably many "bands" $\lambda_n(\hat{\Gamma})$
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Fermi measure

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For each Borel set $B \subset \mathbb{R}$ define the quasi-measure of the Fermi shell by

$$\mu^F(B) \coloneqq \operatorname{meas} \bigcup_{\lambda \in B} F(\lambda) = \operatorname{meas}\{\chi \in \hat{\Gamma} \mid \operatorname{spec} H_\chi \cap B \neq \emptyset\}$$

λ is an atom of $\mu^F \Leftrightarrow \mu^F(\{\lambda\}) > 0 \Leftrightarrow \lambda \in \operatorname{spec}_p H$
Recall: The spectral measure of H at $f \in L^2(X)$ is

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Determinants

- $(H_\chi)_{\chi\in\hat{\Gamma}}$ is a real-analytic operator family
- The family $d_H(\lambda,\chi) := \det^\zeta(H_\chi - \lambda)$ of ζ-regularized determinants is real-analytic in χ, analytic in λ.
- $d_H(\lambda,\chi) = 0 \Leftrightarrow \lambda \in \operatorname{spec} H_\chi$

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Connections

Theorem (G, 2002)

1. $\mu^{d_H} = \mu^F$
2. $\mu^{\mathcal{N}}$ *and* μ^F *have the same null-sets (and atoms).*
3. $\mu_f^H(B) \le \int_{\hat{\Gamma}} \|f_\chi\|^2 \operatorname{tr} P_B^{H_\chi} \, d\chi$

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2. $\mu^{\mathcal{N}}$ *and* μ^F *have the same null-sets (and atoms).*
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Proof.

$\mu^{d_H} = \mu^F$ by definition and the properties of the determinant.

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$$\mu^{\mathcal{N}}(B) = \int_{\hat{\Gamma}} \operatorname{tr} P_B^{H_\chi} \, \mathrm{d}\chi \quad \text{and}$$

$$\operatorname{tr} P_B^{H_\chi} \neq 0 \Leftrightarrow P_B^{H_\chi} \neq 0 \Leftrightarrow B \cap \operatorname{spec} H_\chi \neq \emptyset$$
$$\Leftrightarrow \exists \lambda \in B : d_H(\lambda, \chi) \neq 0$$

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$$\begin{aligned} \mu_{f_\chi}^{H_\chi}(B) &= \left\langle f_\chi \,\middle|\, P_B^{H_\chi} f_\chi \right\rangle \\ &\leq \|f_\chi\|^2 \operatorname{tr} P_B^{H_\chi} \\ \mu_f^H(B) &= \int_{\hat{\Gamma}} \mu_{f_\chi}^{H_\chi}(B) \, \mathrm{d}\chi \leq \int_{\hat{\Gamma}} \|f_\chi\|^2 \operatorname{tr} P_B^{H_\chi} \, \mathrm{d}\chi \end{aligned}$$

□

Spectral consequences

Corollary (G, 2002)

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- n-th band is non-degenerate iff $\lambda_n(k)$ is not constant
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$\Rightarrow$ $\|(H(k) - \lambda)f\| \geq (C(k) - \|V - \lambda\|)\|f\| > 0$ for $k \to \infty$

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The Schrödinger operator on $\mathbb{R}^d$ with periodic electric field has purely absolutely continuous spectrum.

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- Arbitrary periodic a requires construction of a pseudo-differential parametrix (Sobolev 1999, Kuchment/Levendorski 1999/2002)
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- B constant, $V \equiv 0$: what defines the lattice, hence the flux per cell?
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Setup

- $X = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$.
- B arbitrary smooth periodic, $\Phi = \frac{1}{2\pi} \int_{[0,1]^2} B(x,y)\,dx\,dy$

Write B as

$$B = B_c + B_z \quad \text{with}$$
$$B_c = 2\pi\Phi \quad \text{and} \quad B_z = B - B_c.$$

B_z has flux 0, choose vector potential A_z as

$$A_z(x,y) = \begin{pmatrix} \varepsilon_0 A^0(y) \\ \varepsilon_1 A^1(x,y) \end{pmatrix}$$

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Magnetic Laplacian

B_c constant, choose

$$A_c(y) = B_c \begin{pmatrix} y \\ 0 \end{pmatrix}$$

to get the magnetic Laplacian

$$H = \left[\left(\frac{1}{\imath} \frac{\partial}{\partial x} - B_c y - \varepsilon_0 A^0(y) \right)^2 + \left(\frac{1}{\imath} \frac{\partial}{\partial y} - \varepsilon_1 A^1(x, y) \right)^2 \right].$$

$\varepsilon_1 = 0$

Fourier transform on $L^2(\mathbb{R}_x)$, $L^2(\mathbb{R}^2) = \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}_y)\,\mathrm{d}\xi$ with

$$L^2(\mathbb{R}^2) \ni f \mapsto \hat{f}, \quad \hat{f}_\xi(y) = \int_{\mathbb{R}} f(x,y) e^{-2\pi\imath\xi x}\,\mathrm{d}x,$$

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$$H = \int_{\mathbb{R}}^{\oplus} \hat{H}_\xi\,\mathrm{d}\xi, \quad \hat{H}_\xi = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V_\xi(y) \quad \text{with}$$

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$$V_\xi(y) = (2\pi\xi - B_c y)^2 = B_c^2 (y - \beta\xi)^2$$

is a harmonic oscillator potential shifted by $\beta\xi$, $\beta = \frac{2\pi}{B_c} = \frac{1}{\Phi}$.

$\operatorname{spec} H_\xi$ is discrete pure point, independent of ξ (Landau levels), eigenfunctions $\Psi_{\xi,m}(y) = \sqrt[4]{B_c}\, h_m\left(\sqrt{B_c}\,(y - \beta\xi)\right)$, h_m is Weber-Hermite function

$$h_m(y) = \frac{(-1)^m}{\sqrt{\sqrt{\pi}2^m m!}} \exp\left(\tfrac{y^2}{2}\right) \frac{\mathrm{d}^m}{\mathrm{d}y^m} \exp\left(-y^2\right), \quad m \in \mathbb{Z}_+.$$

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"periodic" perturbation of a harmonic oscillator

Spectrum of H_ξ is discrete pure point, possibly depending on ξ.

Eigenvalues differ by at most $C_m \varepsilon_0 \max |A^0|$ from Landau levels.

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H is x-dependent, periodic in x.

Bloch theory in x and Weber-Hermite decomposition in y leads to $L^2(\mathbb{R}^2) = \int_{[0,1]}^{\oplus} L^2([0,1]_x) \otimes \ell^2(\mathbb{N}_0)\,\mathrm{d}\xi$ and $H = \int_{[0,1]}^{\oplus} \hat{H}_\xi\,\mathrm{d}\xi$, $\hat{H}_\xi$ is sum of difference operators, typical coefficient:

$$A^0_{l,m}(p) = B_c^{\frac{3}{2}} \int_{\mathbb{R}} A^0(y+p)\, y\, h_l\left(\sqrt{B_c}\,y\right) h_m\left(\sqrt{B_c}\,y\right) \mathrm{d}y$$

Assumptions:

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Spectrum

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Theorem (G, 2003)

