Regularity of the Density of States[0]

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[0] Joint work with Dhriti Ranjan Dolai and Anish Mallick

The Model(s)

We consider the random operators

$$H^\omega = -\Delta + \sum_{n \in \mathbb{Z}^d} \omega_n u_n,$$

on $L^2(\mathbb{R}^d)$ and

$$h^\omega = h_0 + \sum_{n \in \mathbb{Z}^d} \omega_n P_n$$

on $\ell^2(\mathbb{Z}^d)$ where h_0 is the discrete Laplacian for this talk.
Our models include more general free parts replacing $-\Delta, h_0$, see Dolai-Mallick-Krishna [1].

Integrated Density of States

Denote the spectral projection of a self-adjoint operator A by $E_A(\cdot)$. Then we take the integrated density of states of the above operarors as

$$\mathcal{N}^c(E) = \frac{1}{\int u_0(x)dx} \mathbb{E}\Big(Tr\Big(u_0 E_{H^\omega}((-\infty, E])\Big)\Big)$$

and

$$\mathcal{N}^d(E) = \frac{1}{tr(P_0)} \mathbb{E}\bigg(tr\bigg(P_0 E_{h^\omega}((-\infty, E])\bigg)\bigg)$$

for the above two models.

Operators in finite boxes

We will need the compression of H^ω_Λ and h^ω_Λ of H^ω and h^ω respectively to finite boxes Λ in $\mathbb{Z}^d$ or $\mathbb{R}^d$ in which case we define

$$\mathcal{N}^c_\Lambda(E) = \frac{1}{\int u_0(x)dx}\mathbb{E}\Big(Tr\Big(u_0 E_{H^\omega_\Lambda}((-\infty, E])\Big)\Big)$$

and

$$\mathcal{N}^d_\Lambda(E) = \frac{1}{tr(P_0)}\mathbb{E}\Bigg(tr\Bigg(P_0 E_{h^\omega_\Lambda}((-\infty, E])\Bigg)\Bigg)$$

In the case of $\mathbb{R}^d$ we take Dirichlet boundary conditions to define the compressions. In these cases the expectation is clearly over finitely many variables ω_n.

Assumptions on randomness

- We assume that ω_n are i.i.d random variables with distribution $\rho(x)dx$, where ρ is of compact support in $(0, \infty)$.
- We take $u_n(x) = u_0(x - n), \ \ n \in \mathbb{Z}^d$ with u_0 supported in the unit cube centered at 0 and $\sum_{n \in \mathbb{Z}^d} u_n(x) = 1$.

Our method of proof allows for non-stationary randomness but cannot extend as of now to ρ of unbounded support.

Assumptions on the Spectral Region

We consider the part of the spectrum of h^ω where exponential localization is valid, more precisely an interval J such that

$$\sup_{\Re(z)\in J, \Im(z)>0} \mathbb{E}\left[\|P_n(h^\omega - z)^{-1}P_m\|^s\right] \le Ce^{-\xi d(n,m)} \quad (1)$$

hold for some $0 < s < 1$, for any n, m with $d(n, m) > M$, for large enough M. For the operators h^ω_Λ exponential localization is similarly defined with ξ_Λ replacing ξ in the bound. We also assume for Λ large enough

$$\xi \le \xi_\Lambda.$$

A similar definition for the continuous case using the Operator Kernels $u_n(H^\omega - z)^{-1}u_m$ to define the region of exponential localization gives the energy region J.

IDS in finite boxes

The average IDS in finite boxes Λ satisfies

Lemma

$\mathcal{N}_\Lambda^{(d)} \in C^m(\mathbb{R})$ *if* $\rho \in C^m(\mathbb{R})$.

Proof: This comes from a simple observation that if A is a self-adjoint operator and $\{T_n\}$ is a finite partition of unity by positive operators, $\{\omega_n\}$ independent random variables distributed as $\{\rho_n(x)dx\}$, then for the operators $A^\omega = A + \sum_{n=1}^N \omega_n T_n$ we have $\mathbb{E}g(A^\omega - E)$ is just the convolution of two functions on $\mathbb{R}^N$ evaluated at the point $E\vec{1}, \vec{1} = (1, 1, \dots, 1)^t$, i.e. $\mathbb{E}g(A^\omega - E)$ is of the form

$$\int F(\vec{\omega} - E\vec{1})\Phi(\vec{\omega})d\vec{\omega},$$

where $\Phi(\vec{\omega}) = \prod_{n=1}^N \rho_n(\omega_n)$.

