# Rapidly Rotating Bose Gases
# and
# the Transition to a Giant Vortex State

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# Vorticices in Fluid Dynamics

Consider a fluid with velocity field $\mathbf{v(r)}$. The circulation around a closed loop $\mathcal{C}$ enclosing a domain $\mathcal{D}$ is, by Stokes,

$$\oint_{\mathcal{C}} \mathbf{v} \cdot d\ell = \int_{\mathcal{D}} (\nabla \times v) \cdot \mathbf{n} dS.$$

Hence nonzero circulation requires that the *vorticity*

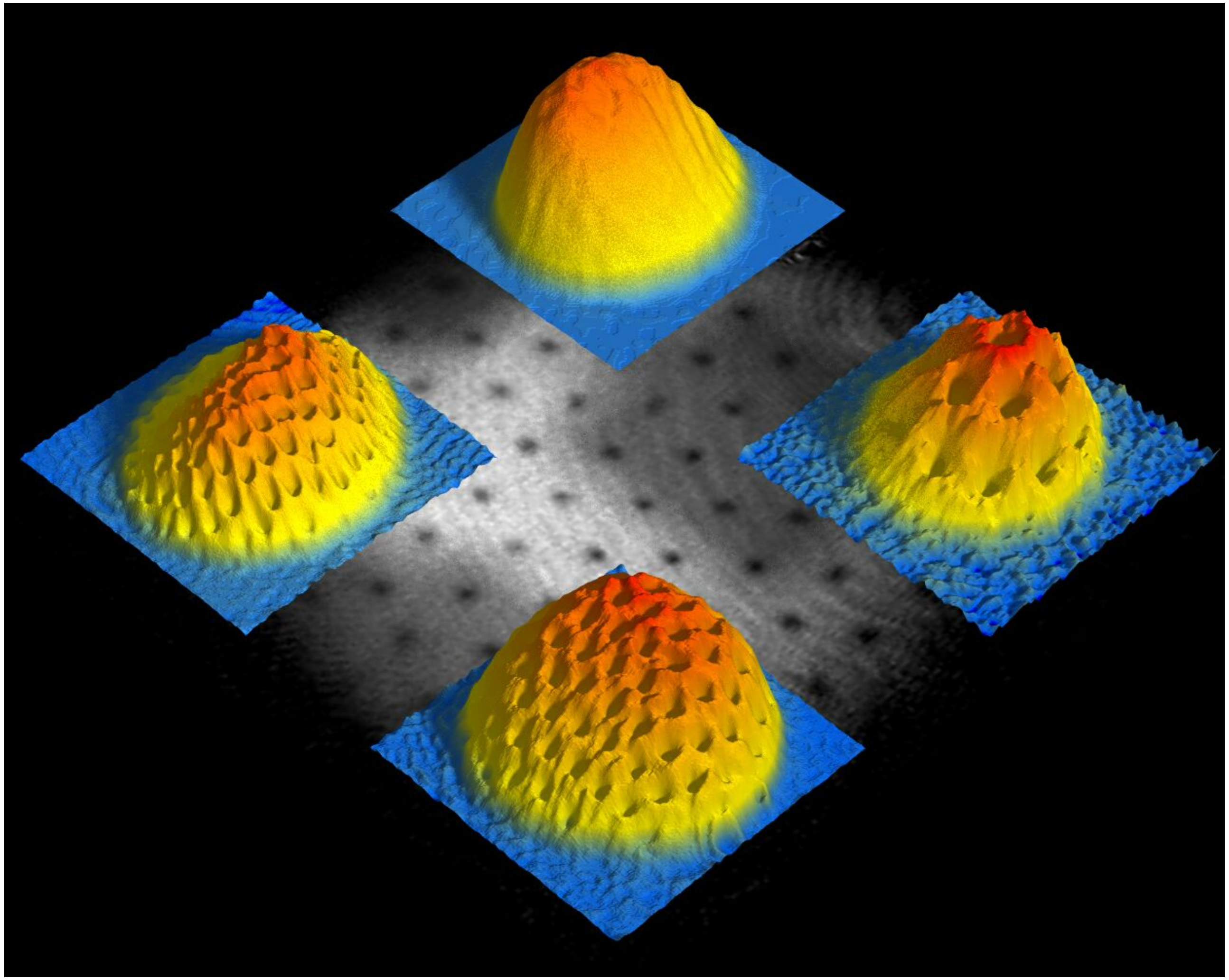
$$\nabla \times v$$

is nonzero somewhere in $\mathcal{D}$.

A region where $\nabla \times v \neq 0$ is called a *vortex*.

# A bathtub vortex

# Quantum vortices



## Why is vorticity quantized in a superfluid?

Describe the superfluid by a complex valued function ("order Parameter") $\psi$ satisfying a nonlinear Schrödinger Equation (Gross-Pitaevskii equation). The phase of $\psi$ determines the velocity: If $\psi = e^{\mathrm{i}\varphi}|\psi|$ then

$$\mathbf{v} = \frac{\hbar}{m}\nabla\varphi.$$

Since $\psi$ is single valued we have $\oint_{\mathcal{C}} \nabla\varphi \cdot d\ell = n\,2\pi$ with $n \in \mathbb{Z}$, so

$$\oint_{\mathcal{C}} \mathbf{v} \cdot d\ell = n\frac{h}{m}.$$

On the other hand, where the phase is nonsingular, i.e., where $|\psi(\mathbf{x})| \neq 0$, we have

$$\nabla \times v = 0.$$

# The Basic Many-Body Hamiltonian

The quantum mechanical Hamiltonian for $N$ spinless bosons with a pair interaction potential $v$ and external potential, $V$, in a rotating frame with angular velocity $\mathbf{\Omega}$ is

$$H = \sum_{j=1}^{N} \left( -\tfrac{1}{2}\nabla_j^2 + V(\mathbf{x}_j) - \mathbf{L}_j \cdot \mathbf{\Omega} \right) + \sum_{1 \le i < j \le N} v(|\mathbf{x}_i - \mathbf{x}_j|).$$

Here $\mathbf{x}_j \in \mathbb{R}^3$, $j = 1, \ldots, N$ are the positions and $\mathbf{L}_j = -\mathrm{i}\,\mathbf{x}_j \times \nabla_j$ the angular momentum operators of the particles. Units have been chosen so that $\hbar = m = 1$ and thus $h/m = 2\pi$. The pair interaction potential $v$ is assumed to be radially symmetric, of short range and repulsive.