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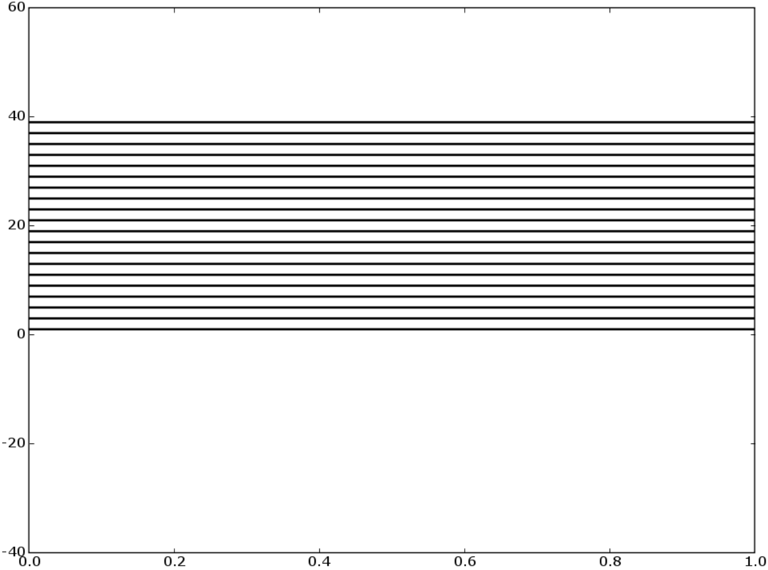
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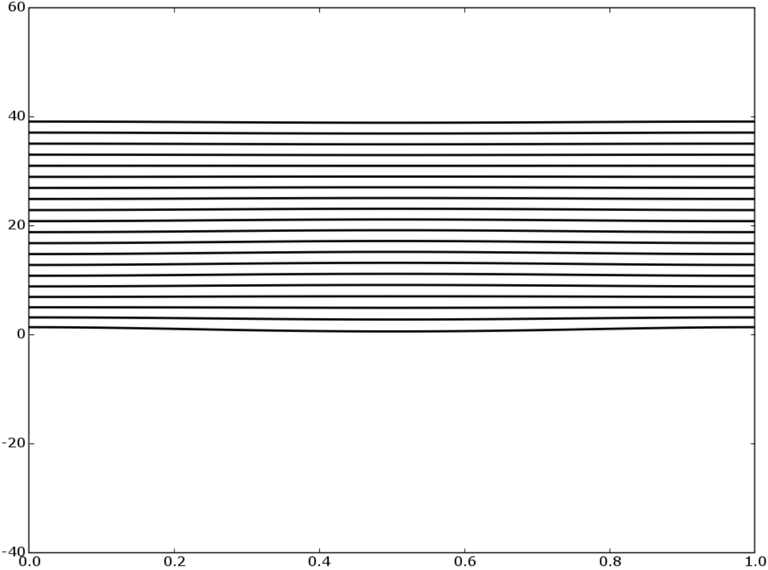
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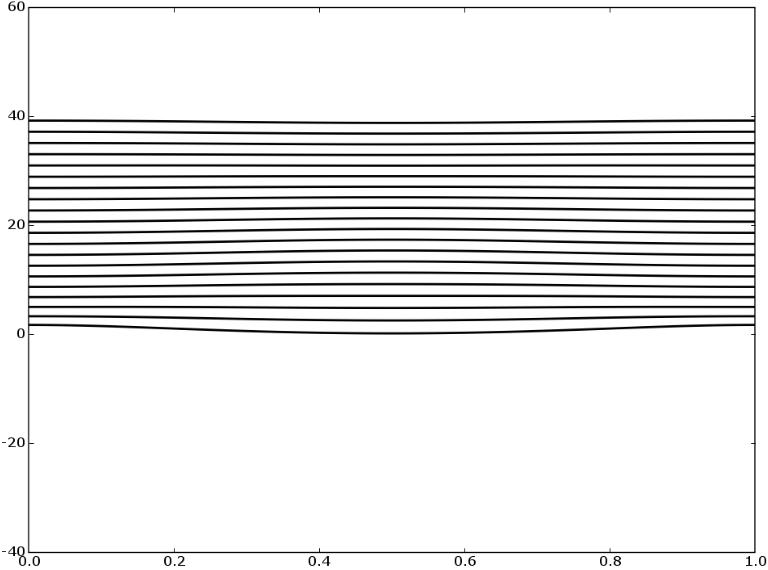
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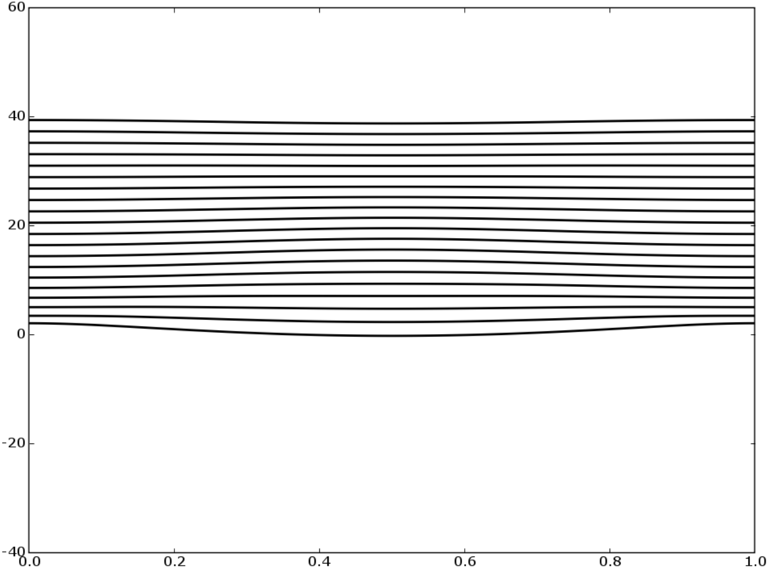
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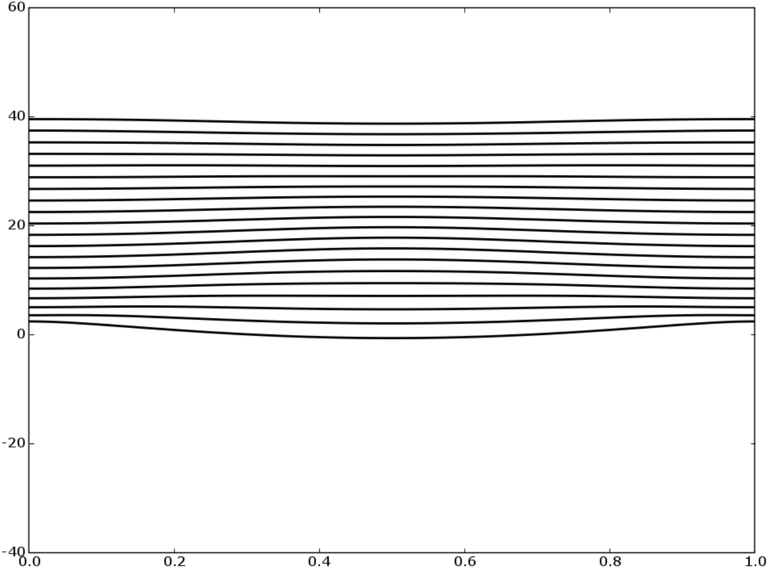
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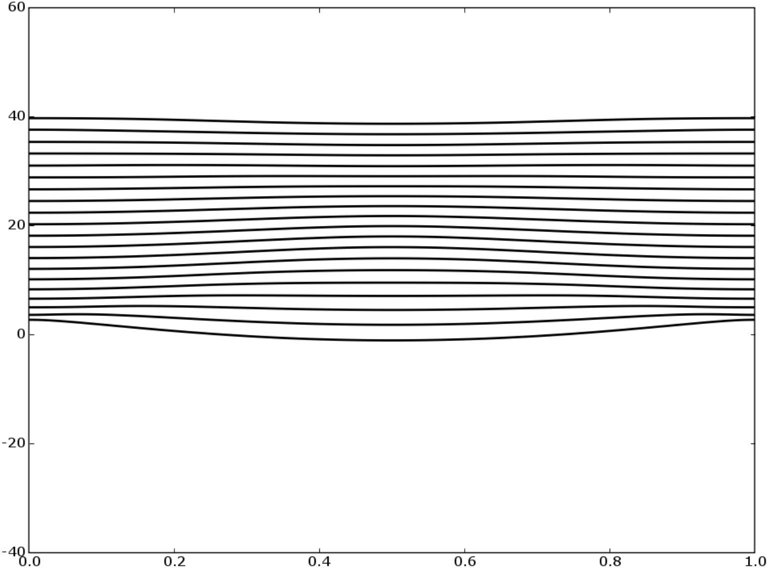
