Introductory homological algebra III - Spectral Sequences

Defn. A (homological) spectral sequence is the following data:

① objects: $\{E^r_{p,q}\}_{p,q\in\mathbb{Z}}^{r\geq a}$

② morphisms: $\partial^r_{p,q}: E^r_{p,q} \to E^r_{p-r,q+r-1}$, $(\partial^r_{p,q})^2=0$

③ isomorphisms: $E^{r+1}_{p,q} \cong H_{p,q}(E^r_{p,q})$

Dually, there are cohomological spectral sequences; $E_r^{p,q} \to E_r^{p+r,q-r+1}$

• can always reindex: given $E^r_{p,q}$, define $'E_r^{p,q} = E^r_{-p,-q}$.

Defn. A spectral sequence is bounded if ~~only finitely many~~ ~~terms~~ $\forall n$, $\exists$ only finitely many (p,q) st $E^a_{p,q}\neq 0$.

Allows definition of $E^\infty_{p,q}$: eventually, for $r>>0$, $\partial^r=0$.

Defn. A bounded spectral sequence converges to a graded object H_n if, $\forall n$, $\exists$ finite filtration

$$0 = F_p H_n \subseteq F_{p+1} H_n \subseteq \dots \subseteq F_{t-1} H_n \subseteq F_t H_n = H_n$$

such that $E^\infty_{p,q} \cong F_p H_{p+q} / F_{p-1} H_{p+q}$.

• cohomological variant: filtration is decreasing,

$$E^{p,q}_\infty \cong F_p H_{p+q} / F_{p+1} H_{p+q}$$

Recall that a filtration on a chain complex is bounded if $\forall n$, $\exists$ $s(n), t(n)$ st. $F_{s(n)} C_n = \{0\}$, $F_{t(n)} C_n = C_n$.

Thm: let C be a bounded filtration. Then, $\exists$ spectral sequence, with $E^0_{p,q} \cong F_p C_{p+q} / F_{p-1} C_{p+q}$,

$$E^1_{p,q} \cong H_{p+q}(F_p C / F_{p-1} C)$$

$$\Rightarrow H_n(C).$$

$\square$

②

Double complexes

A double complex is a family of ~~maps~~ modules $\{D_{p,q}\}_{p,q\in\mathbb{Z}}$ / and maps $\{d^h_{p,q}: D_{p,q}\to D_{p-1,q}\}_{p,q\in\mathbb{Z}}$

$$\{d^v_{p,q}: D_{p,q}\to D_{p,q-1}\}_{p,q\in\mathbb{Z}}$$

Such that $(\partial^h)^2=(\partial^v)^2=0$,

$$\partial^h\partial^v = -\partial^v\partial^h$$

$$\begin{tikzcd} D_{p-1,q} \arrow[d] & D_{p,q} \arrow[l] \arrow[d] \arrow[dl, phantom, "\text{anti-}\oplus"] \\ D_{p-1,q-1} & D_{p,q-1} \arrow[l] \end{tikzcd}$$

why? There are two total complexes we can associate to a double complex: $\mathrm{Tot}^{\oplus}(D)$, $\mathrm{tot}^{\Pi}(D)$.

$$\mathrm{Tot}^{\oplus}(D)_n = \bigoplus_{p+q=n} D_{p,q}, \quad \mathrm{Tot}^{\Pi}(D)_n = \prod_{p+q=n} D_{p,q}.$$

$$\partial = \partial^v+\partial^h \quad ((\partial^v+\partial^h)^2 = (\partial^v)^2+(\partial^h)^2+\underbrace{\partial^v\partial^h+\partial^h\partial^v}_{=0}$$

__Note!__ $D\notin \mathrm{Ch}(\mathrm{Ch})$ ← objects here are $\mathbb{Z}\times\mathbb{Z}$ graded, but

Easy to go between the two:
Given D, double complex, define $C\in\mathrm{Ch}(\mathrm{Ch})$ by

$$C_{p,q}=D_{p,q}, \quad {}^{C}\partial^h_{p,q}=\partial^h_{p,q}, \quad {}^{C}\partial^v_{p,q}=(-1)^p\partial^v_{p,q}.$$

Why in opposite direction.

- With a few exceptions, the above formulas give a good way of going between D and $\mathrm{Ch}(\mathrm{Ch})$.

__Examples:__

① if X, Y complexes, consider $D(X,Y)_{p,q} = X_p\otimes Y_q$

$$\partial^h(m\otimes y) = \partial(m)\otimes y, \quad \partial^v(m\otimes y) = (-1)^p m\otimes\partial(y).$$

call $\mathrm{Tot}^{\oplus}(D(X,Y)) =: X \otimes Y$, tensor product of X and Y.