$H$ operates on *symmetric* functions in $L^2(\mathbb{R}^{3N})$.

## Hamiltonian, Magnetic Version

The Hamiltonian can alternatively be written in the form

$$H = \sum_{j=1}^{N} \left( \tfrac{1}{2}[\mathrm{i}\nabla_j + \mathbf{A}(\mathbf{x}_j)]^2 + V(\mathbf{x}_j) - \tfrac{1}{2}\Omega^2 r_j^2 \right) + \sum_{1\le i<j\le N} v(|\mathbf{x}_i - \mathbf{x}_j|).$$

with

$$\mathbf{A}(\mathbf{x}) = \mathbf{\Omega} \times \mathbf{x} = \Omega r\, \mathbf{e}_\theta$$

and $r$=distance from the rotation axis.

This corresponds to the splitting of the rotational effects into Coriolis and centrifugal forces. The Coriolis forces have formally the same effect as a magnetic field with vector potential $\mathbf{A}(\mathbf{x})$.

# Harmonic vs. Anharmonic Traps

If $V$ is harmonic in the direction $\perp$ to $\Omega$, i.e.,

$$V(\mathbf{x}) = \tfrac{1}{2}\Omega_{\text{trap}} r^2 + V^{\|}(z)$$

then stability requires $\Omega < \Omega_{\text{trap}}$. Rapid rotation means here that

$$\Omega \to \Omega_{\text{trap}}$$

from below.

If $V$ is anharmonic and increases faster than quadratically in the direction $\perp$ to $\Omega$, e.g. $V(\mathbf{x}) \sim r^s + V^{\|}(z)$ with $s > 2$, then rapid rotation means simply $\Omega \to \infty$.

# Gross-Pitaevskii Equation

Basic fact (E. Lieb and R. Seiringer, 2005) about the the many-body Hamiltonian for $N \to \infty$ with $Na$ and $\Omega$ fixed, where $a$ is the scattering length of the interaction potential $v$:

There is Bose-Einstein condensation in the ground state as $N \to \infty$, and the wave function of the condensate satisfies a non-linear Schrödinger equation, the Gross-Pitaevskii equation

$$\left\{\tfrac{1}{2}(\mathrm{i}\nabla + \mathbf{A})^2 + (V - \tfrac{1}{2}\Omega^2 r^2) + 4\pi Na|\psi|^2\right\}\psi = \mu\psi \ .$$

## Gross-Pitaevskii Energy Functional

The GP equation is obtained by minimizing the energy functional

$$\begin{aligned}\mathcal{E}^{\mathrm{GP}}[\psi] &= \int_{\mathbb{R}^3} \left\{ \tfrac{1}{2}|\nabla\psi|^2 + V|\psi|^2 - \psi^* \,\mathbf{\Omega}\cdot\mathbf{L}\psi + 2\pi Na|\psi|^4 \right\} d\mathbf{x} \\ &= \int_{\mathbb{R}^3} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 + (V - \tfrac{1}{2}\Omega^2 r^2)|\psi|^2 + 2\pi Na|\psi|^4 \right\} d\mathbf{x}\end{aligned}$$

with the normalization condition $\int_{\mathbb{R}^3}|\psi|^2 = 1$. A minimizer, i.e., a solution of the GP equation, will be denoted by $\psi^{\mathrm{GP}}$.

## Asymptotic Regimes

The GP minimization problem has two parameters, $\Omega$ and $Na$. In anharmonic traps we shall consider it in the asymptotic regime where both these parameter are large. It is convenient to introduce

$$\varepsilon \equiv (2\pi Na)^{-1/2}$$

which is small if $Na$ is large.

In harmonic traps with $\Omega \to \Omega_{\text{trap}}$ it turns out to be appropriate to restrict the wave functions to the Lowest Landau Level of the magnetic Hamiltonian $\frac{1}{2}(\mathrm{i}\nabla + \mathbf{A})^2$. Discussed in the lecture of Mathieu Lewin!

## Status of GP for Rapid Rotation

Important note: The rigorous derivation of Lieb and Seiringer of the GP equation from the many-body problem is carried out for $\Omega$ and $\varepsilon$ fixed. For rapid rotation the GP description may break down both in harmonic and anharmonic traps. The exact limitations can be conjectured but are still not completely proven. Step in this direction in anharmonic traps:

If $N \to \infty$ and $\Omega \to \infty$ but the gas remains dilute (in the sense that mean density $\ll a^{-3}$) the TF approximation, i.e., GP without the kinetic term $\frac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2$, gives the leading term in the ground state energy as a function of $\Omega$ and $\varepsilon$.

This was proved by M. Correggi, J.-B. Bru, P. Pickl and JY in 2007.