Therefore any derivative over E is the directional derivative of the above convolution along the direction $\vec{1}$ in $\mathbb{R}^N$, so we can integrate by parts and transfer the derivative to the function Φ, namely

$$\frac{d^m}{dE^m}\int F(\vec{\omega}-E\vec{1})\Phi(\vec{\omega})d\vec{\omega} = \int F(\vec{\omega}-E\vec{1})(\nabla^m\Phi)(\vec{\omega})d\vec{\omega}.$$

Since,

$$\mathcal{N}_\Lambda^{(d)}(E) = \int Tr(P_0 E_{h_\Lambda^\omega}(-\infty, E))\prod_{n\in\Lambda}\rho(\omega_n)d\omega_n$$
$$= \int Tr(P_0 E_{h_\Lambda^\omega - E}(-\infty, 0))\prod_{n\in\Lambda}\rho(\omega_n)d\omega_n,$$

is precisely of the above form, the Lemma follows from the above observation.

Stieltjes Transforms

It is known that given a measure $g(x)dx$, then $g(x)$ is obtained as

$$\pi g(x) = \lim_{\epsilon \downarrow 0} \Im \Big(\int \frac{1}{y - x - i\epsilon} \, g(y) \; dy \Big), \quad a.e.x$$

therefore to show that $g \in C^m(J)$, it is enough to show that

$$\sup_{x \in J, \; \epsilon > 0} \Big| \frac{d^m}{dx^m} \Big(\int \frac{1}{y - x - i\epsilon} g(y) \; dy \Big) \Big| < \infty.$$

Approximation by finite boxes

We will use the fact that h^ω_Λ converges in the strong resolvent sense to h^ω and therefore, if $z = x + i\epsilon, \epsilon > 0$,

$$(h^\omega - z)^{-1} = \lim_{\Lambda \uparrow \mathbb{Z}^d} (h^\omega_\Lambda - z)^{-1},$$

implying,

$$\mathbb{E}\,Tr(P_0(h^\omega - z)^{-1}) = \mathbb{E}\,Tr(P_0(h^\omega_M - z)^{-1})$$
$$+ \sum_{K=M}^{\infty} \mathbb{E}\,Tr(P_0(h^\omega_{K+1} - z)^{-1}) - \mathbb{E}\,Tr(P_0(h^\omega_K - z)^{-1})$$

where K parametrize side lengths of the the boxes $\Lambda_K = \{n : |n| \leq K\}$ and take $h^\omega_K = h^\omega_{\Lambda_K}$.

Idea of Proof

If we show that

$$\sup_{\Re(z)\in J} \Big|\frac{d^{m-1}}{dz^{m-1}}\mathbb{E}\, Tr(P_0 \left[(h^{\omega}_{K+1} - z)^{-1}) - (h^{\omega}_{K} - z)^{-1}\right] P_0)\Big| < Ce^{-\gamma K}, \ \ \gamma > 0,$$

we are done. Since P_0 is finite rank projection and so trace class, it is enough to get a norm bound

$$\sup_{\Re(z)\in J} \Big|\frac{d^{m-1}}{dz^{m-1}}\mathbb{E} P_0 \left[(h^{\omega}_{K+1} - z)^{-1}) - (h^{\omega}_{K} - z)^{-1}\right] P_0\Big| < Ce^{-\gamma K}, \ \ \gamma > 0.$$

Idea of Proof

Transferring the derivatives to the measure in the Stieltjes transform as we have setting $g_K(x)$ to be the density of the absolutely continous measure $\mathbb{E}(E_{H_K^\omega}(\cdot))$, we have

$$\frac{d^{m-1}}{dz^{m-1}}\mathbb{E}P_0\left[(h_{K+1}^\omega - z)^{-1}) - (h_K^\omega - z)^{-1}\right]P_0 =$$
$$\int P_0\left[(h_{K+1}^\omega - z)^{-1}) - (h_K^\omega - z)^{-1}\right]P_0\nabla_\omega^{m-1}\prod_{n\in\Lambda_{K+1}}\rho(\omega_n)d\omega_n$$
$$\int P_0\left[(h_{K+1}^\omega - z)^{-1}) - (h_K^\omega - z)^{-1}\right]P_0\Psi_K(\omega)\prod_{n\in\Lambda}d\omega_n.$$

where Ψ_K is sums of derivatives of products of ρ and its L^∞ norm is polynomially bounded in K.