(2) "innerhom" given two complexes X, Y, we can form a third complex $\mathrm{Hom}(X,Y)$.

$$H^n(\mathrm{Hom}(X,Y)) = \mathrm{Hom}_{K(R)}(X, Y[n]).$$

$\mathrm{Hom}(X,Y)$ is the total complex of a complex.

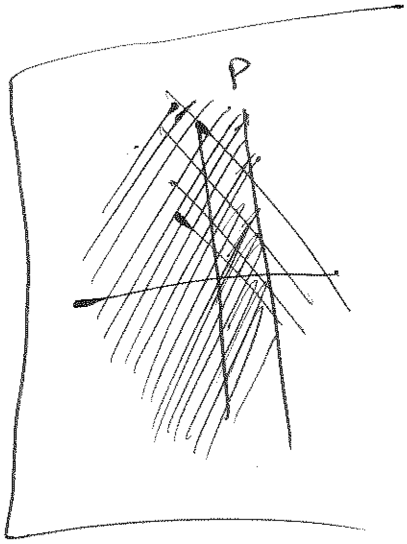
<u>Convention:</u> All double complexes will be bounded: $\forall n$, $\exists$ only finitely many (p,q) st $p+q=n$, $D_{p,q} \neq 0$.

<u>Note:</u> $\Rightarrow$ $\mathrm{Tot}^{\oplus}(D) = \mathrm{Tot}^{\pi}(D)$. Denote by $\mathrm{Tot}(D)$.

2 Filtrations on $\mathrm{Tot}(D)$.

(1) $^{I}F_p(\mathrm{Tot}(D))_n = \bigoplus_{i \leq p} D_{i,n-i}$. Note that $F_p \subseteq F_{p+1}$.

$$^{I}E^0_{p,q} = \bigoplus_{i \leq p} D_{i,p+q-i} \Big/ \bigoplus_{i \leq p-1} D_{i,p+q-i} = D_{p,q}$$



$$^{I}E^0_{p,q} \to {}^{I}E^0_{p,q-1} \; : \; D_{p,q} \to D_{p,q-1}.$$

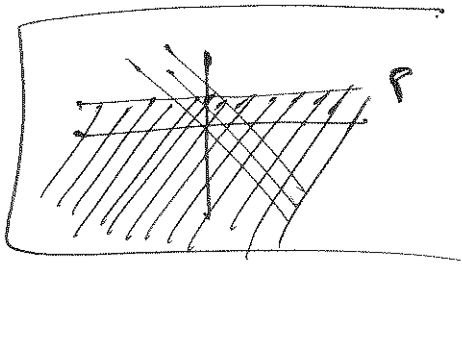
$$\therefore \; {}^{I}E^1_{p,q} := H_q(D_{p,*}).$$

$\exists$ natural map $H_q(D_{p,*}) \to H_q(D_{p-1,*})$.

$$^{I}E^1_{p,q} \to {}^{I}E^1_{p-1,q}.$$

$$^{I}E^2_{p,q} = H_p(H_q(D_{p,*})) \longleftarrow \text{"filter by columns"}.$$

(2) $^{II}F_p(\mathrm{Tot}(D))_n = \bigoplus_{i \leq p} D_{n-i,i}$.



$$^{II}E^0_{p,q} = D_{q,p}$$

$$^{II}E^1_{p,q} = H_q(D_{*,p})$$

$$^{II}E^2_{p,q} = H_p(H_q(D_{*,p})).$$

Cohomological variant?

, maps reverse degree.

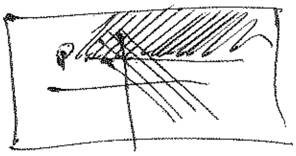
· ${}^{I}F^{p}(\mathrm{Tot}(D))^{n} = \bigoplus_{i\geq p} D_{i,n-i}$, ${}^{II}F^{p}(\mathrm{Tot}(C))^{n} = \bigoplus_{i\geq p} D_{n-i,i}$

${}^{I}E_0^{p,q} = D_{p,q}$

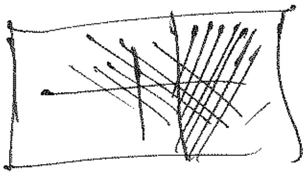
${}^{I}E_1^{p,q} = H_q(D_{p,*})$

${}^{I}E_2^{p,q} = H_p(H^q(D_{p,*}))$ ${}^{II}E_2^{p,q} = H^p(H^q(D_{*,p}))$.