## GP Theory for Rapid Rotation, Anharmonic Traps

Consider, for simplicity, a 2D 'flat', circular trap with radius 1. The GP energy functional is then

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathcal{B}} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 - \tfrac{1}{2}\Omega^2 r^2 |\psi|^2 + \frac{1}{\varepsilon^2}|\psi|^4 \right\} d\mathbf{r}$$

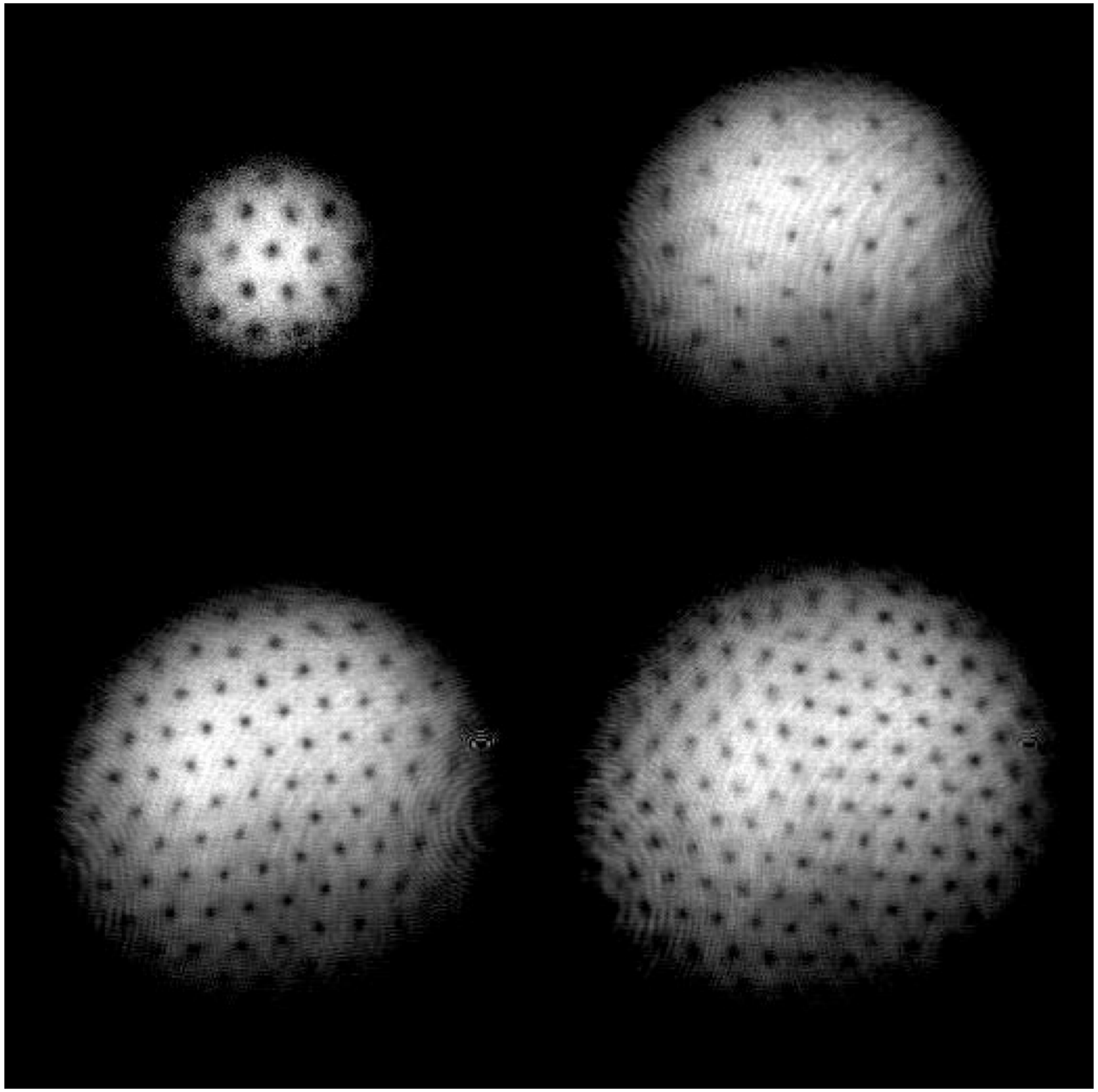
where $\mathcal{B}$ is the unit disc and $\mathbf{A}(\mathbf{r}) = \Omega\, r\, \mathbf{e}_\theta$.

It can be proved that if $\Omega \leq \Omega_1 |\log \varepsilon| + O(\log|\log \varepsilon|)$ there is a finite number of vortices, even as $\varepsilon \to 0$. For larger $\Omega$ the number of vortices is unbounded as $\varepsilon \to 0$.

If $\Omega = O(1/\varepsilon)$ the centrifugal term $-(\Omega^2/2)r^2|\psi(\mathbf{r})|^2$ and the interaction term $(1/\varepsilon^2)|\psi(\mathbf{r})|^4$ are comparable in size.

## Vortices Reduce Kinetic Energy

The kinetic energy term $\frac{1}{2}|(\mathrm{i}\nabla + \mathbf{A}(\mathbf{r}))\psi(\mathbf{r})|^2$ is formally also of order $1/\varepsilon^2$ if $\Omega \sim 1/\varepsilon$. However, it turns out that its contribution to the energy is, in fact, of lower order, namely $\sim \Omega|\log\varepsilon|$, because a lattice of vortices emerges as $\varepsilon \to 0$. The velocity field generated by the vortices compensates partly that generated by $\mathbf{A}(\mathbf{r}) = \Omega r\,\mathbf{e}_\theta$.



