Cohomological Finite Generation

1. Lecture 1

In 1868, Paul Gordan published a paper which is the beginning of this work. Let $G = \mathrm{Sl}_2(\mathbb{C})$, then G is an algebraic group. G acts on the vector space $V = \mathbb{C}^2$ as $g.v = gv$.

General principle: if a group G acts on 2 sets X and Y and if $f : X \to Y$ is a function then G acts on f as: $g.f(x) = g.(f(x))$. Gordan considered this situation in the case of the algebra of polynomial functions $\mathbb{C}[V] = \mathbb{C}[x, y]$, this action respects the graded structure. Let W_d denote the homogeneous degree d part of $\mathbb{C}[x, y]$ (say $W_2 =< ax^2 + bxy + dy^2 >$). Then $\mathbb{C}[W_2] = \mathbb{C}[a, b, c]$. We are interested in the invariants $\mathbb{C}[W_2]^G$. An example of such an invariant is $b^2 - 4ac$. Then Gordan showed (constructively) that $\mathbb{C}[W_d]^G$ is finitely generated as a $\mathbb{C}$-algebra.

The next obvious case is Sl_3. This case was thought to be out of reach since the previous case was already quite difficult. Hilbert showed in 1890 that for $G = \mathrm{Sl}_n(\mathbb{C})$ acting on a finite-dimensional vector space V, the ring of invariants $\mathbb{C}[V]^G$ is finitely generated over $\mathbb{C}$. He used the "Ω process" of Cayley, which allows one to bring the degrees down by induction.

In 1897, Hurwitz considered the orthogonal group of matrices $K = O_n(\mathbb{R})$. This is a compact manifold which has a measure on it which is invariant under translation. Let V be a finite-dimensional real v.s. and consider the averaging map:

$$v \longmapsto \frac{\int_K kvdk}{\int_K dk}, k \in K.$$

This is an invariant, an element of V^K. There exists a map called the Reynold's operator $\phi : V \to V^K$ which respects the K-action. Then $V = V \oplus \ker\phi$. Hurwitz showed that this can take the place of the Ω process of Hilbert and showed finite generation for this case.

In 1897 Mauver showed the following: let $G = \mathbb{G}_a = (\mathbb{C}, +)$; let $\mathbb{G}_a$ act on V algebraically i.e. for $t \in \mathbb{G}_a$, $v \in V$, $gv = v_0 + v_1 t + \dots v_m t^m$. $\mathbb{C}[V]^{\mathbb{G}_a}$ is finitely generated as a $\mathbb{C}$-algebra. Mauver thought he could say $(\mathbb{C}[V]^{\mathbb{G}_a})^m$ is also finitely generated, but this is not true! This prompted Hilbert to frame his 14th problem: if G is a linear algebraic group over C acting on $\mathbb{C}[x_1, \dots, x_n] \cap k(f_1, \dots, f_d)$, (where $f_1, \dots, f_d$ are rational functions over a subfield k of $\mathbb{C}$) then A^G is finitely generated over $\mathbb{C}$. In 1958, Nagata gave a counterexample i.e. an example where the invariant ring is not f.g. The example has $G = \mathbb{G}_a^{13}$, $\dim V = 32$ and this is not f.g. as an algebra. (These numbers have actually come down over the years.)

Enter Emmy Noether. She showed that if k is a noetherian ring (with identity), A is a f.g. commutative k-algebra, G is a finite group acting on A then A^G is f.g. as a k-algebra and A is integral over A^G.

Weyl's work shows that for Hilbert's 14th problem to be solvable, the group should be reductive. Mumford phrased a sufficient condition for finite generation and Nagata proved it. Now we come to the author's conjecture. This work is over fields of finite characteristic. Antoine Touzé proved this conjecture.

Let $t \in \mathbb{G}_a$, consider its action on $\mathbb{C}[x, y, z]/ < xz >$ by $x \mapsto x, y \mapsto y + tx, z \mapsto z$. Claim: $A^{\mathbb{G}_a}$ is not f.g. For example, $A^{\mathbb{G}_a} = \mathbb{C}[x, z, yz, y^z, y^3 z, \dots, yz^2, yz^3, \dots]$. WLOG assume the generators are homogeneous. We use the total xy degree and the z degree for grading. In degree 0 we have $\mathbb{C}[z]$, in degree 1 we have $x, yz, yz^2, \dots$. The degree 1 generators are not f.g. over degree 0 generators, hence $A^{\mathbb{G}_a}$ cannot be f.g.

	xy deg.		
z deg.	1	x	
	z	yz	y^z $\dots$
	z^2	yz^2	
	$\vdots$	yz^3	

(Popov's work...)

2. Lecture 2

Finite generation for A^G: the correct class to consider is the class of reductive groups. If G is reductive there is such a theorem, if not then there isn't one. I.e. if G is not reductive then there is an algebra A which is f.g. but A^G is not f.g.

