

## Tilting Modules and their Applications

Olivier Mathieu

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*0. Introduction:* The aim of the paper is the theory of tilting modules for a reductive algebraic group  $G$  over an algebraically closed field  $K$  of characteristic  $p$ . In order to be convenient to the reader, the paper is essentially self-contained, what explains its length. Also we tried to make it accessible to a "characteristic zero" reader. In the introduction, we will describe the content of the paper.

The first three sections contains the basic material on algebraic groups. We recall some elementary results about restricted Lie algebras (due to Jacobson [J]), the definition of Chevalley groups and the

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main theorems of their representation theory: classification of their simple modules [St], simplicity of the Steinberg's module [St], Kempf's vanishing theorem [Ke]. We included most proofs, but, because of its technical difficulty, it has not been possible to give a complete proof of the linkage principles [J1][A4]. A full account of the subject can be found in Jantzen's book [J3].

In the setting of finite dimensional algebras, the notion of tilting modules has been first introduced by Brenner and Butler [BB1][BB2], and it has been used afterwards by various authors with different meanings. In the context of algebraic groups, this notion has been found by Donkin [D3], following Ringel's work [Ri] on quasi-hereditary algebras. In this setting, the definition of a tilting module is based on the good filtrations. Therefore we give an elementary treatment of these filtrations in Section 4. Then the reader can consult the appendix for the usual cohomological approach of good filtrations [FP].

A crucial fact about tilting modules is their stability under tensor product: this follows from a similar result for good filtrations. The proof of this result uses a refinement of the notion of Frobenius splittings namely the canonical Frobenius splittings [M2]. Therefore, we will define the Frobenius splittings in Section 5. This notion, invented by Metha, Ramanan and Ramanathan [MR][RR], has been first used by these authors to prove some vanishing theorems, from which they deduce the Demazure character formula. The main example of a Frobenius split variety is the flag space  $G/B$ . Indeed, this variety has many Frobenius splittings [MR], but none of them is  $G$ -equivariant. Hence we defined a weaker notion of equivariance, namely the notion of canonical Frobenius splittings [M2]. Our treatment of Frobenius splittings, based on the formula proved in [M1], seems more intuitive than the original presentation [MR]. In Section 6, we give a full proof of the stability by tensor product of good filtrations, in a clearer way than our original paper [M2].

Section 7 is devoted to the definition of tilting modules, following the work of Donkin [D3]. As the tensor product of two tilting modules is a tilting module, we can form a tensor category  $\mathcal{P}$  for which the morphisms are defined by the functor  $T^\Gamma$ , which is investigated in Section 8. In Section 9, we explain, following [GM1], the similarities with a tensor category  $\mathcal{O}_{int}^l$  considered by Moore and Seiberg [MS] in the context of Conformal Field Theory. In  $\mathcal{O}_{int}^l$ , the tensor product multiplicities are given by Verlinde's formula [MS][V]. These similarities have suggested to the authors of [GM1] a formula for certain tensor product multiplicities of tilting modules. For this reason, we call it the *modular Verlinde's formula*. In Section 10, we give a proof of the formula which is simpler

than the original proof [GM2].

The applications of the tilting module theory involves the *stable representations theory*, i.e. the modular representation theory of  $S_n$  and the theory of polynomial representations of  $GL(n)$  with  $n$  large. This subject has only a little intersection with Lusztig conjecture [Lu1] (whose validity domain is  $n \leq p$ ) or the conjectures of Broué and James (whose validity domains are  $n < p^2$ ): up to now, only a small part of this topic is well understood, and there are no general conjectures. To apply the modular Verlinde's formula, we use various dual pairs in the sense of Howe [Ho]. Hence the abstract theory of the commutant algebra of a tilting module is explained in Section 11 and the subsequent sections are devoted to concrete examples. The section 12, concerning  $GL(V)$ -theory, follows closely [MP1]. In Section 13, we show an easy example of a fusion ring, from which we recover a result of Doty [Do] and we give a new proof of a result Benkart, Britten and Lemire [BBL]. The section 14, concerning the symmetric group, follows [M3] but it also contains some unpublished results connected with the works of Erdmann [E] and Kleshchev [K11].

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**1. Lie algebras and algebraic groups in finite characteristics.** Let  $K$  be an algebraically closed field of characteristic  $p$ . In this section, we recall a few facts about Lie algebras and algebraic groups over  $K$ .