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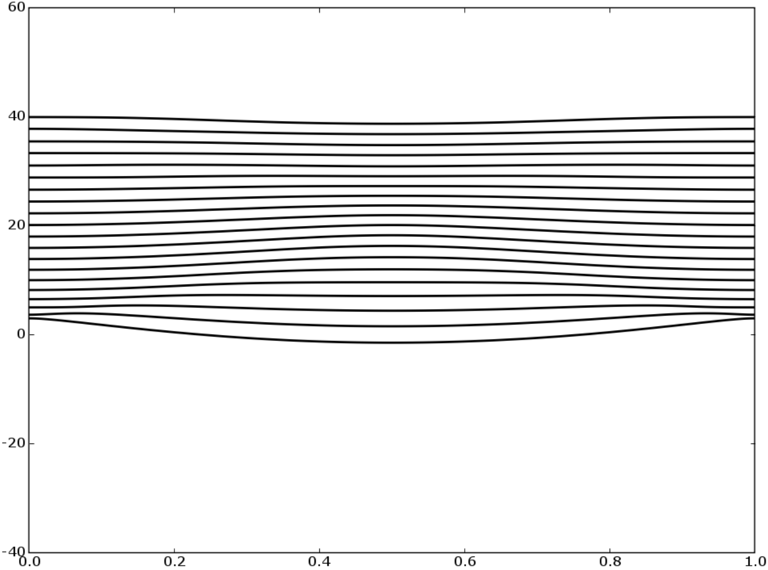
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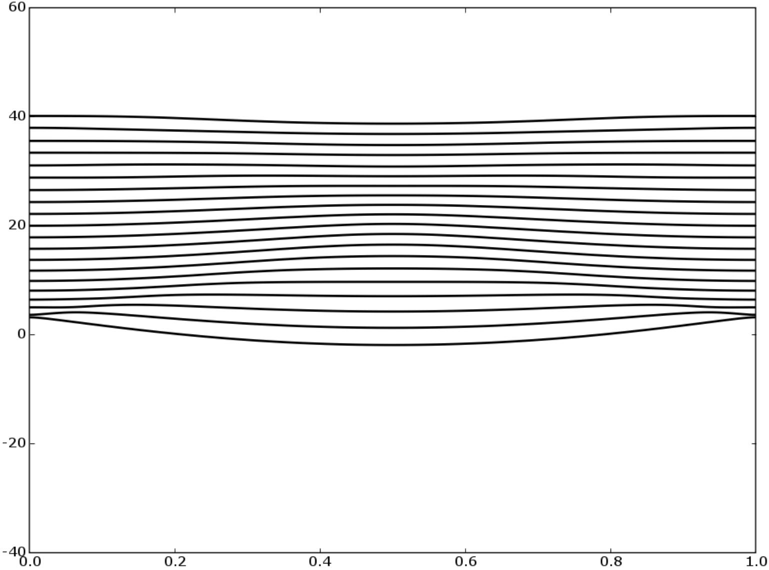
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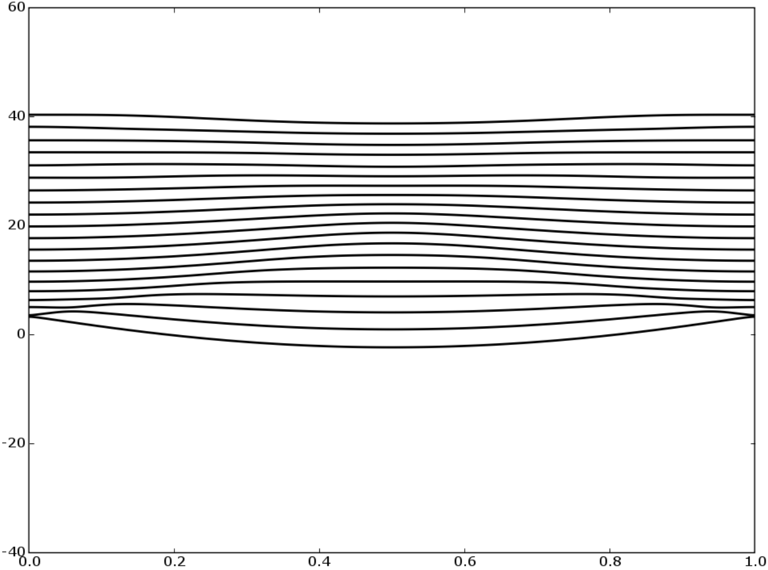
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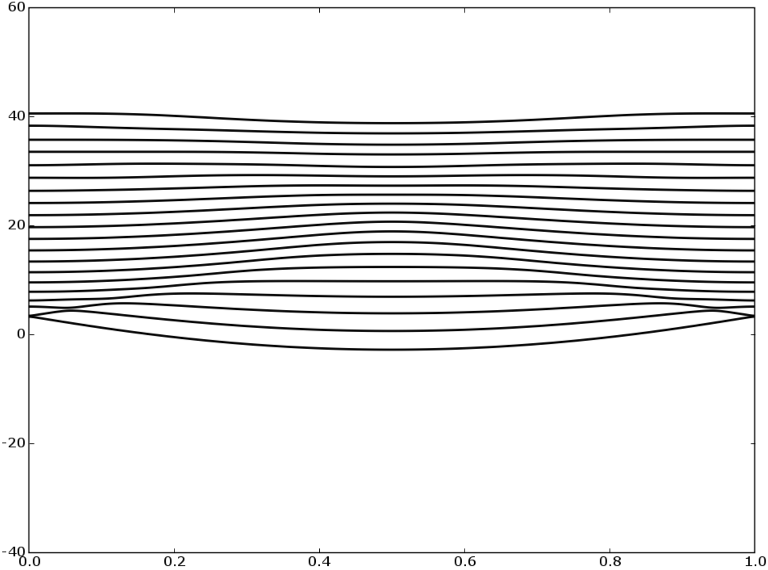
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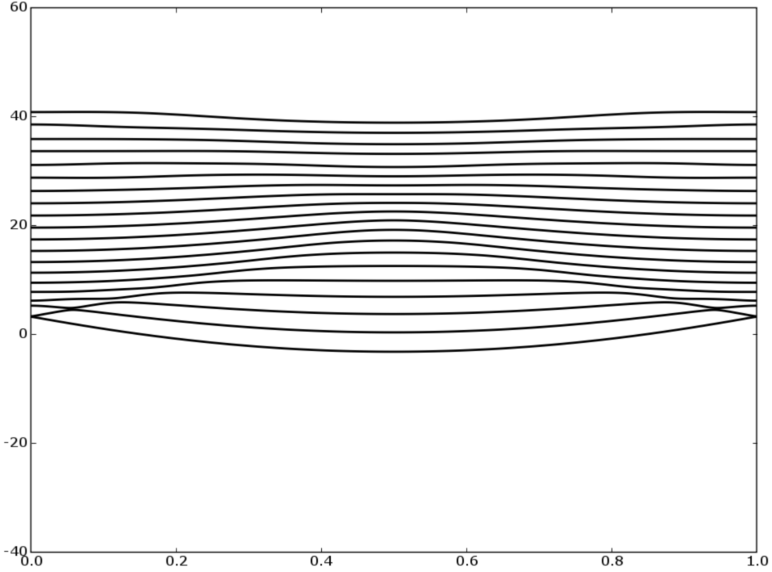
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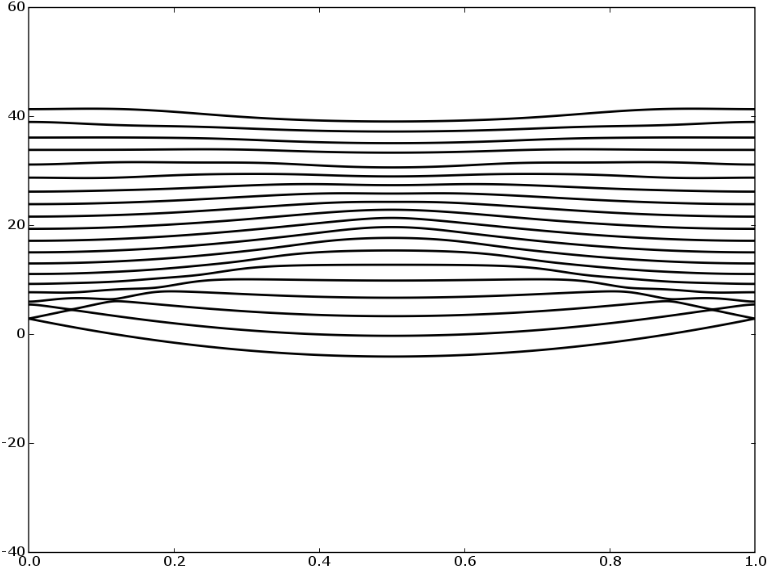
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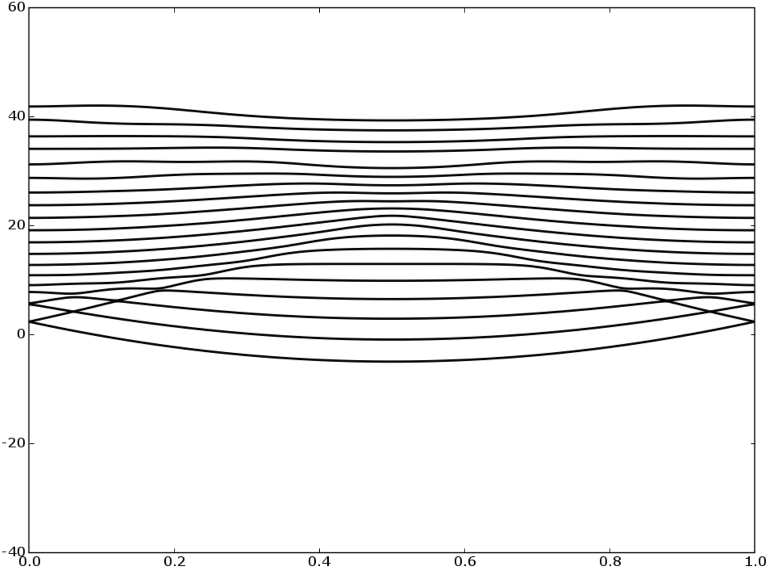