## Energy to Subleading Order

**THEOREM** (M. Correggi and JY, 2008) *Let* $E^{\mathrm{GP}}$ *denote the GP energy, i.e., the minimum of the GP energy functional. Let* $E^{\mathrm{TF}}$ *denote the minimal energy of the GP functional without the kinetic term.*

*If* $|\log \varepsilon| \ll \Omega \ll 1/\varepsilon$, *then*

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \tfrac{1}{2}\Omega|\log(\varepsilon^2\Omega)|(1 + o(1)).$$

*If* $1/\varepsilon \lesssim \Omega \ll 1/(\varepsilon^2|\log \varepsilon|)$ *then*

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \tfrac{1}{2}\Omega|\log \varepsilon|(1 + o(1)).$$

## An Electrostatic Analogy

Write points $\mathbf{r} = (x, y) \in \mathbb{R}^2$ as complex numbers, $\zeta = x + iy$, and consider a lattice of points $\zeta_i$. Placing a vortex of degree 1 at each point $\zeta_i$ leads to a trial function for the GP energy of the form

$$\psi(\mathbf{r}) = f(\mathbf{r})\exp\{\mathrm{i}\varphi(\mathbf{r})\}$$

where $f$ is real valued and

$$\exp\{\mathrm{i}\varphi(\mathbf{r})\} = \prod_i \frac{\zeta - \zeta_i}{|\zeta - \zeta_i|}.$$

Now

$$|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 = |\nabla f|^2 + f^2|\mathbf{A} - \nabla\varphi|^2$$

and by some elementary complex analysis we can write

$$|\mathbf{A} - \nabla\varphi|^2 = |\Omega\, r\mathbf{e}_r - \nabla\chi|^2$$

where

$$\chi(\mathbf{r}) = \sum_i \log|\mathbf{r} - \mathbf{r}_i|.$$

But

$$\mathbf{E}(\mathbf{r}) \equiv \Omega\, r\mathbf{e}_r - \nabla\chi(\mathbf{r})$$

has a simple physical interpretation: It can be regarded as an 'electric field' generated by a uniform charge distribution of density $\Omega/\pi$ together with unit 'charges' of opposite sign at the positions of the vortices, $\mathbf{r}_i$.

## Vortex Lattice

We now distribute the vortices over the unit disk so that the vorticity per unit area is $\Omega/\pi$. (This is really $2\Omega \cdot m/h$.) Thus every vortex $\mathbf{r}_i$ sits at the center of lattice cell $Q_i$ of area $|Q_i| = \pi/\Omega$, surrounded by a uniform charge distribution of the opposite sign so that the total charge in the cell is zero. *If* the cells were disc-shaped, then Newton's theorem would imply that the 'electric field' generated by the cell would vanish outside the cell, i.e, there would be no interaction between the cells.

## Why hexagonal cells are optimal

The cells can, of course, not be disc shaped, but the closest approximation to that are hexagonal cells, giving the optimal energy. The vortices then sit on a triangular lattice. The interaction between the cells, although not zero, is small because the cells have only a quadrupole moment or higher and no dipole moment.

# Emergence of a 'Giant Vortex'

If $\Omega \gg 1/\varepsilon$ the centrifugal forces dominate the repulsive interaction forces and the density becomes concentrated in a thin annulus of size $\sim (\varepsilon\Omega)^{-1}$. This is $\ll 1$ if $\Omega \gg 1/\varepsilon$. Nevertheless, this annulus may still contain a lattice of vortices. In fact, as long as $\Omega \ll (\varepsilon^2 |\log \varepsilon|)^{-1}$ it can be shown that, for $\varepsilon \to 0$, vorticity is uniformly distributed in the annulus with density $\Omega/\pi$.

For

$$\Omega > \frac{\Omega_c}{\varepsilon^2 |\log \varepsilon|}$$

with a certain $\Omega_c > 0$, however, a phase transition occurs: Vortices disappear completetely from the thin layer where the density is concentrated and all the vorticity is concentrated in the 'hole'. This is called a Giant Vortex.

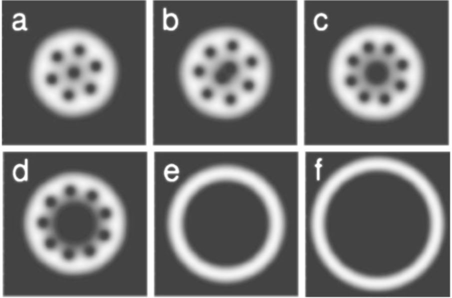
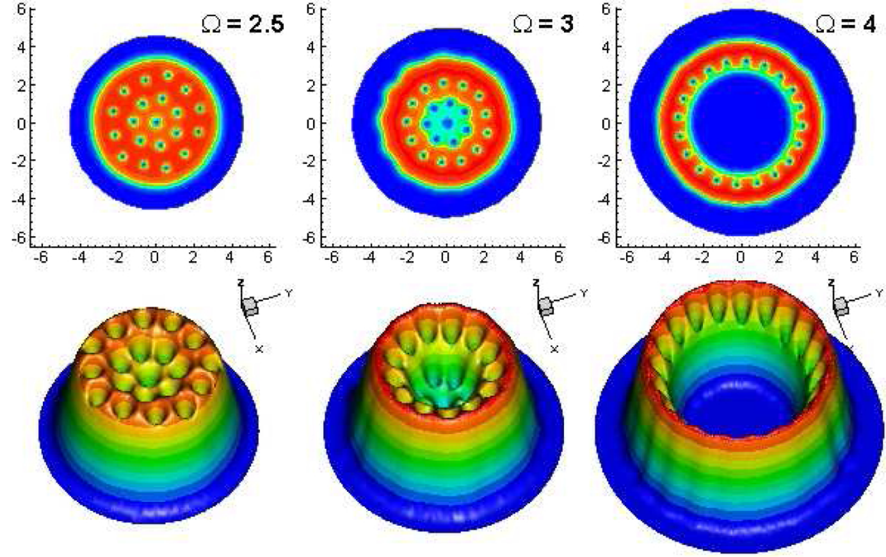



FIG. 1. Density profiles of a rotating condensate at $g = 80$ and $\lambda = 0.5$, for $\Omega =$ (a) 2.0, (b) 2.1, (c) 2.25, (d) 2.5, (e) 3.0, and (f) 3.5. The scale of each figure is $4 \times 4$ in units of $d_\perp$.