What to do?

Taking resolvent differences we have

$$\int P_0 \left[(h^\omega_{K+1} - z)^{-1} \Big((h_0)_{K+1} - (h_0)_K \Big) (h^\omega_K - z)^{-1} \right] P_0 \Psi_K(\omega) \prod_{n \in \Lambda} d\omega_n$$

The above expression appears to want to decay in norm since the resolvent difference has matrix elements of the form $P_0(h^\omega_X - z)^{-1} P_n$, with $|n| \geq K$ and we are in the localized part of the spectrum. But the only known exponential decay estimates in the average come from the fractional moment bounds, so we have to extract fractional moments out of this expression. We do this using the following resolvent equation.

A non-standard formula

Given a self-adjoint operator A and a bounded positive operator F, We have a non-standard resolvent formula that we use, namely

$$F^{\frac{1}{2}}(A + xF - z)^{-1}F^{\frac{1}{2}} = \frac{1}{x}I - \frac{1}{x^2}(\frac{1}{x}I + F^{\frac{1}{2}}(A - z)^{-1}F^{\frac{1}{2}})^{-1},$$

where I is the identity on the range of F. A similar two parameter formula for two self-adjoints $A, B,\ F_1, F_2$ and $2F = F_1 + F_2, 2G = F_1 - F_2$ is

$$F^{\frac{1}{2}}(A + x_1F_1 + x_2F_2 - z)^{-1}F^{\frac{1}{2}} = F^{\frac{1}{2}}(A + xF + yG - z)^{-1}F^{\frac{1}{2}}$$
$$= \frac{1}{x}I - \frac{1}{x^2}(\frac{1}{x}I + F^{\frac{1}{2}}(A + yG - z)^{-1}F^{\frac{1}{2}})^{-1},$$

where we take $x = \frac{1}{2}(x_1 + x_2), y = \frac{1}{2}(x_1 - x_2)$. A can be unbounded here, only F needs to be bounded.

A Remarkable Inequality

With the above notation we have the following:

Lemma

Let A, B are self-adjoint operators, F a positive operator and $\Im(z) > 0$. Suppose ρ has compact support in $(0, \infty)$ and has α holder continuous derivative for positive α. Then

$$\| \int \left(F^{\frac{1}{2}}(A + xF - z)^{-1}F^{\frac{1}{2}} - F^{\frac{1}{2}}(B + xF - z)^{-1}F^{\frac{1}{2}} \right) \rho(x)dx \|$$
$$\leq C \int \|F^{\frac{1}{2}}(A + xF - z)^{-1}F^{\frac{1}{2}} - F^{\frac{1}{2}}(B + xF - z)^{-1}F^{\frac{1}{2}}\|^{s} \rho(x)dx,$$

for some $0 < s < 1$ depending upon α.

A, B can be unbounded.

The Idea of proof of the Remarkable Inequality

We use the resolvent equation written earlier to write the difference

$$\begin{aligned}
&\| \int \left(F^{\frac{1}{2}}(A + xF - z)^{-1} F^{\frac{1}{2}} - F^{\frac{1}{2}}(B + xF - z)^{-1} F^{\frac{1}{2}} \right) \rho(x) dx \| \\
&= \| \int \int_0^\infty dt \left(e^{i\frac{t}{x} + itF^{\frac{1}{2}}(A-z)^{-1}F^{\frac{1}{2}}} - e^{i\frac{t}{x} + itF^{\frac{1}{2}}(B-z)^{-1}F^{\frac{1}{2}}} \right) \frac{1}{x^2} \rho(x) dx \| \\
&\leq \int_0^\infty \| \left(e^{itF^{\frac{1}{2}}(A-z)^{-1}F^{\frac{1}{2}}} - e^{itF^{\frac{1}{2}}(B-z)^{-1}F^{\frac{1}{2}}} \right) \| \tilde{\rho}(t) dt, \\
&\leq 2^{1-s} \| F^{\frac{1}{2}}(A - z)^{-1} F^{\frac{1}{2}} - F^{\frac{1}{2}}(B - z)^{-1} F^{\frac{1}{2}} \|^s \int_0^\infty |t|^s \ \tilde{\rho}(t) dt,
\end{aligned}$$

where $\tilde{\rho}(x) = \rho(\frac{1}{x})$.