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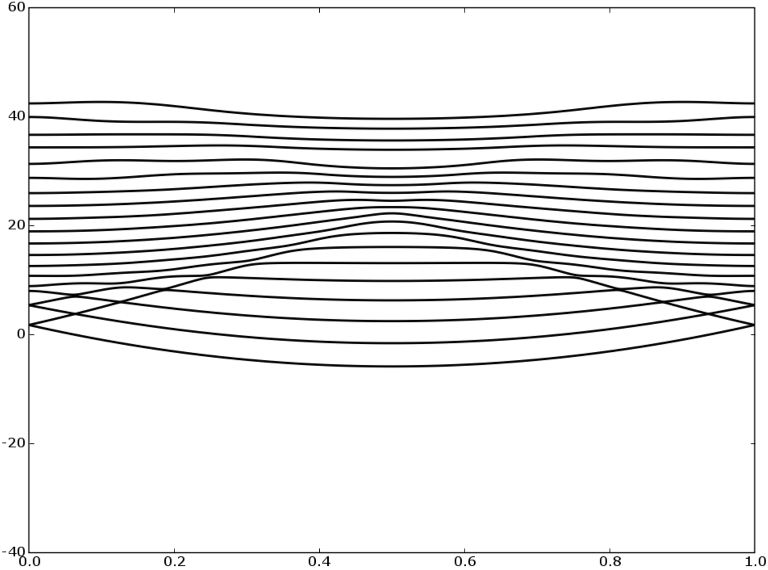
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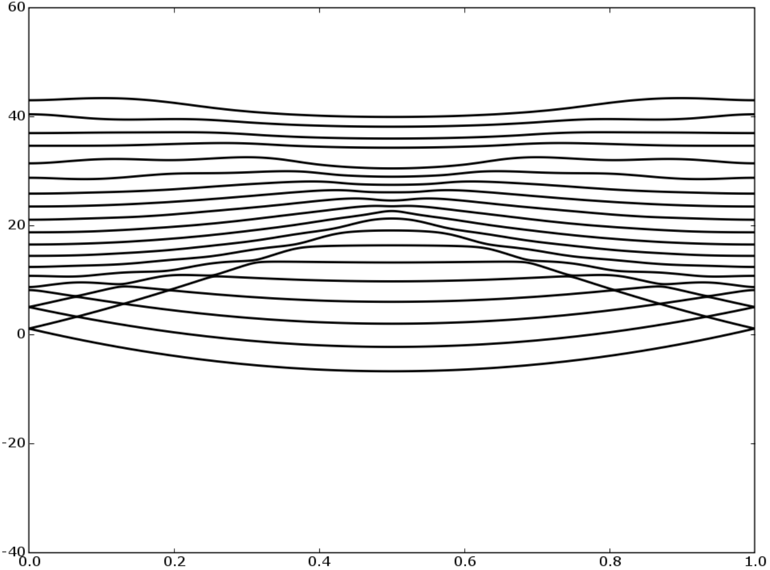
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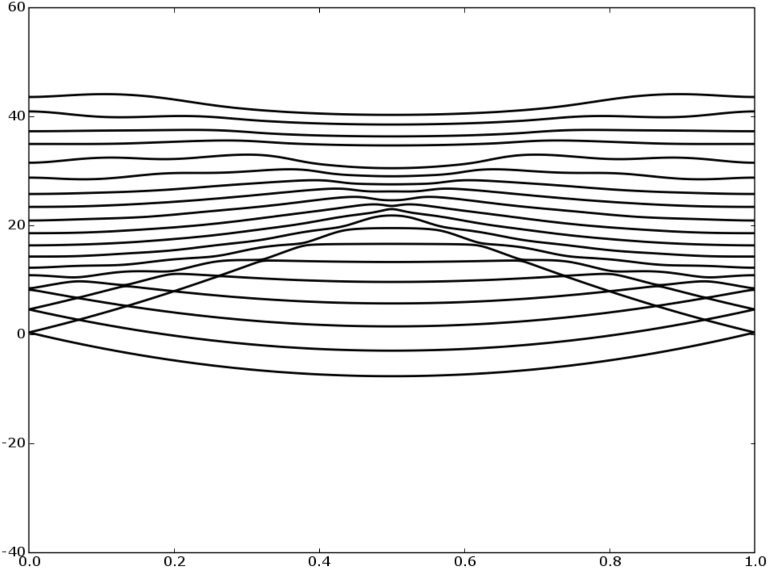
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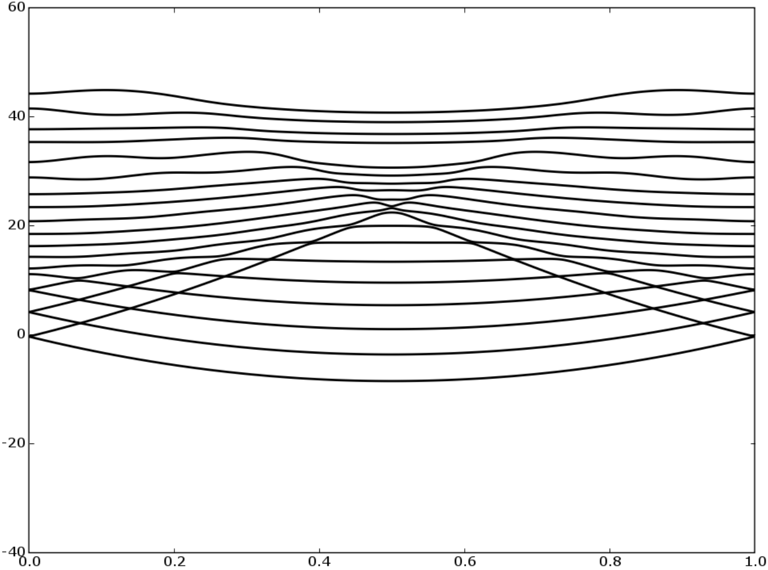
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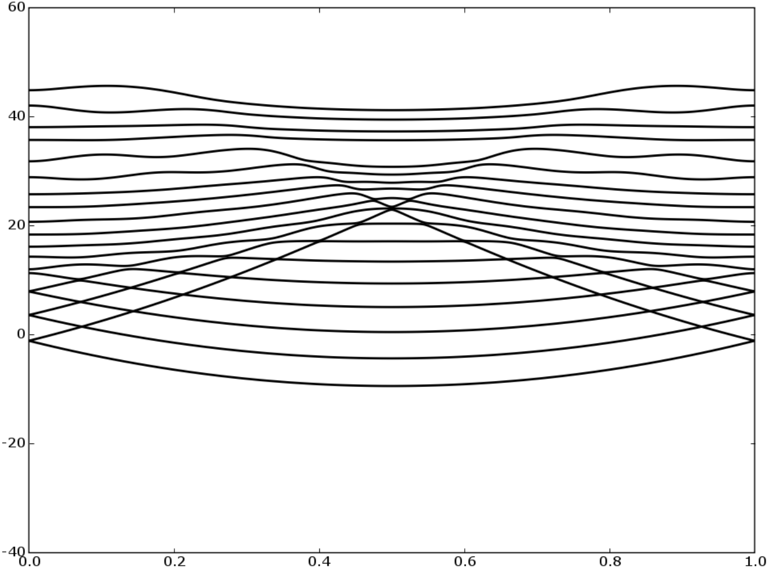
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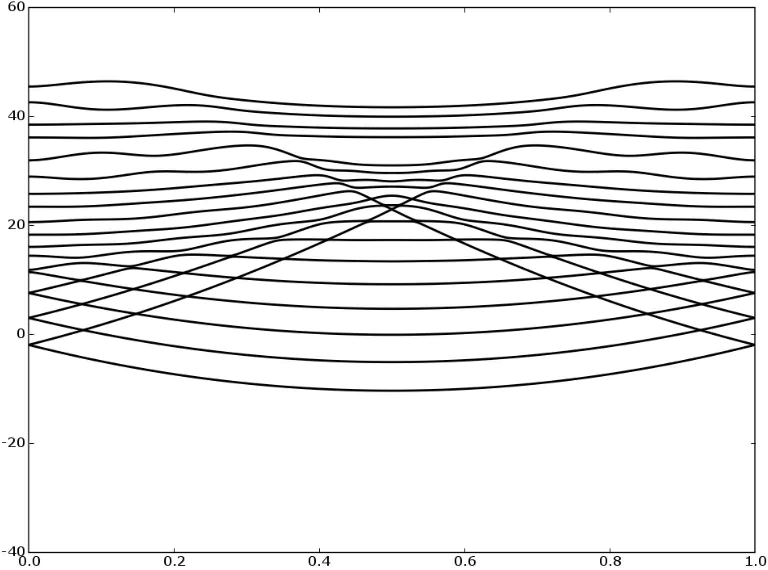