Ω = 2.5
Ω = 3
Ω = 4
6
4
2
0
-2
-4
-6
Z
Y
X

A heuristic explanation for the transition at $\Omega \sim 1/(\varepsilon^2|\log \varepsilon|)$ can be given by exploiting the electrostatic analogy:

A variational *ansatz* for $\psi$ of the form

$$\psi(\mathbf{r}) = f(\mathbf{r})\exp(\mathrm{i}\hat{\Omega}\theta)$$

with a real valued function $f$ is optimal if

$$\hat{\Omega} = \Omega - O(\varepsilon^{-1}).$$

The 'electric field' generated by a charge $\hat{\Omega}$ at the origin exactly cancels, in the annulus of thickness $(\varepsilon\Omega)^{-1}$, the 'electric field' generated in the annulus by the uniform charge density $\Omega/\pi$ of the ‘hole’(by Newton’s theorem). However, the ‘charge’ corresponding to $\mathbf{A}$ in the annulus is not cancelled, and this ‘residual charge’ is

$$\text{charge density x area of annulus } \sim \Omega \times (\varepsilon\Omega)^{-1} = \varepsilon^{-1}.$$

The electrostatic energy of the residual charge distribution is $\sim \varepsilon^{-2}$. Creating a vortex in the annulus neutralizes one charge unit and thus reduces the electrostatic energy by $\varepsilon^{-1}$. In other words,

$$\text{gain by creating a single vortex} \sim \frac{1}{\varepsilon}.$$

On the other hand, the *cost* of a vortex is $\sim f^2 \,|\log \varepsilon|$, and we have $f^2 \sim (\varepsilon\Omega)$, so

$$\text{cost of a single vortex} \sim \varepsilon\Omega \,|\log \varepsilon|\,.$$

Gain and cost are thus comparable if

$$\Omega \sim \frac{1}{\varepsilon^2 |\log \varepsilon|}$$

If $\Omega$ is smaller it still pays to have vortices also in the annulus, but if $\Omega$ is larger, the cost outweighs the gain and there are no vortices in the annulus. In other words: If

$$\Omega > \frac{\text{const.}}{\varepsilon^2 |\log \varepsilon|}$$

all vorticity originates in the region where the density is vanishingly small.

A mathematical proof of this is, however, far from simple.

## The 'Giant Vortex Theorem'

Recall the GP energy functional

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathcal{B}} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 - \tfrac{1}{2}\Omega^2 r^2 |\psi|^2 + \frac{1}{\varepsilon^2}|\psi|^4 \right\} d\mathbf{r}$$

with $\mathbf{A}(\mathbf{r}) = \Omega\, r\, \mathbf{e}_\theta$. Let $\psi^{\mathrm{GP}}$ be a minimizer. Define

$$\mathcal{A}_{\mathrm{bulk}} := \{\mathbf{r} :\ R_{\mathrm{h}} + \varepsilon |\log \varepsilon|^{-1} \le r \le 1\}$$

where $R_{\mathrm{h}} = 1 - c(\Omega\varepsilon)^{-1}$ is the radius of the 'hole' of the TF density $\rho^{\mathrm{TF}}(\mathbf{r}) = [\frac{1}{4}\Omega^2 r^2 - \mu^{\mathrm{TF}}]_+$.

**THEOREM** (M. Correggi, N. Rougerie, JY, 2010)
Suppose $\Omega = \Omega_0(\varepsilon^2|\log\varepsilon|)^{-1}$. If $\Omega_0 > (3\pi)^{-1}$, then $\psi^{\mathrm{GP}}$ has no zeros in $\mathcal{A}_{\mathrm{bulk}}$ for small $\varepsilon$. More precisely, for $\mathbf{r} \in \mathcal{A}_{\mathrm{bulk}}$

$$\left| |\psi^{\mathrm{GP}}(\mathbf{r})|^2 - \rho^{\mathrm{TF}}(\mathbf{r}) \right| \le C\varepsilon^{-7/8} |\log\varepsilon|^3 \ll \rho^{\mathrm{TF}}(\mathbf{r}) = O(\varepsilon^{-1}|\log\varepsilon|^{-1}).$$

## Steps of the proof

1. Determining the optimal giant vortex ansatz
2. Splitting of the energy functional
3. Concentration of the density
4. Simple energy bound
5. Division into cells and vortex ball construction
6. Jacobian estimate and improved energy bounds
7. Gradient estimate and completion of proof

# Determining the optimal giant vortex ansatz

Optimizing a variational ansatz of the form $\psi(\mathbf{r}) = g(\mathbf{r})\,\mathrm{exp}(\mathrm{i}\widehat{\Omega}\theta)$ leads to a coupled minimization problem: The nonnegative function $g$ minimizes the functional

$$\widehat{\mathcal{E}}^{\mathrm{GP}}[g] = \int_{\mathcal{B}} \left\{ \tfrac{1}{2} \left[ |\nabla g|^2 - \Omega^2 r^2 |g|^2 + B^2 |g|^2 \right] + \varepsilon^{-2} |g|^4 \right\} d\mathbf{r}$$

under the normalization condition $\int_{\mathcal{B}} g^2 = 1$ with

$$\mathbf{B}(r) \equiv \left( \widehat{\Omega}/r - \Omega r \right) \mathbf{e}_\theta$$

and

$$\widehat{\Omega} = \Omega \left( \int_{\mathcal{B}} g(r)^2 r^{-2} d\mathbf{r} \ \right)^{-1}.$$

