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Lieb-Robinson Bounds and the Existence of Quantum Oscillator Dynamics

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based on joint work with

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General Set-Up

We consider a collection of quantum systems labeled by $x \in \Gamma$. Associated to each system is a Hilbert space $\mathcal{H}_x$ and a densely defined self-adjoint operator H_x. For finite $\Lambda \subset \Gamma$, the Hilbert space of the composite system is

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x \,,$$

and the algebra of observables for the composite system is

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x) = \mathcal{B}(\mathcal{H}_\Lambda) \,.$$

If $X \subset \Lambda$, then by identifying $A \in \mathcal{A}_X$ with $A \otimes \mathbb{1}_{\Lambda \setminus X} \in \mathcal{A}_\Lambda$, we see that $\mathcal{A}_X \subset \mathcal{A}_\Lambda$.
In general, these systems may describe spins, qudits, oscillators, atoms . . . etc.

Two Typical Examples

1. A quantum spin system has a Hilbert space of states given by

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^{n_x}$$

   where $n_x \geq 2$ is an integer, and e.g. the Hamiltonian may be $H_x = S_x^j$, where for $j = 1, 2, 3$, S_x^j is a spin matrix.

2. A quantum oscillator system has composite Hilbert space

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} L^2(\mathbb{R}),$$

   and the on-site Hamiltonians H_x may be position q_x or momentum $p_x = -i\frac{d}{dq_x}$.

Of course, each of these on-site Hamiltonians may be regarded as acting on the composite system by tensoring with an appropriate identity.

Interactions

In general, a bounded interaction for these quantum systems is a map Φ from the set of finite subsets of Γ to the algebra of observables such that for all finite $X \subset \Gamma$, $\Phi(X)^* = \Phi(X)$ and $\Phi(X) \in \mathcal{A}_X$.

For finite $\Lambda \subset \Gamma$, local Hamiltonians are given by

$$H_\Lambda = \sum_{x \in \Lambda} H_x + \sum_{X \subset \Lambda} \Phi(X),$$

and the corresponding Heisenberg dynamics, $\{\tau_t^\Lambda\}_{t \in \mathbb{R}}$, is defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad \text{for } A \in \mathcal{A}_\Lambda.$$

Examples of Models

1. (Quantum Spin) Take $\mathcal{H}_x = \mathbb{C}^2$. A Heisenberg Hamiltonian over $\Lambda \subset \Gamma = \mathbb{Z}^d$ is given by

$$H_\Lambda = \sum_{x\in\Lambda} hS_x^3 + \sum_{1\le|x-y|\le R} J_{xy}\mathbf{S}_x\cdot\mathbf{S}_y\,,$$

   where $\mathbf{S}_x = (S_x^1, S_x^2, S_x^3)$ has components the Pauli spin matrices. Here $h\in\mathbb{R}$, $1\le R<\infty$, and $\{J_{xy}\}$ are parameters.

2. (Quantum Oscillator) An anharmonic Hamiltonian over $\Lambda\subset\Gamma=\mathbb{Z}^d$ is given by

$$H_\Lambda = \sum_{x\in\Lambda} p_x^2+\omega^2 q_x^2+V(q_x)+\sum_{|x-y|=1}\lambda(q_x-q_y)^2+\Phi(q_x-q_y)\,.$$

   Here $\omega^2\ge 0$, $\lambda\ge 0$, V is chosen so that $H_x = p_x^2+\omega^2 q_x^2 + V(q_x)$ is a densely defined, self-adjoint operator on $\mathcal{H}_x = L^2(\mathbb{R})$, and $\Phi\in L^\infty(\mathbb{R})$.

Observables and Support

As we have seen, for any finite sets $X \subset \Lambda \subset \Gamma$, each $A \in \mathcal{A}_X$ can be identified with a unique element $A \otimes \mathbb{1}_{\Lambda \setminus X} \in \mathcal{A}_\Lambda$.

The support of an observable A is the smallest set X such that $A \in \mathcal{A}_X$. We denote this by $\mathrm{supp}(A)$, and $A \in \mathcal{A}_X$ if and only if $\mathrm{supp}(A) \subset X$.

Let $X \subset \Lambda \subset \Gamma$ and consider H_Λ a s.a. operator on $\mathcal{H}_\Lambda$ with, e.g., nearest neighbor interactions. For general $A \in \mathcal{A}_X$, $\mathrm{supp}(\tau_t^\Lambda(A)) = \Lambda$ for all $t \neq 0$.