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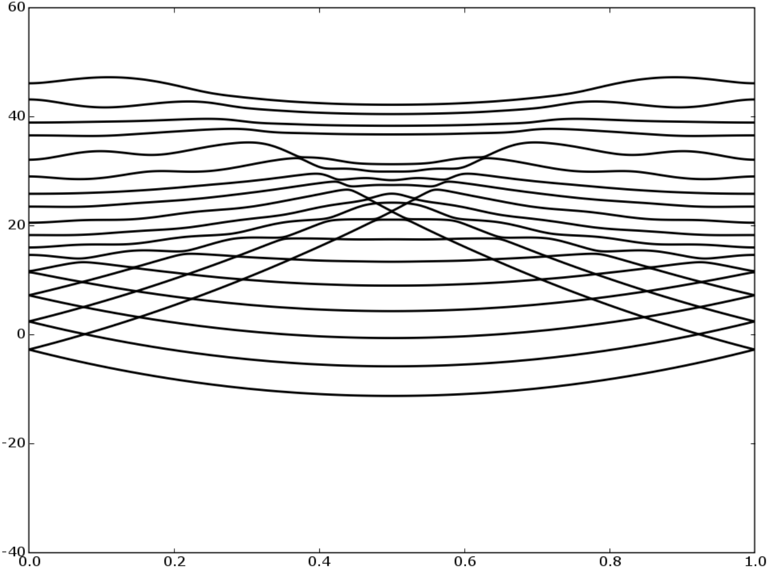
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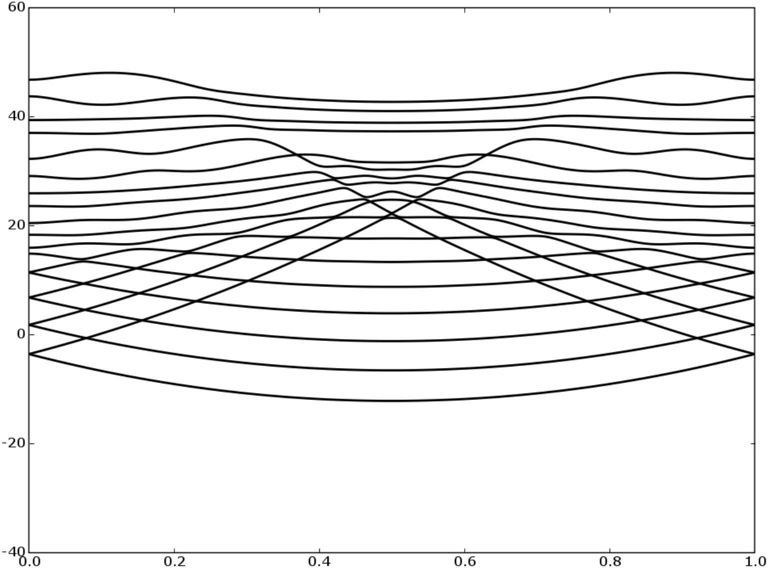
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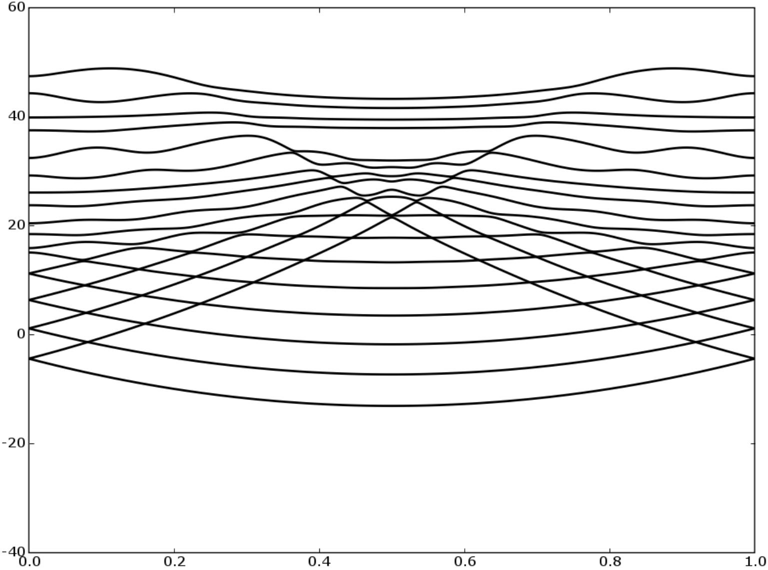
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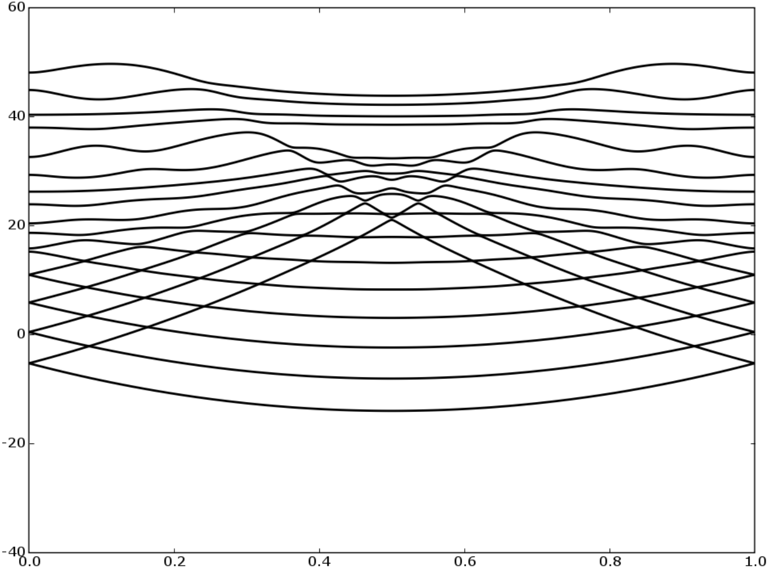
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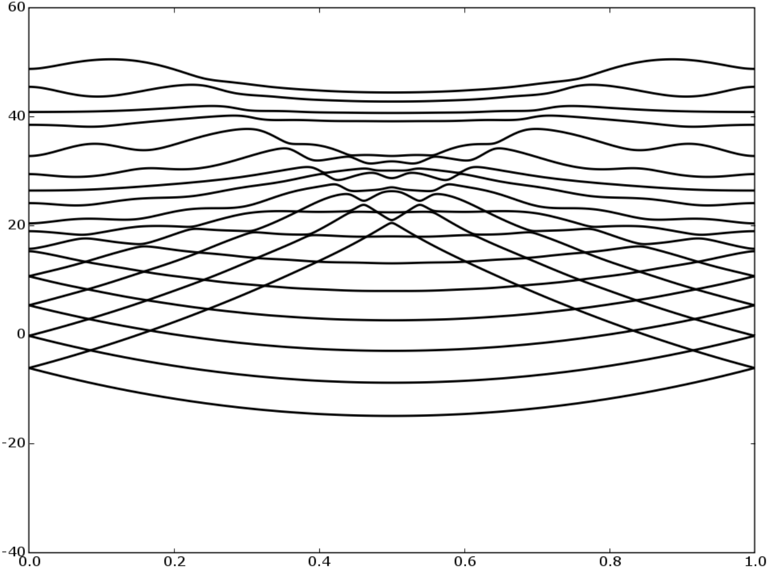
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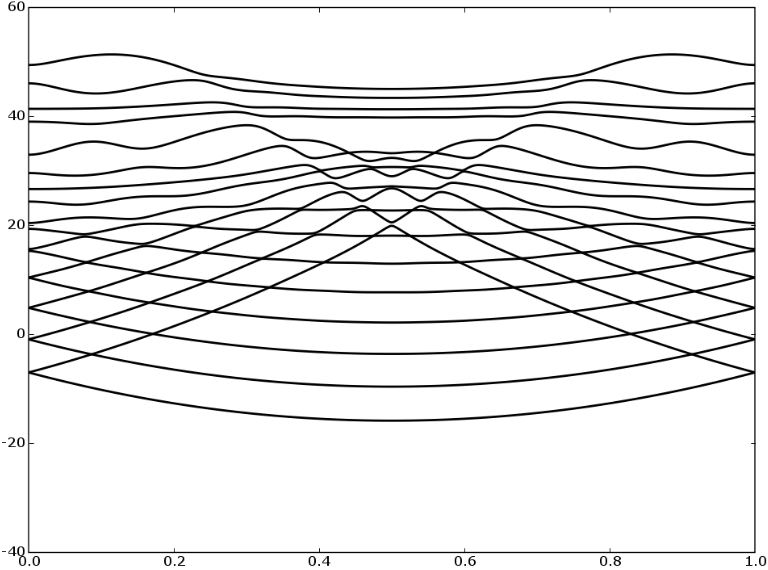