## Splitting of the energy functional

Write $\psi(\mathbf{r}) = u(\mathbf{r})g(r)\,\exp(\mathrm{i}\widehat{\Omega}\theta)$. Then, using the variational equation for $g$, we obtain the splitting

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \widehat{E}^{\mathrm{GP}} + \mathcal{E}_g[u]$$

with $\widehat{E}^{\mathrm{GP}}$ the g.s.e. of $\widehat{\mathcal{E}}^{\mathrm{GP}}$ and

$$\mathcal{E}_g[u] = \int_{\mathcal{B}} \left\{ |\nabla u|^2 - \mathrm{i}\,\mathbf{B}\cdot u^*\nabla u + g^2\varepsilon^{-2}(1-|u|^2)^2 \right\} g^2 d\mathbf{r}.$$

Minimizing $\mathcal{E}^{\mathrm{GP}}$ w.r.t. $\psi$ is equivalent to minimizing the weighted Ginzburg-Landau-type functional $\mathcal{E}_g$ w.r.t. $u$. Henceforth $u$ denotes the minimizer with normalization $\int_{\mathcal{B}} |u|^2 g^2 = 1$.

## Concentration of the density

The 'shape function' $g$ depends on the small parameter $\varepsilon$. As $\varepsilon \to 0$ it gets concentrated in an annulus $\mathcal{A}$ of thickness $\ell \sim (\varepsilon\Omega)^{-1}$ which is $O(\varepsilon|\log\varepsilon|)$ if $\Omega = \Omega_0\varepsilon^{-2}|\log\varepsilon|^{-1}$. The same holds for the full GP minimizer $\psi^{\mathrm{GP}} = ug$. More precisely:

$$\text{If} \quad r \leq R_{\mathrm{h}} - \varepsilon^{7/6} \quad \text{then} \quad g(r)^2 \leq C\,(\varepsilon\Omega)\exp\{-1/\varepsilon^{1/6}\}.$$

Here $R_{\mathrm{h}} = 1 - \ell$ is the radius of the 'hole'.

Analogous estimate holds for $|\psi^{\mathrm{GP}}|^2$.

On $\mathcal{A}$, on the other hand, $g^2$ can be shown to be close to a function that rises from zero at $r = R_{\mathrm{h}}$ to $O(\varepsilon\Omega)$ at $r = 1$.
**Important**: On $\mathcal{A}_{\mathrm{bulk}} = \{\mathbf{r} : \; R_{\mathrm{h}} + \varepsilon|\log\varepsilon|^{-1} \leq r \leq 1\}$

$$g^2(r) \geq c\frac{\varepsilon\Omega}{|\log\varepsilon|} \sim \frac{1}{\varepsilon|\log\varepsilon|^2}.$$

## Simple energy bound

Write

$$\mathcal{E}_g[u] = \mathcal{F}_g[u] - g^2 \mathrm{i}\, \mathbf{B} \cdot u^* \nabla u$$

with

$$\mathcal{F}_g[u] = \int_{\mathcal{B}} \left\{ |\nabla u|^2 + g^2 \varepsilon^{-2} (1 - |u|^2)^2 \right\} g^2 d\mathbf{r}.$$

In the annulus $\mathcal{A}$ we have $B(r) \sim \Omega(1/r - r) = O(\varepsilon^{-1})$. The upper bound $\mathcal{E}_g[u] \leq 0$, together with Cauchy-Schwarz and the normalization condition, leads to

$$\mathcal{F}_g[u] \leq \frac{C}{\varepsilon^2} \quad \text{and} \quad \mathcal{E}_g[u] \geq -\frac{C}{\varepsilon^2}.$$

The simple estimate for $\mathcal{F}_g[u]$ is not enough to conclude that $u$ has no zeros. Estimates of this type have, however, been used in Ginzburg-Landau theory (e.g. by E. Sandier and S. Sherfaty) to show that eventual zeros can be isolated in little balls ('vortex balls') and to obtain lower bounds on 'magnetic' functionals of the type $\mathcal{E}_g$ in terms of winding numbers of $u$ around these balls. By 'jacobian estimates' one would then like to show that nonzero winding numbers are too costly if $\Omega$ is too large. There are two problems, however:

- The relevant domain $\mathcal{A}_{\text{bulk}}$ is not fixed but depends on $\varepsilon$.

- The function $g$ is not constant in $\mathcal{A}_{\text{bulk}}$.

The former is more serious than the latter.

It is instructive to pretend for a moment that $g^2$ is constant $\sim (\varepsilon\Omega) \sim (\varepsilon|\log\varepsilon|)^{-1}$ on $\mathcal{A}_{\text{bulk}}$ and strictly zero in the 'hole'. By scaling, writing $\mathbf{r} = \ell\tilde{\mathbf{r}}$, $\tilde{u}(\tilde{\mathbf{r}}) = u(\mathbf{r})$ and $\tilde{\mathbf{B}}(\tilde{r}) = \ell\mathbf{B}(r)$, we have

$$\int_{\mathcal{A}_{\text{bulk}}} \left\{|\nabla u|^2 - \mathrm{i}\,\mathbf{B}\cdot u^*\nabla u + g^2\varepsilon^{-2}(1-|u|^2)^2\right\} d\mathbf{r} =$$

$$\int_{\tilde{\mathcal{A}}_{\text{bulk}}} \left\{|\tilde{\nabla}\tilde{u}|^2 - \mathrm{i}\,\tilde{\mathbf{B}}\cdot \tilde{u}^*\tilde{\nabla}\tilde{u} + \tilde{\varepsilon}^{-2}(1-|\tilde{u}|^2)^2\right\} d\tilde{\mathbf{r}}$$