Question: Does the dynamics of such a Hamiltonian satisfy some weaker form of locality?

Some Comments

Clearly, if $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $X \cap Y = \emptyset$, then $[A, B] = 0$, i.e., observables with disjoint supports commute.

Moreover, if $A \in \mathcal{A}_\Lambda$ and

$$[A, B] = 0 \quad \text{for all } B \in \mathcal{A}_Y \,,$$

then $\text{supp}(A) \subset \Lambda \setminus Y$.

In fact, a more general statement is true.

If A almost commutes with all $B \in \mathcal{A}_Y$, then A is approximately supported in $\Lambda \setminus Y$.

A Lemma

Lemma

Let $\epsilon \geq 0$ *and take* $A \in \mathcal{A}_\Lambda$. *If there is* $Y \subset \Lambda$ *with*

$$\|[A, B]\| \leq \epsilon \|B\|, \quad \textit{for all } B \in \mathcal{A}_Y \tag{1}$$

then there exists $A' \in A_{\Lambda \setminus Y}$ *such that*

$$\|A' \otimes \mathbb{1} - A\| \leq c\epsilon$$

with $c = 1$ *if* $\dim \mathcal{H}_Y < \infty$, *and* $c = 2$ *in general.*

In this case, we can approximate $\text{supp}(\tau_t^\Lambda(A))$ by estimating $[\tau_t^\Lambda(A), B]$ for $B \in \mathcal{A}_Y$.

This is the basic idea of a Lieb-Robinson bound.

Quasi-Locality of the Dynamics

Under some conditions (to be made precise below), one may prove quasi-locality of the dynamics. Rigorously, this notion is expressed by an estimate on commutators of the form

$$\|[\tau_t^\Lambda(A), B]\| \leq C(A, B)e^{-\mu(d(X,Y)-v|t|)}.$$

Here $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$. Crucially, the numbers $C(A, B)$ and v are independent of Λ.

Note that if $d(X, Y) > 0$, then $[\tau_0^\Lambda(A), B] = [A, B] = 0$. The above bound then indicates that the commutator remains exponentially small for times

$$t \sim \frac{d(X, Y)}{v}.$$

The first such estimates were proven by Lieb and Robinson in 1972. A variety of interesting generalizations have recently been derived.

Assumptions 1

Let Γ be a set equipped with a metric. If Γ has infinite cardinality, we need to assume there is a non-decreasing function $F : [0, \infty) \to (0, \infty)$ satisfying:

i) $\|F\| = \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x, y)) < \infty$

ii) there exists $C > 0$ such that for all $x, y \in \Gamma$,

$$\sum_{z\in\Gamma} F(d(x, z))F(d(z, y)) \leq C\,F(d(x, y)).$$

Example: If $\Gamma = \mathbb{Z}^\nu$, take $F(r) = (1 + r)^{-(\nu+\epsilon)}$.

Note that, in general, if F satisfies i) and ii) above, then so does $F_a(r) = e^{-ar}F(r)$ for all $a \geq 0$. Moreover, $\|F_a\| \leq \|F\|$ and also $C_a \leq C$.

Assumption 2

We need an assumption on the interactions being considered.

Recall that we already have assumed that there is a collection of on-site Hamiltonians denoted by $\{H_x\}_{x\in\Gamma}$.

Let Γ be a set equipped with a metric and a function F as above. For any $a \geq 0$, we consider those interactions Φ for which

$$\|\Phi\|_a = \sup_{x,y\in\Gamma} \frac{1}{F_a(d(x,y))} \sum_{\substack{X\subset\Gamma:\\ x,y\in X}} \|\Phi(X)\| < \infty.$$

If $\Gamma = \mathbb{Z}^\nu$ and $F(r) = (1+r)^{\nu+1}$, then clearly finite range interactions satisfy $\|\Phi\|_a < \infty$ for all $a \geq 0$.

A Lieb-Robinson Bound

Let Γ be a set equipped with a metric and a function F as described above.

Theorem (Nachtergaele, Raz, Schlein, S. 09)

Fix a collection of on-site Hamiltonians $\{H_x\}_{x \in \Gamma}$. *Let* $a \geq 0$, *take* Φ *such that* $\|\Phi\|_a < \infty$, *and consider finite subsets* $X, Y \subset \Gamma$. *For any finite* $\Lambda \subset \Gamma$ *with* $X \cup Y \subset \Lambda$, *any* $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, *and* $t \in \mathbb{R}$, *one has that*

$$\|[\tau_t^\Lambda(A), B]\| \leq C e^{-a(d(X,Y) - v_\Phi |t|)},$$

where

$$C = \frac{2\|A\|\|B\|\|F\|}{C_a} \min[|\partial_\Phi X|, |\partial_\Phi Y|] \text{ and } v_\Phi = \frac{2\|\Phi\|_a C_a}{a}.$$

Sketch of the Proof

First, suppose there are no on-sites Hamiltonians.