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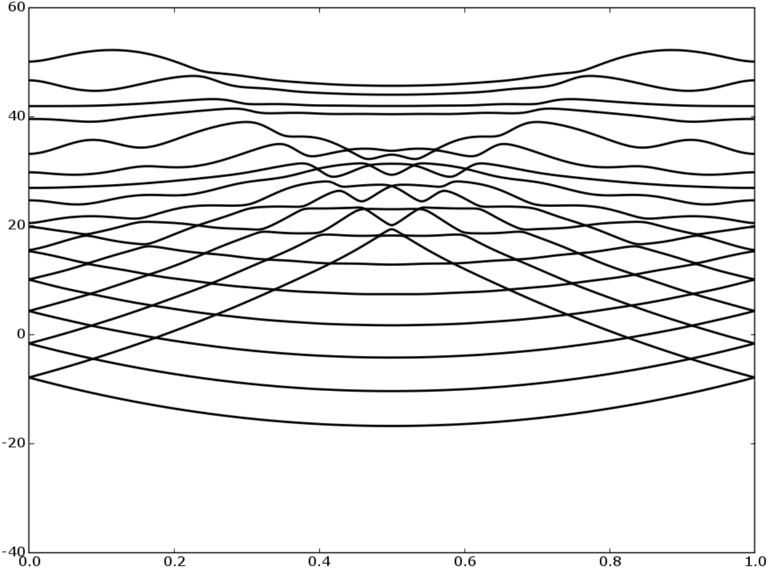
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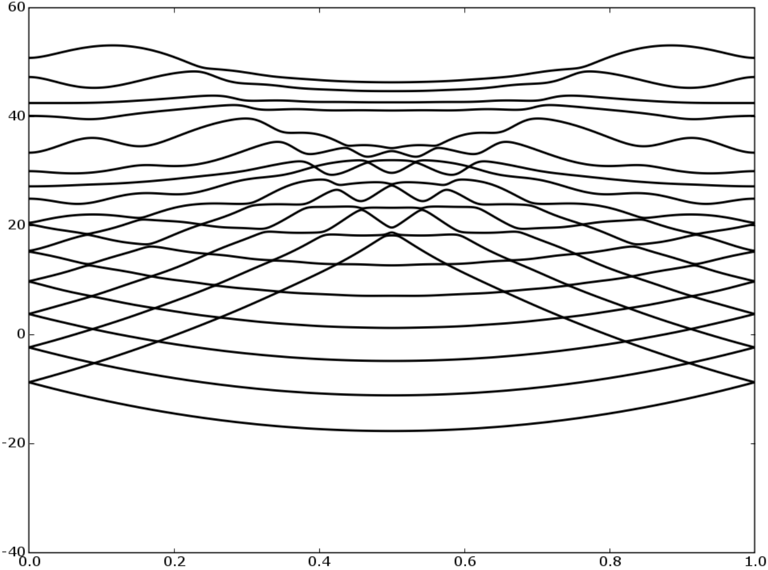
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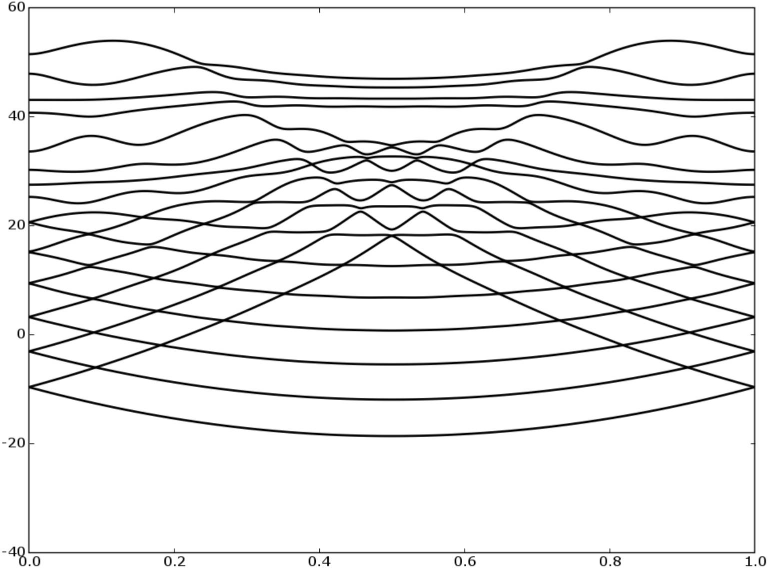
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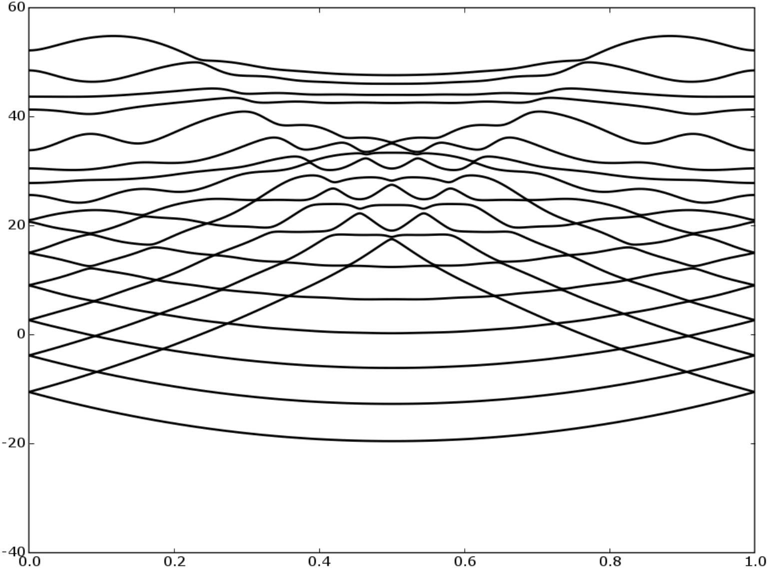
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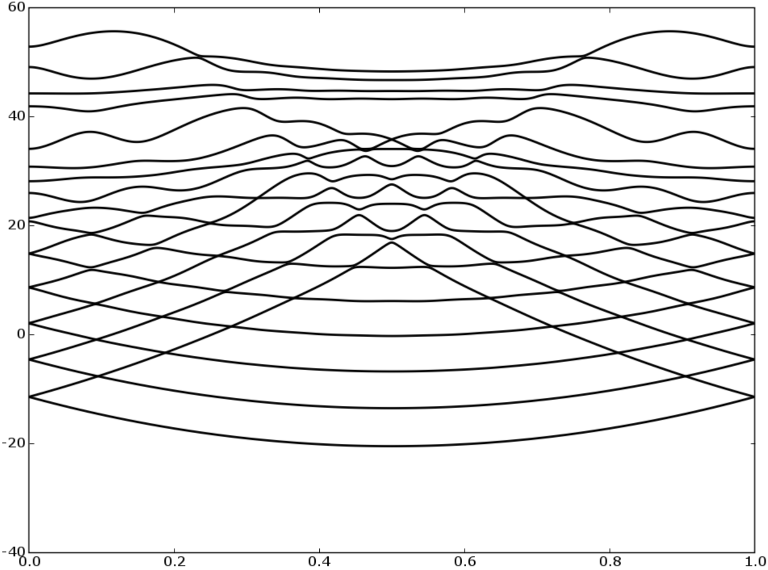
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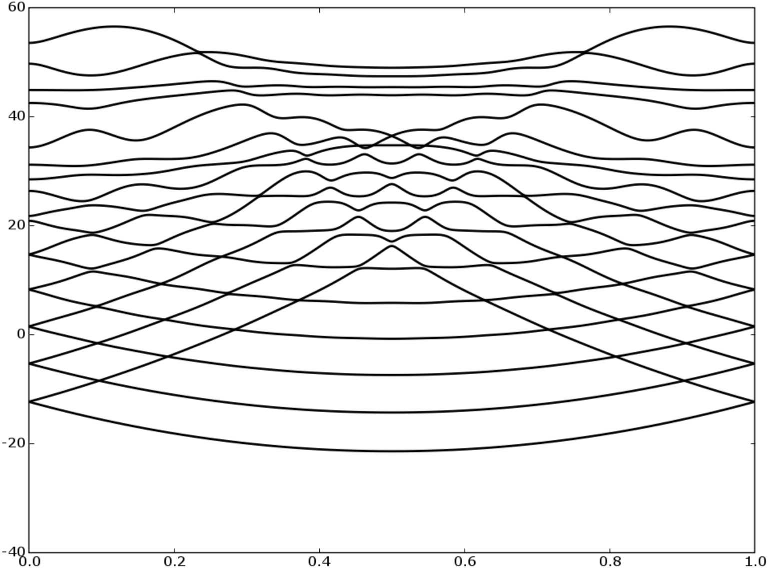