with $\tilde{\varepsilon} \sim \varepsilon^{3/2}|\log\varepsilon|^{1/2}$. Since $|B| \sim \varepsilon^{-1}$ one sees that the scaled vector potential satisfies

$$|\tilde{B}| \sim \Omega_0^{-1}|\log\tilde{\varepsilon}|.$$

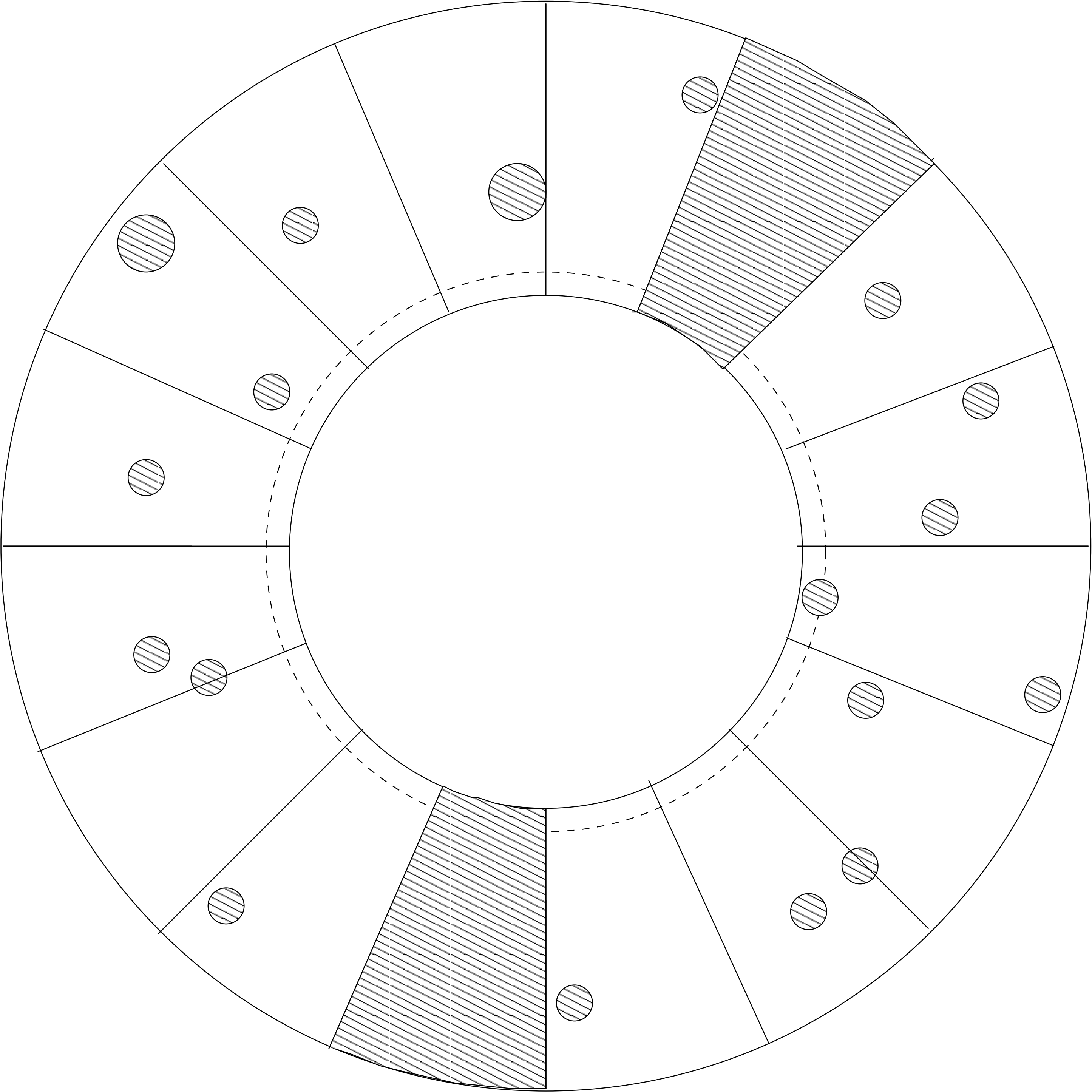
Hence after this scaling the effective vector potential gets *smaller* as $\Omega_0$ increases. Moreover, since $|\log\tilde{\varepsilon}|$ is the order of the cost of a vortex (by previous heuristics), vortices should dissappear for $\Omega_0$ large enough. The problem is that $\tilde{\mathcal{A}}_{\text{bulk}}$ still depends on $\tilde{\varepsilon}$!