*The Jacobson polynomial:* A Lie polynomial  $H(x, y)$  in the two variables  $x, y$  is an element of the free Lie algebra generated by  $x$  and  $y$ . An ordinary commutative polynomial can be evaluated in any commutative algebra. Similarly the Lie polynomial  $H$  can be evaluated in any Lie algebra: if  $a, b$  are two elements of a Lie algebra  $\mathfrak{g}$ ,  $H(a, b)$  is a well-defined element of  $\mathfrak{g}$ .

**Theorem 1.1:** (Jacobson's identity) *There is a two variables Lie*

polynomial  $J_p$  such that:  $(x + y)^p = x^p + y^p + J_p(x, y)$ , for any elements  $x, y$  in an associative  $K$ -algebra  $A$ .

*Proof:* Set  $R = \mathbf{Q} \cap \mathbf{Z}_p$ , i.e.  $R$  is the ring of rational numbers  $t \in \mathbf{Q}$  such that  $v_p(t) \geq 0$ . Let  $\mathcal{A}$  be the free associative  $R$ -algebra in two variables  $x, y$ , and let  $\mathfrak{g} \subset \mathcal{A}$  be its Lie  $R$ -subalgebra generated by  $x$  and  $y$ . The Campbell-Hausdorff formula:

$$\exp x \exp y = \exp \sum_{k \geq 1} \frac{1}{k} H_k(x, y)$$

is an identity which holds in a suitable completion of  $\mathbf{Q} \otimes \mathcal{A}$ , where each  $H_k \in \mathbf{Q} \otimes \mathfrak{g}$  is a homogenous Lie polynomial of degree  $k$ . Note that  $H_1(x, y) = x + y$ . Identifying the degree  $p$  components in the previous formula, one gets:

$$\frac{1}{p!}(x^p + y^p) = \frac{1}{p!}(x + y)^p + \frac{1}{p}H_p(x, y) + \tau,$$

where  $\tau$  is a non-commutative polynomial into  $H_1(x, y), \dots, H_{p-1}(x, y)$  with coefficients in  $R$ . Moreover  $H_k$  belongs to  $\mathfrak{g}$  for any  $k \leq p$ . Therefore  $\tau$  belongs to  $\mathcal{A}$  and one gets:

$$x^p + y^p = (x + y)^p + (p - 1)! H_p(x, y) \text{ modulo } p,$$

from which Jacobson's identity follows. Q.E.D.

*Remark:* Indeed we have  $J_p = H_p$  modulo  $p$ , thanks to Stirling's identity:  $(p - 1)! = -1$  modulo  $p$ . For examples, we have  $J_2(x, y) = [x, y]$ ,  $J_3(x, y) = [x, [x, y]] + [y, [y, x]]$ . A convenient reference for the original proof, based on identities of binomial coefficients, is [J].

*Restricted Lie algebras.* Another special feature of characteristic  $p$  is the following: if  $\delta$  is a derivation of a  $K$ -algebra, then  $\delta^p$  is again a derivation. By definition, a Lie algebra  $\mathfrak{g}$  is called *restrictable* if  $\text{ad}(x)^p$  is an inner derivation for all  $x \in \mathfrak{g}$ . If  $\mathfrak{g}$  is restrictable, a  $p$ -structure is a map,  $x \in \mathfrak{g} \mapsto x^{[p]} \in \mathfrak{g}$  such that:

- (i)  $\text{ad}(x^{[p]}) = \text{ad}(x)^p$ ,
- (ii)  $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ ,
- (iii)  $(x + y)^{[p]} = x^{[p]} + y^{[p]} + J_p(x, y)$ ,

for all  $x, y \in \mathfrak{g}$ ,  $\lambda \in K$ . A restrictable Lie algebra endowed with a  $p$ -structure is called *restricted*. Let  $\mathfrak{g}$  be a restricted Lie algebra and let  $U(\mathfrak{g})$  be its enveloping algebra. For any  $x \in \mathfrak{g}$ ,  $x^p - x^{[p]}$  is a central element of  $U(\mathfrak{g})$ . By definition, the *restricted enveloping algebra*  $u(\mathfrak{g})$  is the quotient of  $U(\mathfrak{g})$  by the ideal generated by all the elements  $x^p - x^{[p]}$ . For a vector space  $V$ , let  $\overline{SV}$  be the quotient of  $SV$  by the ideal generated by all  $x^p, x \in V$ . Then  $\overline{SV}$  is the restricted enveloping algebra of the abelian Lie algebra  $V$  endowed with a trivial  $p$ -structure. If  $x_1, \dots, x_n$  is a basis of  $V$ , then the elements  $x_1^{m_1} \dots x_n^{m_n}$ , where  $m_k < p$  for any  $k$ , form a basis of  $\overline{SV}$ . In particular  $\overline{SV}$  has dimension  $p^n$ . Let  $\mathfrak{g}$  be