Consider the function f defined by

$$f(t) := [\tau_t^{\Lambda}(A), B]. \tag{2}$$

Differentiate to see that f satisfies the following differential equation

$$f'(t) = -i\left[f(t), \tau_t^{\Lambda}(H_X)\right] - i\left[\tau_t^{\Lambda}(A), \left[\tau_t^{\Lambda}(H_X), B\right]\right], \tag{3}$$

with the notation

$$H_Y = \sum_{\substack{Z \subset \Lambda: \\ Z \cap Y \neq \emptyset}} \Phi(Z), \tag{4}$$

for any subset $Y \subset V$. The first term in (3) above is norm-preserving, and therefore we have

$$\| [\tau_t^{\Lambda}(A), B] \| \leq \|[A, B]\| + 2\|A\| \int_0^{|t|} \| [\tau_s^{\Lambda}(H_X), B] \| \, ds \tag{5}$$

Define the quantity

$$C_B(X,t) := \sup_{A\in\mathcal{A}_X} \frac{\|[\tau_t^\Lambda(A), B]\|}{\|A\|}, \tag{6}$$

then (5) implies that

$$C_B(X,t) \le C_B(X,0) + 2 \sum_{\substack{Z\subset\Lambda: \\ Z\cap X\neq\emptyset}} \|\Phi(Z)\| \int_0^{|t|} C_B(Z,s)ds. \tag{7}$$

Clearly, one has that

$$C_B(Z,0) \;\le\; 2\,\|B\|\,\delta_Y(Z), \tag{8}$$

where $\delta_Y(Z) = 0$ if $Z \cap Y = \emptyset$ and $\delta_Y(Z) = 1$ otherwise. Using this fact, iterate (7) and find that

$$C_B(X,t) \;\le\; 2\|B\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} a_n, \tag{9}$$

where the coefficients are given by

$$a_n = \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \emptyset}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \emptyset}} \cdots \sum_{\substack{Z_n \subset \Lambda: \\ Z_n \cap Z_{n-1} \neq \emptyset}} \prod_{i=1}^{n} \|\Phi(Z_i)\| \, \delta_Y(Z_n). \tag{10}$$

Using the properties of the function F_a and the norm $\|\Phi\|_a$ one can estimate a_n:

$$a_n \leq C_a \left[\frac{\|\Phi\|_a}{C_a} \right]^n \sum_{x \in X} \sum_{y \in Y} F_a(d(x,y)) \, ,$$

This completes the proof.

If there are on-site Hamiltonians, re-work the above argument for the function $\tilde{f}$ defined by

$$\tilde{f}(t) := [\tau_t^{\Lambda}(\tau_{-t}^{\Lambda;\mathrm{loc}}(A)), B] \, . \tag{11}$$

Applications

In the context of quantum spin systems, there have recently been a variety of applications: Exponential Clustering, a multi-dimensional Lieb-Schultz Mattis Theorem, an area law for one-dimensional gapped systems, quantization of hall conductance, stability of topologically ordered systems under local perturbations . . . to name a few.

Due to time constraints, I will only discuss one application:

the existence of a limiting dynamics

Thermodynamic Limit

Question: For infinite Γ, does there exist a suitable limiting dynamics as $\Lambda \to \Gamma$?

The Set-Up

Suppose Γ has infinite cardinality. Let $\{\Lambda_n\}_{n \geq 1}$ be an increasing sequence of finite subsets with $\Lambda_n \to \Gamma$.

Take $X \subset \Gamma$ finite, $A \in \mathcal{A}_X$, and $t \in \mathbb{R}$.

What can be said about

$$\lim_{n \to \infty} \tau_t^{\Lambda_n}(A) \, ?$$

Result

Let

$$\mathcal{A}_\Gamma = \bigcup \mathcal{A}_\Lambda$$

where the union is over all finite subsets of Γ.