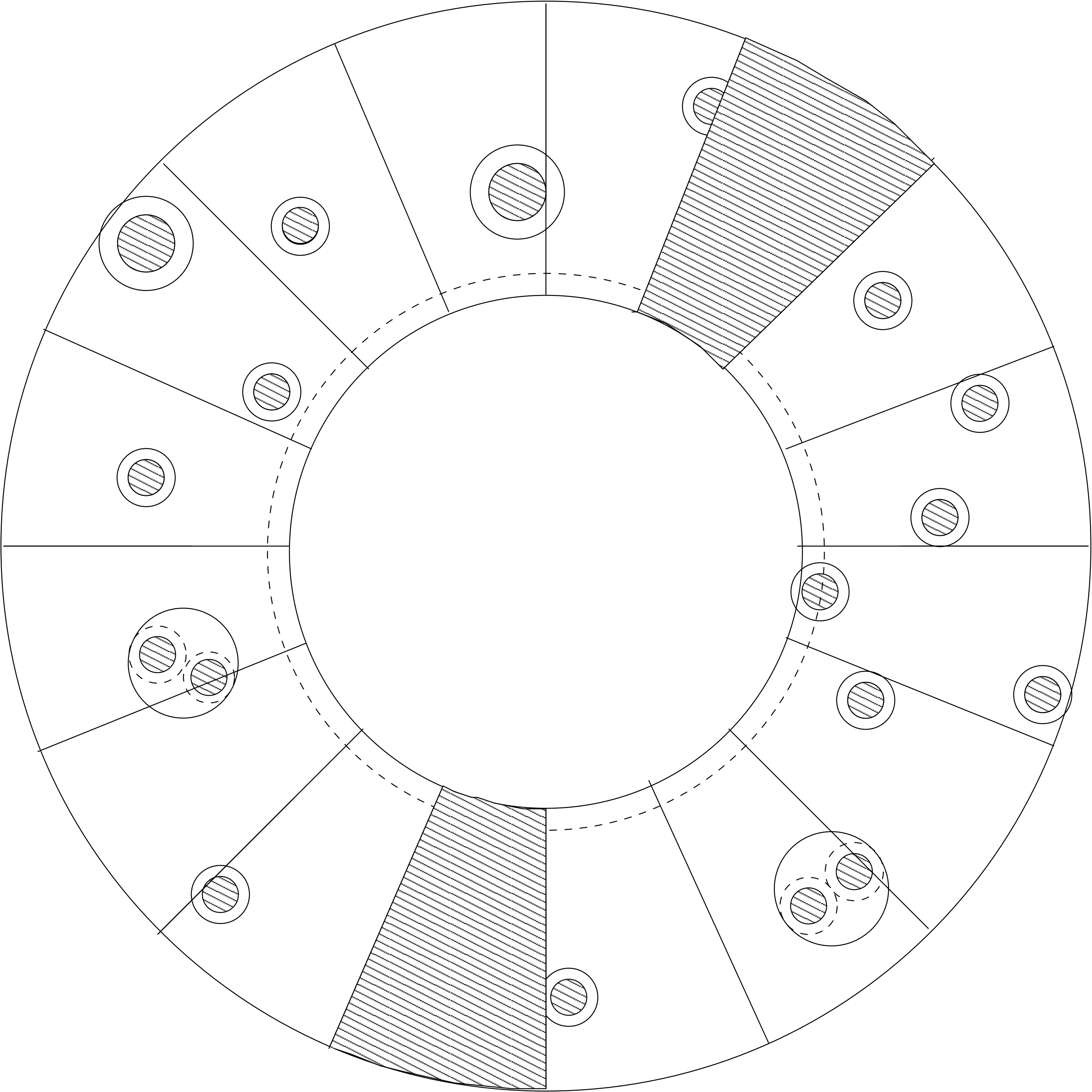
## Division into cells, vortex ball construction, jacobian estimate

The first problem is overcome by a division of $\mathcal{A}_{\text{bulk}}$ into 'almost rectangular' cells and identifying 'good' cells where the vortex ball construction can be applied. Using an iteration process the number of 'good' cells can be increased until they cover the whole of $\mathcal{A}_{\text{bulk}}$. An important ingredient is a 'jacobian estimate', that relates the curl of $\mathrm{i}u^*\nabla u$ to the degrees of the winding numbers around the vortex balls. In this way one obtains (after considerable amount of work!) an improved bound:

**THEOREM** *If* $\Omega = \Omega_0(\varepsilon^2|\log\varepsilon|)^{-1}$ *and* $\Omega_0 > (3\pi)^{-1}$, *then*

$$\mathcal{F}_g[u] \leq \frac{C}{\varepsilon^{1/2}}|\log\varepsilon|^{5/2} \quad and \quad \mathcal{E}_g[u] \geq -\frac{C}{\varepsilon^{1/2}}|\log\varepsilon|^{3/2}$$





## Gradient estimate and completion of proof

The last property needed is an estimate on the gradient of $u$:

**Lemma** *With* $\Omega = \Omega_0(\varepsilon^2|\log\varepsilon|)^{-1}$ *and* $\Omega_0 > (3\pi)^{-1}$*we have on* $\mathcal{A}_{\text{bulk}}$

$$|\nabla u(\mathbf{r})| \leq C\frac{|\log\varepsilon|^{3/2}}{\varepsilon^{3/2}}.$$

For the proof of the Lemma the variational equation for $u$ and the Gagliardo-Nirenberg inequality are used.

The proof that $u$ has no zeros in $\mathcal{A}_{\text{bulk}}$ follows from this and the improved upper bound on $\mathcal{F}_g$.

Indeed, suppose for some $\mathbf{r} \in \mathcal{A}_{\text{bulk}}$

$$|u(\mathbf{r})| \leq (1 - \varepsilon^{1/8} |\log \varepsilon|^3).$$

Then $|u(\mathbf{r})| \leq (1 - \frac{1}{2}\varepsilon^{1/8} |\log \varepsilon|^3)$ in a disc of radius $\sim \varepsilon^{13/8} |\log \varepsilon|^{3/2}$ around $\mathbf{r}$. This implies

$$\mathcal{F}_g[u] \geq \int_{\text{Disc}} \frac{g^4}{\varepsilon^2}(1 - |u|^2)^2 \geq \frac{C|\log \varepsilon|^3}{\varepsilon^{1/2}}$$

which contradicts the bound

$$\mathcal{F}_g[u] \leq \frac{C}{\varepsilon^{1/2}} |\log \varepsilon|^{5/2}$$

that holds for $\Omega_0 > (3\pi)^{-1}$.

# References

M. Correggi, T. Rindler-Daller, JY, *J. Math. Phys.* **48**, 042104 (2007)

M. Correggi, JY, *J. Phys. A, Math. Theor.* **41**, 445002 (2008)

M. Correggi, N. Rougerie, JY, arXiv:1005.0686

Extension to Dirichlet bd. cond. and more general trapping potentials:

M. Correggi, F. Pinsker, N. Rougerie, JY, in preparation