${}^{II}F$ —



${}^{I}F_p$ —



Examples:

(1) "Balancing" Tor.

Let A, B be right, left module. Let $P \to A$, $Q \to B$ be projective resolutions. Consider the ~~following~~ double complex

$P \otimes Q$.

$$\begin{tikzcd}
\downarrow & \downarrow & \downarrow \\
P_0\otimes Q_1 \arrow[d] & P_1\otimes Q_1 \arrow[l] \arrow[d] & P_2\otimes Q_1 \arrow[l] \arrow[d] \\
P_0\otimes Q_0 & P_1\otimes Q_0 \arrow[l] & P_2\otimes Q_0 \arrow[l]
\end{tikzcd}$$

(I) ~~${}^{I}F_p(\mathrm{Tot}\,P\otimes Q)$~~ $P_p \otimes Q_q = {}^{I}E^0_{p,q}$.

${}^{I}E^1_{p,q} = H(P_p\otimes Q_q \to P_p \otimes Q_{q-1} \cdots)$

$\cong P_p \otimes H_q(Q)$, since P_p projective.

since H_q Q is resolution of B.

${}^{I}E^1_{p,q} = 0$ if $q \neq 0$, ${}^{I}E^1_{p,0} = P_p \otimes B$.

$\therefore$ ${}^{I}E^2_{p,q} \cong H_p(\to P_p\otimes B \to P_{p-1}\otimes B \to)$.

(II) ${}^{II}E^0_{p,q} = P_q \otimes Q_p$

$\therefore$ ${}^{II}E^1_{p,q} = H_q(P)\otimes Q_p$

${}^{II}E^1_{p,0} = A \otimes Q_p$, $\Rightarrow$ ${}^{II}E^2_{p,0} = H_p(A\otimes Q_p \to A \otimes Q_{p-1}$

(5)

$^{I}E^2_{p,q} = {}^{I}E^\infty_{p,q} \Rightarrow H_n(\mathrm{Tot}(P \otimes Q)) \cong H_n(P_n \otimes B \to P_{n-1} \otimes B)$

$^{II}E^2_{p,q} = {}^{II}E^\infty_{p,q} \Rightarrow H_n(\mathrm{Tot}(P \otimes Q)) \cong H_n(A \otimes Q_n \to A \otimes Q_{n-1})$

$\therefore$ we can compute $\mathrm{Tor}^R_n(A,B)$ with either resolution.

(2) Recall isomorphism, given modules M_R, ${}_RN_S$, L_S

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(N,L)) \cong \mathrm{Hom}_S(M \otimes_R N, L).$$

Let $P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M$ be a projective resolution of M, $0 \to L \to I^0 \to I^1 \to \cdots$ injective resolution of L.

Then, consider double complex

$$D^{p,q} = \mathrm{Hom}_R(P_p, \mathrm{Hom}_S(N, I^q)) = \mathrm{Hom}_S(P_p \otimes_R N, I^q).$$

(a) $^{I}E^0_{p,q} = \mathrm{Hom}_R(P_p, \mathrm{Hom}_S(N, I^q)) = \mathrm{Hom}_S(P_p \otimes N, I^q)$.

Use first side of isomorphism

$\Rightarrow {}^{I}E^1_{p,q} \cong \mathrm{Hom}_R(P_p, \cancel{\mathrm{Hom}_S(N} \;\mathrm{Ext}^q_S(N,L))$

$\Rightarrow {}^{I}E^2_{p,q} \cong \mathrm{Ext}^p_R(M, \mathrm{Ext}^q_S(N,L))$.

(b) $^{II}E^0_{p,q} = \mathrm{Hom}(P_q \otimes_R N, I^p)$

$^{II}E^1_{p,q} = \mathrm{Hom}_S(\mathrm{Tor}^R_q(M,N), I^p)$

$^{II}E^2_{p,q} = \mathrm{Ext}^p_S(\mathrm{Tor}^R_q(M,N), L)$.

$\therefore$ Thm: let M_R, ${}_RN_S$, L_S. Then, $\exists$ two s.s,

$^{I}E^2_{p,q} = \mathrm{Ext}^p_R(M, \mathrm{Ext}^q_S(N,L))$

$^{II}E^2_{p,q} = \mathrm{Ext}^p_S(\mathrm{Tor}^R_q(M,N), L)$.

converging to the same graded object. (6)

Cor: If $f: R \to S$ is a ring homomorphism, $\exists$ s.s

$$E_2^{p,q}: \mathrm{Ext}_S^p(\mathrm{Tor}_q^R(M,S), L)$$

$$\Rightarrow \mathrm{Ext}_R^{p+q}(M,L).$$

Illy, if $f: S \to R$, $\exists$ ss

$$E_2^{p,q}: \mathrm{Ext}_R^p(M, \mathrm{Ext}_S^q(R,L))$$

$$\Rightarrow \mathrm{Ext}_S^{p+q}(M,L).$$

(3) Duality: Ischebeck spectral sequence.