a restricted Lie algebra. As for ordinary enveloping algebras, there is a natural filtration:

$$K = u_0(\mathfrak{g}) \subset u_1(\mathfrak{g}) \subset u_2(\mathfrak{g}) \subset \dots$$

of the algebra  $u(\mathfrak{g})$ , where  $u_n(\mathfrak{g})$  the linear subspace generated by products of less than  $n$  elements of  $\mathfrak{g}$ . The version of Poincaré-Birkoff-Witt Theorem for restricted enveloping algebras is the following:

**Theorem 1.2:** (Jacobson) *The graded algebra associated with the filtered algebra  $u(\mathfrak{g})$  is isomorphic to  $\overline{S}\mathfrak{g}$ . In particular, we have:*

$$\dim u(\mathfrak{g}) = p^{\dim \mathfrak{g}}.$$

The proof, which is standard, is omitted. Although there is no Haar measure in finite characteristics, a certain linear form  $L$  on  $u(\mathfrak{g})$  is quite similar.

**Corollary 1.3:** *Let  $\mathfrak{g}$  be a restricted Lie algebra of dimension  $n$ .*

(i) *There is an isomorphism:  $L : u(\mathfrak{g})/u_k(\mathfrak{g}) \rightarrow (\wedge^n \mathfrak{g})^{\otimes p-1} \simeq K$ , where  $k = n(p-1) - 1$ .*

(ii) *The bilinear form  $x, y \in u(\mathfrak{g}) \mapsto L(xy)$  is non-degenerated.*

*Proof:* For a vector space  $V$  of dimension  $n$ ,  $\overline{S}^{(p-1)n} V$  has dimension 1 and it is isomorphic with  $(\wedge^n V)^{\otimes p-1}$  as a  $GL(V)$ -module. Moreover  $\overline{S}^m V = 0$  for  $m > (p-1)n$ . Therefore,  $u(\mathfrak{g})/u_k(\mathfrak{g})$  is isomorphic to  $(\wedge^n \mathfrak{g})^{\otimes p-1}$  by Theorem 1.2. Moreover, as the corresponding bilinear form on  $\overline{S}V$  is non-degenerated, the bilinear form  $x, y \mapsto L(xy)$  is non-degenerated. Q.E.D.

*Simple algebraic groups.* A detailed account of the theory of algebraic groups can be found in [J3]. By *group scheme*, we mean a finitely generated affine  $K$ -scheme  $G$  endowed with a group structure. This definition is equivalent to require that  $K[G]$  is a commutative Hopf algebra which is finitely generated as a commutative algebra. An *algebraic group*  $G$  is a reduced group scheme. Indeed, any group scheme over a field of characteristic zero is reduced. In contrast, in finite characteristics, there are non-reduced schemes with a group structure: the most simple example is  $\text{Spec } K[t]/(t^p)$ . Therefore the hypothesis that  $G$  is reduced is part of the definition of an algebraic group. As it is defined, an algebraic group is automatically smooth.

Let  $G$  be a connected algebraic group. Its *hyperlgebra*  $\mathcal{H}_G$  is the algebra of left  $G$ -invariant differential operators on  $G$  or, equivalently, the convolution algebra of distributions supported at 1 (i.e. the linear maps  $K[G] \rightarrow K$  whose kernel contains a power of the maximal ideal defining 1). The Lie algebra  $\mathfrak{g}$  of  $G$  consists of left  $G$ -invariant derivations of  $K[G]$ . It is a restricted Lie algebra, whose  $p$ -structure is  $x \mapsto x^p$ . It

follows easily that the subalgebra of  $\mathcal{H}_G$  generated by  $\mathfrak{g}$  is isomorphic to  $u(\mathfrak{g})$ . In contrast with the characteristic zero case,  $\mathcal{H}_G$  is not generated by  $\mathfrak{g}$ , because  $\mathcal{H}_G$  is infinite dimensional whenever  $\dim G > 0$ . Things are even worse: if  $\dim G > 0$ , the algebra  $\mathcal{H}_G$  is not finitely generated.