Theorem (Nachtergaele, Schlein, S., Starr, Zagrebnov)

Let Γ, F, and $\{H_x\}$ be as before. Take $a \geq 0$ and Φ with $\|\Phi\|_a < \infty$. For each $t \in \mathbb{R}$ and any $A \in \mathcal{A}_\Gamma$, the norm limit

$$\lim_{n \to \infty} \tau_t^{\Lambda_n}(A)$$

exists. The limit defines a one parameter group of $*$-automorphisms on the completion of $\mathcal{A}_\Gamma$. Moreover, the convergence is uniform for t varying in compact sets.

Comments

1) If all the on-site Hamiltonians $\{H_x\}$ are bounded, then the limiting dynamics is strongly continuous.

2) If this is not the case, then we only have weak continuity of the limiting dynamics. This is true, for example, in any product state whose factors correspond to normalized eigenvectors of H_x.

Sketch of the proof

Assume there are no on-site Hamiltonians.

Let $X \subset \Lambda_m \subset \Lambda_n$ all finite subsets of Γ. Observe that for any strictly local $A \in \mathcal{A}_X$ and $t \in \mathbb{R}$, the norm-estimate

$$\left\| \tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \right\| \leq \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{Z \ni z} \int_0^t \| [\tau_s^{\Lambda_m}(A), \Phi(Z)] \| \, ds \,,$$

readily follows. By our Lieb-Robinson bound, the above commutator is small (in distance between X and Z), independently of Λ_m. This shows that the sequence $\tau_t^{\Lambda_n}(A)$ is Cauchy .

A similar trick applies in the case of non-trivial on-site Hamiltonians.

What about unbounded interactions?

In all of the previous discussion, it was assumed that the interactions are bounded. What can be said about systems where this assumption is lifted?

Only a few models have been analyzed so far.

The Harmonic Model

Take $\Lambda \subset \mathbb{Z}^d$, $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} L^2(\mathbb{R})$, and consider

$$H_\Lambda^h = \sum_{x\in\Lambda} p_x^2 + \omega^2 q_x^2 + \sum_{j=1}^{d} \lambda_j (q_x - q_{x+e_j})^2,$$

introduce annihilation and creation operators

$$a_x = \frac{1}{\sqrt{2}}(q_x + ip_x) \quad \text{and} \quad a_x^* = \frac{1}{\sqrt{2}}(q_x - ip_x)$$

which satisfy the CCR and for each $f : \Lambda \to \mathbb{C}$ let

$$a(f) = \sum_{x\in\Lambda} \overline{f(x)} a_x \quad \text{and} \quad a(f)^* = \sum_{x\in\Lambda} f(x) a_x^*$$

and define a Weyl Operator by

$$W(f) = \exp\left[\frac{i}{2}\left(a(f) + a(f)^*\right)\right].$$

Note that the Weyl relation holds

$$W(f+g) = e^{-i\mathrm{Im}[\langle f,g\rangle]} W(f) W(g).$$

By going to Fourier variables,

$$H_\Lambda^h = \sum_{k \in \Lambda^*} \gamma(k)\,(2 b_k^* b_k + 1)$$

where

$$\gamma(k) = \sqrt{\omega^2 + 4 \sum_{j=1}^{d} \lambda_j \sin^2(k_j/2)}\,.$$

In this case, the dynamics is trivial

$$\tau_t^{h,\Lambda}(b_k) = e^{-2i\gamma(k)t}\, b_k \quad \text{and} \quad \tau_t^{h,\Lambda}(b_k^*) = e^{2i\gamma(k)t}\, b_k^*$$

Moreover, the b's can be obtained from the a's (and vice versa)

$$b(f) = a(Uf) + a(Vf)^* \quad \text{and} \quad a(f) = b(U^*f) - b(V^*f)^*$$

where U and V are real-linear maps (Bogoliubov) which satisfy

$$U^*U - V^*V = \mathbb{1} \quad \text{and} \quad V^*U - U^*V = 0\,.$$

Hence

$$\tau_t^{h,\Lambda}(W(f)) = W(T_t f)$$

where

$$T_t f = (U+V)\mathcal{F}^{-1} M_{e^{2i\gamma t}} \mathcal{F}(U^* - V^*) = f * K_t^{(1)} + \overline{f} * K_t^{(2)}$$

and for every $\mu > 0$,

$$\max\{|K_t^{(1)}(x)|, |K_t^{(2)}(x)|\} \leq C_\mu e^{-\mu(|x| - v_h(\mu)|t|)} .$$