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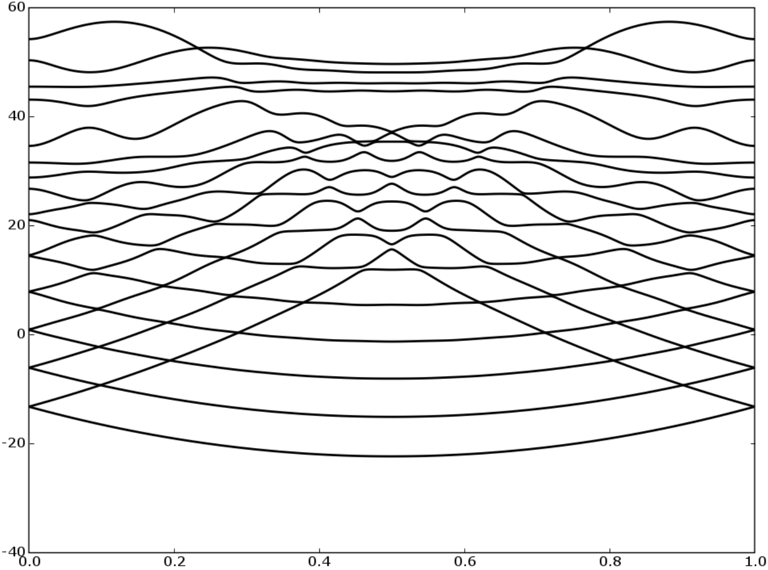
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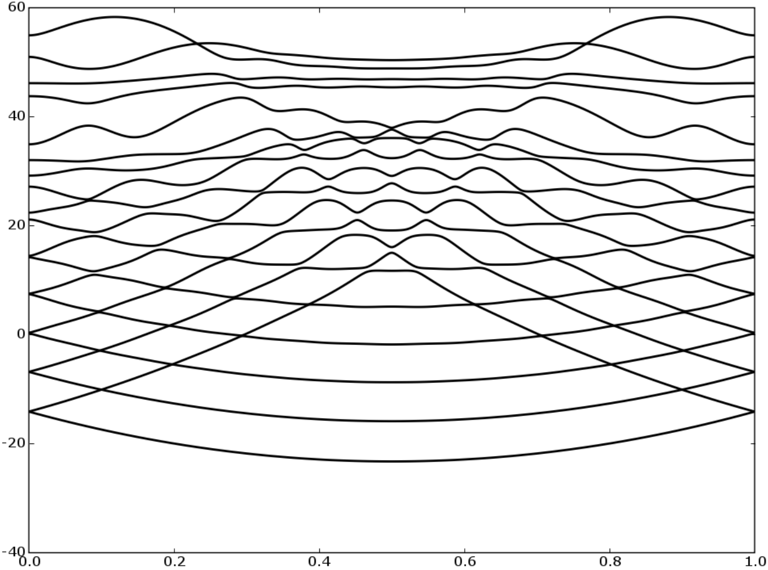
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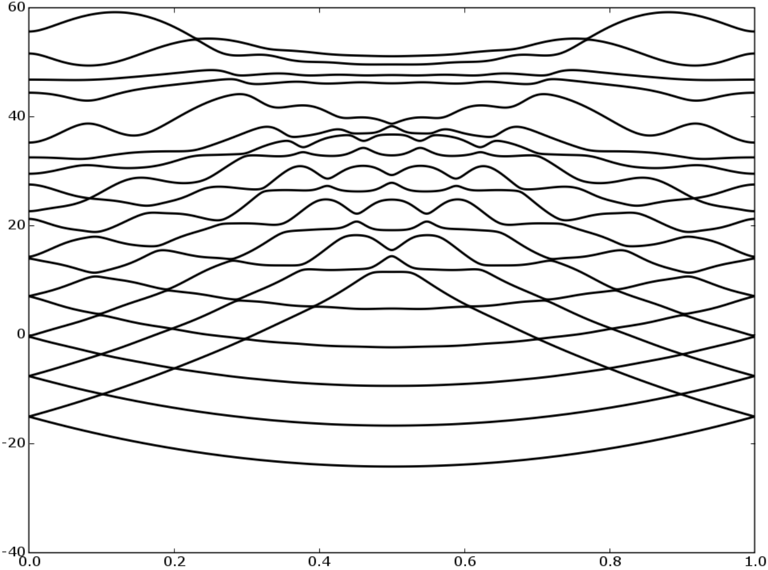
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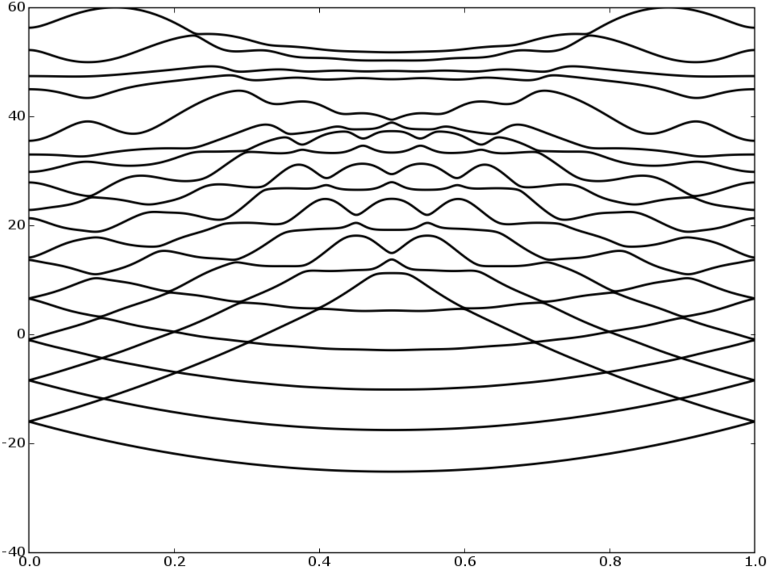
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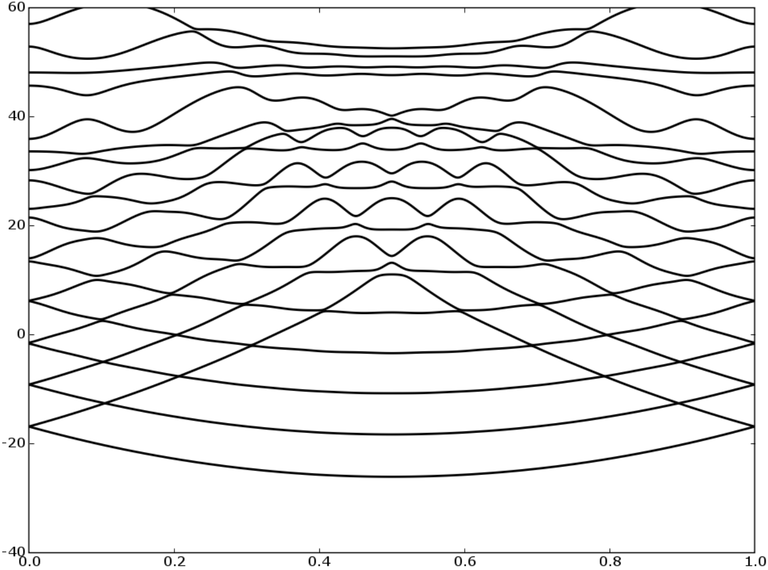
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Strategy for genericity