The most basic example is the additive group  $\mathbf{A}_1$ . We have  $K[\mathbf{A}_1] = K[t]$ . For all  $n \geq 0$ , define the differential operator  $e^{(n)} : K[t] \rightarrow K[t]$  by  $e^{(n)} t^m = \binom{m}{n} t^{m-n}$ . As  $e^{(n)}$  is the reduction modulo  $p$  of  $(1/n!) d^n/dt^n$ , the elements  $e^{(n)}$  are called the *reduced powers* of  $e = d/dt$ . The set  $\{e^{(n)}\}$  is a basis of the hyperalgebra  $\mathcal{H}_{\mathbf{A}_1}$ ; it follows that  $\mathcal{H}_{\mathbf{A}_1}$  is the commutative algebra generated by the infinite set of variables  $(x_n)_{n \geq 1}$  and defined by the relations  $x_n^p = 0$  for all  $n \geq 1$ . To find such a presentation of  $\mathcal{H}_{\mathbf{A}_1}$ , set  $x_n = e^{(p^n)}$ .

We would like to define the iterated Frobenius kernels, following [J2]. Let  $G$  be an algebraic group. It turns out that the subalgebra  $(K[G])^p$  is a Hopf subalgebra of  $K[G]$  and therefore it defines an algebraic group  $G_p$ . The corresponding map  $Fr : G \rightarrow G_p$  is called the *absolute Frobenius map*. The kernel of this morphism is called the *Frobenius kernel*. It is a group scheme whose underlying Hopf algebra is  $u(\mathfrak{g})^*$ . At the set theoretical level,  $Fr$  is bijective and the Frobenius kernel is an infinitesimal group scheme, i.e. its unique point is 1. In general  $G_p$  is not isomorphic to  $G$  as an algebraic group: this occurs exactly when  $G$  is defined over  $\mathbf{F}_p$ . More generally, one gets the iterated Frobenius kernels by using the subalgebras  $(K[G])^q \subset K[G]$ ; where  $q$  is any power of  $p$ .

Recall that in characteristic zero, the connected simple algebraic groups are not simple, but they are simple up to a finite center. In characteristic  $p$ , the notion of a simple algebraic group is more involved. Any algebraic group contains a lot of invariant group subschemes, like the iterated Frobenius kernels. Thus it is more difficult to give an axiomatic to define the simple algebraic groups: roughly speaking a simple algebraic group is a connected algebraic group whose smooth invariant subgroups are all contained in a finite center. Indeed, Chevalley and Steinberg [St] defined the connected simply connected simple algebraic groups as reductions modulo  $p$  of some  $\mathbf{Z}$ -forms of the corresponding objects in characteristic zero. Hence they are classified by the indecomposable Dynkin diagrams.

The simplest way to describe a connected simply connected simple group  $G$  is to start by defining its Hopf algebra  $K[G]$ . Let  $I$  be an indecomposable Dynkin diagram and let  $(a_{i,j})_{i,j \in I}$  be its Cartan matrix. Let  $\mathfrak{g}_{\mathbf{Q}}$  be the simple split Lie  $\mathbf{Q}$ -algebra with Dynkin diagram  $I$ , i.e. the Lie  $\mathbf{Q}$ -algebra generated by the symbols  $(e_i, f_i, h_i)_{i \in I}$  and defined

by the standard relations:

- (i)  $[h_i, h_j] = 0$
- (ii)  $[e_i, f_i] = h_i,$
- (iii)  $[e_i, f_j] = 0$
- (iv)  $[h_i, e_j] = a_{i,j}e_j,$
- (v)  $[h_i, f_j] = -a_{i,j}e_j,$
- (vi)  $ad(e_i)^{1-a_{i,j}}(e_j) = 0,$
- (vii)  $ad(f_i)^{1-a_{i,j}}(f_j) = 0,$