Using the Weyl relation

$$\left[\tau_t^{h,\Lambda}(W(f)), W(g)\right] = \left(1 - e^{2i\operatorname{Im}[\langle T_t f, g\rangle]}\right) W(T_t f) W(g)$$

we have a Lieb-Robinson bound for the harmonic model, i.e.

$$\left\| \left[\tau_t^{h,\Lambda}(W(f)), W(g)\right] \right\| \leq C_\mu \sum_{x,y} |f(x)||g(y)| e^{-\mu(|x-y| - v_h(\mu)|t|)} .$$

Remarks

i) Again, it is important to note here that the constants C_μ and $v_h(\mu)$ are independent of Λ.

ii) Moreover, optimizing the velocity $v_h(\mu)$ over the rate of spatial decay μ, one finds the bound

$$v_h(\mu_0) \leq 4\sqrt{\omega^2 + 4\sum_{j=1}^{\nu} \lambda_j}.$$

Locality for Anharmonic Systems

Let V be real valued and satisfy

$$\kappa_V = \int |k|^2 |\hat{V}(k)| \, dk < \infty .$$

Define

$$H_\Lambda^{ah} = H_\Lambda^h + \sum_{x \in \Lambda} V(q_x) ,$$

Theorem (Nachtergaele-Raz-Schlein-S. 09)

For every $\mu \geq 1$ and $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\|[\tau_t^{ah,\Lambda}(W(f)), W(g)]\| \leq C \sum_{x,y} |f(x)||g(y)| e^{-\mu(|x-y| - v_{ah}(\mu,\epsilon)|t|)}$$

for all functions $f, g \in \ell^2(\Lambda)$.

Here C and v_{ah} are independent of Λ with

$$v_{ah}(\mu, \epsilon) \leq \left(1 + \frac{\epsilon}{\mu}\right) \left[v_h(\mu + \epsilon) + \frac{\tilde{C}\kappa_V}{\mu + \epsilon} \right] .$$

The Thermodynamic Limit?

Some problems:

In the quantum spin context, τ_t^Λ is a strongly continuous, 1-parameter group of $*$-automorphisms; so too is the limit.

The harmonic dynamics $\tau_t^{h,\Lambda}$ is not strongly continuous w.r.t. time since $\|W(f) - W(g)\| = 2$ if $f \neq g$.

Moreover, due to the unbounded terms in the Hamiltonian, the old proof does not directly apply.

Solutions:

L. Amour, P. Levy-Bruhl, and J. Nourrigat '09 introduce a modified norm and prove convergence of the anharm. dynamics as $\Lambda \to \mathbb{Z}^d$.

We start with the infinite volume harmonic dynamics and prove convergence as the perturbation grows from $\Lambda \to \mathbb{Z}^d$.

∞-Volume Harmonic Dyamics

Consider a subspace $\mathcal{D} \subset \ell^2(\mathbb{Z}^d)$ and let $\mathcal{W}(\mathcal{D})$ be the Weyl Algebra generated by $W(f)$ for $f \in \mathcal{D}$.

Examples: $\mathcal{D} = \ell^2(\Lambda)$, $\mathcal{D} = \ell^1(\mathbb{Z}^d)$, $\mathcal{D} = \ell^2(\mathbb{Z}^d)$.

By replacing Riemann sums with integrals, we can formally define the harmonic dynamics on $\mathcal{W}(\mathcal{D})$ by setting

$$\tau_t^{(0)}(W(f)) = W(T_t f) \quad \text{for all } f \in \mathcal{D}\,,$$

where

$$T_t f = (U+V)\mathcal{F}^{-1} M_{2i\gamma t} \mathcal{F}(U^* - V^*) = f * K_t^{(1)} + \overline{f} * K_t^{(2)}\,.$$

One can easily check that for every $\mu > 0$,

$$\max\{|K_t^{(1)}(x)|, |K_t^{(2)}(x)|\} \leq C_\mu e^{-\mu(|x|-v_h(\mu)|t|)}$$

still holds.

Moreover, one verifies that

1. For $\mathcal{D}$ as above, $T_t : \mathcal{D} \to \mathcal{D}$.
2. $T_0 = \mathbb{1}$ and $T_{s+t} = T_s \circ T_t$.
3. $\mathrm{Im}[\langle T_t f, T_t g\rangle] = \mathrm{Im}[\langle f, g\rangle]$.

In this case, the previously defined $\tau_t^{(0)}$ is a 1-parameter group of $*$-automorphisms on $\mathcal{W}(\mathcal{D})$.