Let $R_u(G)$ be the largest connected unipotent normal subgroup of G. U is called unipotent if it has a faithful representation by matrices of the following shape (upper triangular matrices with 1's on the diagonal):

$$\begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ 0 & \dots & 1 & * \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

G is called reductive if the unipotent radical is trivial.

In Mumford's book on geometric invariant theory (1956) he formulates a conjecture which consists of two parts (in characteristic $p > 0$); the first part was solved by Nagata (1964) and the second part was proved by Haboush (1975). Mumford introduced a notion which came to be known as geometric reductivity; his conjecture was that reductivity $\implies$ geometric reductivity (Haboush) $\implies$ finite generation (Nagata). Nagata needed the idea of "power reductivity" (Franjou, VDK). G is said to be *power reductive* if whenever G acts algebraically on a commutative K-algebra A with an ideal J that is invariant and a map $A^G \to (A/J)^G$ then there exists $n \geq 1$ such that $\overline{f}^n \in (A/J)^G$ lifts to A^G.

In Emmy Noether's work she considers a finite group G, k commutative noetherian, A commutative f.g. as a k-algebra and shows that A^G is also f.g. as a k-algebra.

In 1961, Evens showed that $H^*(G, A)$ is f.g. as a k-algebra. Recall that $H^*(G, A)$ is a graded commutative algebra with $xy = (-1)^{\deg(x)\deg(y)}$ if x and y are homogeneous. (Venkov had this result for $A = k$). Coming back to k with characteristic p, Quillen determined the Krull dimension of $H^*(G, k)$. (Note that if p is odd and $\deg(x)$ is odd then $x \cup x = 0$.) Here Krull dimension is the rank of the largest elementary abelian p-subgroup.

Chouinard (1976): If M is a f.d. G-module then one can detect if M is a projective module by inspecting the M/E E-elementary abelian p-subgroups.

Dade's lemma (1978): Let J be the kernel of the map $H^*(G, k) = \mathrm{Ext}^*_G(k, k) \to \mathrm{Ext}^*_G(M, M)$. $\dim(H^*(G, k)/J$ is the dimension of the support of M. Then $\dim(H^*(G, k)/J = 0$ if and only if M has finite projective dimension. (Note that $\mathrm{Spec}(H^*(G, k)/J)$ is the support of M.)

A generalization of Evens' theorem was desired for G being a finite group scheme. This was done by Friedlander and Suslin in 1997. They introduced what is called as *strict polynomial functors.*

In 1980, Alperin and Evens showed that the dimension of the support of M reflects the complexity of M. Carlson showed that one can define a 'rank variety' without cohomology and conjectured that the 'two pictures agree'. Avrunin and Scott proved this conjecture in 1982. Such results triggered the interest in the finite generation problem for finite group schemes.

Example 2.1. char $p > 0$

$$R \longmapsto \mu_{p^n}(R) := \{r \in R \mid r^{p^n} = 1\}.$$

This defines a group scheme. When R *is a field* F, $\mu_{p^n}(F) = \{1\}$. *In this case* $H^i(\mu_{p^n}, k) = 0$ *for* $i > 0$.

Let k be an algebraically closed field. Let $G = \mathrm{Gl}_n(k)$. Then $k[G] = k[g_{11}, \dots, g_{nn}, \frac{1}{\det(g_{ij})}]$. $G \subset M_n(k)$, the algebra of $n \times n$ matrices. A representation of G may actually come from a representation fo the matrix algebra i.e. a left $M_n(k)$-module. Call a finite-dimensional representation M of G *polynomial* if it extends to $M_n(k)$. Equivalently, if $v \in M$ and $gv = f_1(g)v_1 + \cdots + f_r(g)v_r$ with $< v_i >$ a basis of M, then the f_i should be in the polynomial ring $k[g_{11}, \dots, g_{nn}]$. If all f_i are homogeneous of degree d, we call the representation a polynomial representation of degree d.

Example 2.2. $V = k^n$ *is a polynomial representation of degree* 1. $\Gamma_3 V = (V^{\otimes 3})^{\Sigma_3}$ *(divided powers) is an example of a polynomial representation of degree* 3. *Another example is* $S_3 V = (V^{\otimes 3})_{\Sigma_3}$ *(symmetric powers).*

Recall Yoneda's lemma: let $\mathcal{F} : \mathcal{C} \to Sets$ be any functor and h^C be the functor which maps $X \longmapsto \mathrm{Hom}_{\mathcal{C}}(C, X)$. Then

$$\mathrm{NatTr}(h^C, F) = F(C).$$

In particular $\mathrm{NatTr}(h^C, h^D) = h^D(C) = \mathrm{Hom}(D, C)$.

I.Schur: M_n-modules and polynomial Gl_n-modules are 'same'. Those of degree d form a category equivalent to the category of $S(n, d)$-modules ($S(n, d)$ is the Schur algebra).

Example 2.3. $S_k^*(M)$ *is a* k*-algebra so that* h^C *is* $R \mapsto \mathrm{Hom}_k(M, R)$.