for any  $i, j \in I, i \neq j$ . Let  $\mathcal{H}^{\mathbf{Z}}$  be the  $\mathbf{Z}$ -subalgebra of  $U(\mathfrak{g}_{\mathbf{Q}})$  generated by all divided powers  $e_i^{(n)}$  and  $f_i^{(n)}$  (where the symbol  $x^{(n)}$  means  $x^n/n!$ ). By definition,  $\mathbf{Z}[G_{\mathbf{Z}}]$  is the Hopf  $\mathbf{Z}$ -algebra dual of  $\mathcal{H}^{\mathbf{Z}}$ , i.e. it consists of linear forms  $L : \mathcal{H}^{\mathbf{Z}} \rightarrow \mathbf{Z}$  whose kernel contains a right ideal  $I$  such that  $\mathcal{H}^{\mathbf{Z}}/I$  is a finitely generated  $\mathbf{Z}$ -module. Equivalently: there is an integer  $N$  such that  $L(e_i^{(n)} u) = L(f_i^{(n)} u) = 0$  for any  $u \in \mathcal{H}^{\mathbf{Z}}, i \in I$  and  $n > N$ .

For any field  $K$ , set  $G_K = \text{Spec } K \otimes \mathbf{Z}[G_{\mathbf{Z}}]$ . By definition,  $K[G_K]$  is a commutative Hopf algebra: however it requires some work to prove that  $K[G_K]$  is a finitely generated domain and that  $\mathcal{H}_{G_K} = K \otimes \mathcal{H}^{\mathbf{Z}}$ . By definition,  $G_K$  is the connected simply connected simple group over  $K$  with Dynkin diagram  $I$ .

When there is no ambiguity on the ground field  $K$ , we simply set  $G = G_K$ . As usual, the group  $G$  contains a series of remarkable subgroups (the Borel subgroups, their radicals, and so on), but their definition requires some care because of the involved definition of  $G$ . By definition the group  $U$  (respectively  $U^-$ ) is the group whose associated hyperalgebra  $\mathcal{H}_U$  is the subalgebra generated by all  $e_i^{(n)}$  (respectively by all  $f_i^{(n)}$ ). The Borel subgroups  $B, B^-$  are the normalizers of  $U, U^-$ . As usual  $U$  is the radical of  $B$ . The Cartan subgroup is  $H = B \cap B^-$ . Denote by  $\mathfrak{g}, \mathfrak{u}, \mathfrak{u}^-$  and  $\mathfrak{h}$  be the Lie algebras of the algebraic groups  $G, U, U^-$  and  $H$ . As usual, there is a Cartan decomposition  $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h} \oplus \mathfrak{u}^-$ . Similarly, there is a PBW decomposition of  $\mathcal{H}_G$ :

**Lemma 1.4:** (PBW Decomposition) *The multiplication induces a linear isomorphism:*

$$\mathcal{H}_G \simeq \mathcal{H}_U \otimes \mathcal{H}_H \otimes \mathcal{H}_{U^-}.$$

The proof of this classical lemma can be found in Bourbaki [Bo]. The definition of the set of roots  $\Delta$ , the set of positive roots  $\Delta^+$ , the Weyl group  $W$ , the root lattice  $Q$ , the monoid  $Q^+ = \mathbf{N}\Delta^+$ , the weight lattice  $P$  and the set of dominant weights  $P^+$  are easy and all these discrete objects are independent of the ground field  $K$ . For example  $P$  is the group of characters of  $H$ . It should be noted that  $P \simeq \mathbf{Z}^l$ , where

$l = \dim H$ , hence the natural map  $P \rightarrow \mathfrak{h}^*$  is not injective. Therefore the roots should be understood as elements of  $P$ , not as elements of  $\mathfrak{h}^*$ . Similarly the coroots are elements of  $\text{Hom}(P, \mathbf{Z})$ . Simple roots, coroots are denoted by  $\alpha_i, h_i$ , for any  $i \in I$ .

*From now on,  $K$  will denote an algebraically closed field of characteristic  $p > 0$  and  $G$  will be a simple algebraic group as defined before.*

**2. Representations of simple algebraic groups.** In this section, we will recall some basic facts, due to Steinberg [St], about representations of algebraic groups.