Representing the Dynamics

We represent the dynamics in, e.g. the vacuum state for the b-operators, i.e.,

$$\rho(W(f)) = e^{-(1/4)\|(U^*-V^*)f\|^2},$$

which is clearly regular and $\tau_t^{(0)}$ invariant by definition. A calculation shows that

$$t \mapsto \rho\left(W(g_1)W(T_t f)W(g_2)\right) \quad \text{for } g_1, g_2, f \in \mathcal{D}$$

is continuous, and hence $\tau_t^{(0)}$ is weakly continuous in the GNS representation, denoted by $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$, of ρ. By invariance, the dynamics can be extended to $\mathcal{M}_\rho = \overline{\mathcal{W}(\mathcal{D})}$ and thereby $(\mathcal{M}_\rho, \tau_t^{(0)})$ is a W^*-dynamical system.

Finite Volume Perturbations

The perturbation theory of W^*-dynamical systems is well-understood.

In fact, if $(\mathcal{M}, \alpha_t)$ is a W^*-dynamical system and $P = P^* \in \mathcal{M}$, then

$$\alpha_t^P(A) = \alpha_t(A) + \sum_{n\geq 1} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \times [\alpha_{t_n}(P), [\cdots, [\alpha_{t_1}(P), \alpha_t(P)]] \cdots] ,$$

defines a 1-parameter group of $*$-automorphisms on $\mathcal{M}$ which is also weakly continuous.

Now for $(\mathcal{M}_\rho, \tau_t^{(0)})$ as before and

$$P = P_\Lambda = \sum_{x\in\Lambda} V(q_x) ,$$

we have a well-defined, weakly-continuous dynamics $\tau_t^{(\Lambda)}$.

Lieb-Robinson Bounds Let V be real valued and satisfy

$$\kappa_V = \int |k|^2 |\hat{V}(k)|\, dk < \infty\,.$$

Define $\tau_t^{(\Lambda)}$ as a finite volume perturbation of the infinite volume harmonic dynamics with

$$P = P_\Lambda = \sum_{x \in \Lambda} V(q_x)\,.$$

Theorem (Nachtergaele-Schlein-S.-Starr-Zagrebnov 09)

For every $\mu \geq 1$, there exist numbers C and v_{ah} for which

$$\| [\tau_t^{(\Lambda)}(W(f)), W(g)] \| \leq C \sum_{x,y} |f(x)||g(y)| e^{-\mu(|x-y| - v_{ah}(\mu)|t|)}$$

holds for all functions $f, g \in \ell^2(\mathcal{D})$.

Again, the numbers C and v_{ah} are independent of Λ.

This result holds for $\mathcal{D} = \ell^2(\mathbb{Z}^d)$.

Existence of the Dynamics

Let V be real valued and satisfy

$$\kappa_V = \max \left\{ \int |k||\hat{V}(k)|\, dk,\ \int |k|^2 |\hat{V}(k)|\, dk \right\} < \infty$$

Theorem (Nachtergaele-Schlein-S.-Starr-Zagrebnov 09)

Let $\tau_t^{(0)}$ *be the harmonic dynamics defined on* $\mathcal{W}(\ell^1(\mathbb{Z}^d))$*. Let* $\{\Lambda_n\}$ *denote a non-decreasing, exhaustive sequence of finite subsets of* $\mathbb{Z}^d$*. Denote* P_{Λ_n} *as before. Then for each* $f \in \ell^1(\mathbb{Z}^d)$ *and* $t \in \mathbb{R}$ *fixed, the limit*

$$\lim_{n \to \infty} \tau_t^{(\Lambda_n)}(W(f))$$

exists in norm. The limiting dynamics is weakly continuous.

The Proof

The old proof now works:

Take $m \le n$, then by iteratively perturbing

$$\tau_t^{(\Lambda_n)}(W(f)) = \tau_t^{(\Lambda_m)}(W(f)) + i \int_0^t \tau_s^{(\Lambda_n)} \left(\left[P_{\Lambda_n \setminus \Lambda_m}, \tau_{t-s}^{(\Lambda_m)}(W(f)) \right] \right) \, ds \, .$$

The new Lieb-Robinson bounds complete the argument.

Weak continuity follows by an $\epsilon/3$ argument; since it was true for the finite volume perturbations.

Conclusion

We continue to improve our understanding of the dynamics corresponding to quantum many body systems. In particular, investigating its locality properties has lead to a variety of new results.

Improved knowledge of the dynamics can be used to better understand correlations, excitations, and more general results in perturbation theory.