- Want to show genericity of AC spectrum for continuous periodic perturbations V of the Landau operator
- Have to show that $\lambda_n(\xi)$ (1-dim. perturbation) resp. $\lambda_n(k)$ (2-dim. perturbation) is not constant

Lemma (Klopp 2009)

For all k_0, V_0, n and all $\varepsilon > 0$ there is a pair $(k_\varepsilon, V_\varepsilon)$ and $\delta > 0$ such that $\forall (k, V) \in U_\delta(k_\varepsilon, V_\varepsilon) : \lambda_n(k, V)$ is analytically degenerate.

Lemma (G 2010)

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Theorem (Klopp 2009)

For a generic periodic potential V, the spectrum of the Landau operator perturbed by V (with rational flux) is absolutely continuous.

Lemma (G 2010)

For all $\varepsilon > 0$ *there is* $V \in U_\varepsilon(V_0)$ *such that* $\xi \mapsto \lambda_n(\xi, V)$ *is not constant.*

Theorem (G 2010)

For a generic 1-dimensional periodic potential V*, the spectrum of the Landau operator perturbed by* V *is absolutely continuous.*

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For all $\varepsilon > 0$ there is $V \in U_\varepsilon(V_0)$ such that $\xi \mapsto \lambda_n(\xi, V)$ is not constant.

Theorem (G 2010)

For a generic 1-dimensional almost-periodic potential V, the spectrum of the Landau operator perturbed by V is absolutely continuous.

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Theorem (G 2010)

For a generic 1-dimensional almost-periodic potential V*, the spectrum of the Schrödinger operator with 1-dimensional periodic field perturbed by* V *is absolutely continuous.*

Lemmata ⇒ Theorem

Proof.

- $NC_n := \{V : \lambda_n(\cdot, V) \text{ is not constant}\}$
- By Lipshitz continuity in V: NC_n is open
- Given V_0, n, ε, apply Lemma 1 and find (k, V_1) nearby (resp. take $(\xi, V_1 = V_0)$ such that λ_n is analytically degenerate
- Apply Lemma 2 to find V_2 nearby such that $\lambda_n(\cdot, V_2)$ is not constant

⇒ $NC = \bigcap_{n \in \mathbb{N}_0} NC_n$ is G_δ □

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Feynman-Hellmann ⇒ Lemma 2

Proof.

- Assume $\exists \varepsilon > 0 : \forall V \in U_\varepsilon(V_0) : \lambda_n(\cdot, V)$ is constant
- Take arbitrary U of norm 1, $V := V_0 + tU$, φ eigenfunction
- $\partial_t \lambda = \langle U\varphi, \varphi \rangle$ by Feynman-Hellmann

⇒ By constancy of λ_n in k, $\nabla_k (|\varphi|)^2 = 0$

- Analysing nodal sets, this gives a contradiction

⇒ By constancy of λ_n in ξ, $\partial_\xi(|\varphi|^2) = \beta\partial_y(|\varphi|)^2$

- As a function of ξ, φ solves (for each y) a periodic Sturm-Liouville problem with eigenvalue $\beta^2\lambda_n - (2\pi)^2 y^2$ □

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Further research line

- Analyze matrix elements of periodic V with respect to h_n
- Control special Hermite functions
- For large B, replace $V(y + \beta\xi)$ by $V(\beta\xi)$

Theorem (G 2010)

Let $V \in \mathbb{C}^\infty(\mathbb{R})$ be non-constant and smooth enough, $M \in \mathbb{N}$. Then, for B (constant) large enough, the first M bands are non-degenerate; in particular, they consist of AC spectrum only.

Similar for non-zero (flux zero) 1-dimensional periodic perturbations of B.

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- Magnetic fields can change the spectrum drastically
- Non-vanishing and rationality of flux are key
- Methods differ case by case

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- Non-abelian groups, irrational flux: non-commutative geometry
- Open questions even for integral/rational flux

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Thank you

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