First start with general definitions. Let  $L$  be any group scheme whose Lie algebra is denoted by  $\mathfrak{l}$ . A finite dimensional  $L$ -module  $M$  is a finite dimensional vector space  $M$  together with an group scheme morphism  $L \rightarrow GL(M)$ . Another equivalent definition is given by a structural map  $\Delta_M : M \rightarrow M \otimes K[L]$ , which satisfies an obvious cocycle condition (i.e. the two induced maps  $M \rightarrow M \otimes K[L] \otimes K[L]$  coincide). It turns out that the second definition allows to define infinite dimensional  $L$ -modules: a typical example of infinite dimensional  $L$ -module is  $K[L]$ , when  $\dim L > 0$ . Indeed any  $L$ -module  $M$  is the union of its finite dimensional submodules. *Unless stated otherwise, we will assume from now on that any  $L$ -module is finite dimensional.*

We denote by  $\text{Mod}(L)$  the category of  $L$ -modules of arbitrary dimension. Let  $M \in \text{Mod}(L)$ . Using the structural map  $\Delta_M : M \rightarrow M \otimes K[L]$  together with the natural pairing  $\mathcal{H}_L \times K[L] \rightarrow K$ , one gets a map  $\mu_M : \mathcal{H}_L \times M \rightarrow M$  and it is easy to see that  $\mu_M$  endows  $M$  with a structure of  $\mathcal{H}_L$ -module. In particular  $M$  has a natural structure of  $u(\mathfrak{l})$ -module. For any  $M \in \text{Mod}(L)$ , we denote by  $H^0(L, M)$  the subspace of invariant vectors, i.e. the subspace of all  $m \in M$  such that  $\Delta_M(m) = m \otimes 1$ . Also set  $H^0(\mathfrak{l}, M) = \{m \in M \mid x.m = 0, \forall x \in \mathfrak{l}\}$ . Unlike the characteristic zero case, there are connected group schemes  $L$  and  $L$ -modules  $M$  with  $H^0(L, M) \neq H^0(\mathfrak{l}, M)$ . For example, relative to the left action, we have  $H^0(L, K[L]) = K$  but  $H^0(\mathfrak{l}, K[L]) = K[L]^p$ . However the inclusion  $H^0(L, M) \subset H^0(\mathfrak{l}, M)$  always holds. As  $M \otimes K[L]$  is an injective  $L$ -module and the structural map  $\Delta_M : M \rightarrow M \otimes K[L]$  is one-to-one, the category  $\text{Mod}(L)$  has enough injective objects. Therefore one can derive the left exact functor  $H^0(L, -) : M \mapsto H^0(L, M)$ : its series of derived functors will be denoted by  $H^k(L, -)$ . Similarly, we denote by  $H_0(L, M)$  the space of co-invariant vectors of  $M$ . In general,  $\text{Mod}(L)$  does not have enough projective objects, and there are technical difficulties to define the series of derived functors associated to the right exact functor  $H_0(L, -)$ : however, we will not use them and henceforth

we will skip their definition.

Let  $L'$  be another group scheme and let  $\phi : L \rightarrow L'$  be a morphism of group schemes. It induces a functor  $\phi_* : \text{Mod}(L') \rightarrow \text{Mod}(L)$ , namely  $\phi_*$  is the restriction functor from  $L'$  to  $L$  (for any  $M \in \text{Mod}(L')$ ,  $\phi_* M = M$  as a vector space). The morphism  $\phi$  also determines a functor  $\phi^* : \text{Mod}(L) \rightarrow \text{Mod}(L')$ , namely  $\phi^*$  is the induction functor from  $L$  to  $L'$ . For any  $M \in \text{Mod}(L)$ ,  $\phi^* M = H^0(L, K[L'] \otimes M)$ , where  $K[L']$  is viewed as an  $L$ -module relative to the right action, and the  $L'$ -module structure on  $\phi^* M$  is induced by the left action on  $K[L']$ . When  $L$  is a subgroup of  $L'$  and  $\phi$  is the corresponding inclusion map, the functor  $\phi^*$  will be denoted by  $\text{Ind}_L^{L'}$ . By contrast with the case of finite groups or Lie algebras, the functor  $\text{Ind}_L^{L'}$  is not always exact. However it is always left exact.