Contd..

The last inequality comes from using the fact that $F^{\frac{1}{2}}(B + xF - z)^{-1}F^{\frac{1}{2}}$ is a bounded operator with a positive imaginary part, so genrates a contraction semigroup and this semigroup of operators have their norm uniformly bounded by 1. We use interpolation to get the estimate, for a pair of bounded operators X, Y with $\|e^{itX}\| \leq 1, \ \|e^{itY}\| \leq 1$,

$$\|e^{itX} - e^{itY}\| \leq \int_0^t \|e^{i(t-w)X}(X - Y)e^{iwY}\|dw \leq 2^{1-s}|t|^s\|X - Y\|^s.$$

If we use a dummy variable r and write $xF = (x - r)F + rF$ in the above inequalities and integrate over r finally, we will get the integral over $\rho(x)dx$ in the final expression.

The Exponential Bound

The above analysis combined with the fractional moment bounds of Aizenman-Molchanov [2] gives us

$$\begin{aligned}
&\| \int P_0 \left[(h^\omega_{K+1} - z)^{-1}) - (h^\omega_K - z)^{-1} \right] P_0 \nabla^{m-1}_\omega \prod_{n \in \Lambda_{K+1}} \rho(\omega_n) d\omega_n \| \\
&\leq \| \int P_0 \left[(h^\omega_{K+1} - z)^{-1}) - (h^\omega_K - z)^{-1} \right] P_0 \Psi_K(\omega) \prod_{n \in \Lambda} d\omega_n \| \\
&\leq CP(K) \mathbb{E} \left(\| P_0 \left[(h^\omega_{K+1} - z)^{-1}) - (h^\omega_K - z)^{-1} \right] P_0 \|^s \right) \\
&\leq CP(K) e^{-\gamma K}, \quad \gamma > 0,
\end{aligned}$$

because the difference of resolvents has terms of the form P_n with $|n| \approx K$.

The ideas for the continuous case

For the continuous case on $L^2(\mathbb{R}^d)$ essentially the same ideas work with the required fractional moment bounds on the resolvent kernels $u_x(H^\omega - z)^{-1}u_y$, which are uniform in the real part of z in an interval in the region of localization, coming from the results in Aizenman-Elgart-Naboko-Shenker-Stolz [3].

There are however, some technical issues to be addressed, primarily coming from the fact that the operator

$$u_0(H^\omega - z)^{-1}$$

is not trace class. Otherwise the method outlined earlier goes through.

We premultiply by a finite dimensional projection Q in the range of u_0 and finally obtain bounds uniform in Q.
We use the fact that if x, y are far away sites in $\mathbb{Z}^d$, then the kernel $u_x(H_0 + a)^{-1}u_y$, $u_x((H_0)_\Lambda + a)^{-1}$ are trace class in any dimension for a in the resolvent set of H_0. Therefore we subtract the free resolvents from the difference

$$Qu_0\Big((H^\omega_{K+1} - z)^{-1} - (H^\omega_K - z)^{-1}\Big)u_0$$

which gives us a similar term with a factor $((H_0)_K + a)^{-1}u_0$ in the Shatten $d/2-$ class multiplying it plus a good term. Repeating this we pick up a trace class factor multiplying the resolvent difference.

Some techinical steps involved

Here are some technical results from Aizenman-Elgart-Naboko-Shenker-Stolz [3] that we use, the first two from Chapter 3 of their paper. These are:

- The boundedness of $\int \|D_1(A+x)^{-1}D_1\|^s g(x)dx$ for a compactly supported bounded g, Hilbert-Schmidt D_j s and a dissipative operator A.
- The above statement implying $\mathbb{E}\|u_x(H^\omega - z)^{-1}u_y\|^s < \infty$, the bound uniform in $Re(z)$ in a bounded set. Note that the bound does not depent on which part of the spectrum $Re(z)$ is in.
- Exponential fractional moment bounds for i.i.d single site random potentials.

Using these we can show that replacing the potential at finitely many points does not significantly change the exponential decay bounds on fractional moments.

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