Let M_R, ${}_SN_R$, ${}_SL$.

There is a natural map $M \otimes_R \mathrm{Hom}_S(N,L) \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(M,N),L)$

$$: (m \otimes f)(\psi) = f(\psi(m))$$

iso if M is fg projective.

Lets resolve M, but we are only working with. modules:

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$$

Let $L \to I^0 \to I^1 \to \ldots$ be a resolution of L by injectives.

Consider the double complex:

$$D^{p,q} = P_p \otimes_R \mathrm{Hom}(N, I^q)$$

Problem: not necessarily bounded. Assume that either

a.) M has finite projective dimension, OR

b.) L has finite injective dimension. Then.

$${}^{I}E_1^{p,q} = P_p \otimes \mathrm{Ext}_S^q(N,L)$$

$${}^{I}E_2^{p,q} = \mathrm{Tor}_p^R(M, \mathrm{Ext}_S^q(N,L))$$

To get the second sequence, we need to further assume that M has a resolution by finitely generated projectives. ⑦

$${}^{II}E_0^{p,q} = \mathrm{Hom}_S(\mathrm{Hom}_R(P_q, N), I^p)$$

$${}^{II}E_1^{p,q} = \mathrm{Hom}_S(\mathrm{Ext}_R^{-q}(M,N), I^p)$$

$${}^{II}E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_R^{-q}(M,N), L).$$

Thm: Let M_R, ${}_SN_R$, ${}_SL$. Suppose M has a resolution by f.g. projectives, and either

① M has finite projective dimension

② L has finite injective dimension.

Then, $\exists$ two spectral sequences converging to the same graded object:

$${}^{I}E_2^{p,q} = \mathrm{Tor}_R^{-p}(M, \mathrm{Ext}_S^q(N,L))$$

$${}^{II}E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_R^{-q}(M,N), L).$$

Cor. If $R=S=N$, then ${}^{I}E$ collapses at ${}^{I}E_2$, and we have iso

$$H^n(\mathrm{Tot}(D)) \cong \mathrm{Tor}_{-n}^R(M,L).$$

$\therefore \exists$ SS $\mathrm{Ext}_R^{p}(\mathrm{Ext}_R^{-q}(M,R), L) \Rightarrow \mathrm{Tor}_{-p-q}(M,L)$

ie $\mathrm{Ext}_R^p(\mathrm{Ext}_R^q(M,R), L) \Rightarrow \mathrm{Tor}_{q-p}^R(M,L)$.

If M_R, ${}_SN_R$, L_S, then there is an iso $L \otimes_S \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M, L\otimes N)$

$(l \otimes f)(m) = l \otimes f(m)$.

gives another SS.

④ All our examples so far involve bifunctors. What if we don't have 2 variables?

§ Cartan-Eilenberg resolutions.

Defn: Let C be a complex. A Cartan-Eilenberg resolution is the following:

- A upper half plane complex compound of injective modules $\{I^{p,q}\}_{p\in\mathbb{Z},\, q\in\mathbb{N}}$.
- A chain map $C \to I^{*,0}$.

Such that:

① $C^n = 0 \Rightarrow \{I^{n,*}\} = 0$.

②
- $0 \to C^n \to I^{n,0} \to I^{n,1} \to \cdots$ injective resolution
- $0 \to \ker \partial^n \to \ker \partial^{n,0} \to \ker \partial^{n,1} \to \cdots$ " "
- $0 \to \operatorname{im} \partial^{n-1} \to \operatorname{im} \partial^{n-1,0} \to \cdots$ " "
- $0 \to H^n(C) \to H^n(I^{*,0}) \to H^n(I^{*,1}) \to \cdots$ "

Thm: Every complex has a CE resolution.

Thm: Let $F: A \to B$, $G: B \to C$. Suppose F takes injectives to G acyclics. Then, $\exists$ spectral sequence

$$E_2^{p,q} = (R^pG)(R^qF)(X) \Rightarrow R^{p+q}(GF)(X)$$

$\forall\ X \in \operatorname{Ob}(A)$.

Pf: ? if time.