We will now give some specific definitions for the simple algebraic groups. Let  $G$  be a simple algebraic group. For any  $\mu \in P$ , we will denote by  $K(\mu)$  the corresponding one dimensional  $H$ -module. Sometimes  $K(\mu)$  will be considered as a  $B$ -module or as a  $B^-$ -module with a trivial action of  $U$  or  $U^-$ . For any  $H$ -module  $M$ , we will denote by  $M_\mu$  its  $\mu$ -weight space. A weight  $\lambda$  of  $M$  is called a *highest weight* of  $M$  if  $M_\mu = 0$  for any  $\mu > \lambda$  (i.e. for any  $\mu \neq \lambda$  with  $\mu - \lambda \in Q^+$ ). For any  $B$ -module  $N$ , denote by  $\mathcal{L}(N)$  the sheaf of sections of the bundle  $G \times_B N \rightarrow G/B$  (we will use the simplified notation  $\mathcal{L}(\mu)$  for  $\mathcal{L}(K(\mu))$ ), and set  $DN = \Gamma(G/B, \mathcal{L}(N))$ . Then  $N \mapsto DN$  is a left exact functor  $D : \text{Mod}(B) \rightarrow \text{Mod}(G)$ , which is the induction functor  $\text{Ind}_B^G$ . Indeed the functor  $D$  satisfies the following universal property.

- First, the evaluation of a global section on  $G/B$  at the point  $B/B$  provides a natural transformation of  $B$ -modules  $DN \rightarrow N$  for any  $N \in \text{Mod}(B)$ .

- Next, for any  $M \in \text{Mod}(G)$ ,  $N \in \text{Mod}(B)$ , any  $B$ -morphism  $M \rightarrow N$  factors through a unique  $G$ -morphism  $M \rightarrow DN$ .

For  $\lambda \in P^+$ , we set  $\nabla(\lambda) = H^0(G/B, \mathcal{L}(w_0\lambda))$ , where  $w_0$  is the longest element of  $W$ . Set  $\Delta(\lambda) = \nabla(-w_0\lambda)^*$ . With these definitions,  $\lambda$  is the unique highest weight of the modules  $\nabla(\lambda)$  and  $\Delta(\lambda)$  and their character is given by Weyl character formula, as we will see later: therefore, the  $G$ -modules  $\Delta(\lambda)$ ,  $\nabla(\lambda)$  are called the *Weyl module*, the *dual of the Weyl module*<sup>1</sup> with highest weight  $\lambda$ .

**Lemma 2.1:** *Let  $\lambda \in P^+$ .*

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<sup>1</sup>This well-established terminology could be a bit confusing, because  $\nabla(\lambda)$  is only the dual of  $\Delta(\lambda)$  twisted by the Cartan involution  $\omega : G \rightarrow G$ . Recall that  $\omega(h) = h^{-1}$  for any  $h \in H$ , what implies  $\omega(U) = U^-$ .

- (i) As  $H$ -modules,  $H^0(U, \nabla(\lambda)) \simeq K(\lambda)$ .
- (ii) The weights of  $\nabla(\lambda)$  are in the convex hull of  $W\lambda$ .
- (iii)  $\lambda$  is the unique highest weight of  $\nabla(\lambda)$ , and  $\dim \nabla(\lambda)_\lambda = 1$ .
- (iv)  $\nabla(\lambda)$  contains a unique simple submodule  $L(\lambda)$ , and  $\dim L(\lambda)_\lambda = 1$ .

*Proof:* By semi-continuity principle,  $\nabla(\lambda)$  is "at least as big" as its characteristic zero counterpart. Therefore  $H^0(U^-, \nabla(\lambda))$  contains a non-zero vector of weight  $w_0\lambda$ . As in characteristic zero, Bruhat Decomposition  $G = \coprod_{w \in W} BwB$  holds. The  $U^-$ -orbit  $\Omega$  of  $B/B$  is a dense open subset of  $G/B$ , and therefore  $H^0(U^-, \nabla(\lambda))$  is one dimensional. It follows that all weights of  $\nabla(\lambda)$  are in  $w_0\lambda + Q^+$ . Then the first three assertions follows by  $W$ -invariance of  $\nabla(\lambda)$ . Let  $L(\lambda)$  be the  $G$ -module generated by a non-zero highest weight vector  $v_\lambda$  of  $\nabla(\lambda)$ . By Lie's theorem, any non-zero  $B$ -submodule of  $\nabla(\lambda)$  contains  $v_\lambda$ . Therefore any non-zero  $G$ -submodule contains  $L(\lambda)$ , what amounts to the fact that  $L(\lambda)$  is the unique simple  $G$ -submodule of  $\nabla(\lambda)$ . Q.E.D.

**Theorem 2.2:** (Steinberg) *The map  $\lambda \in P^+ \mapsto L(\lambda)$  is a bijection from  $P^+$  to the set of simple  $G$ -module.*

*Proof:* Let  $S$  be a simple  $G$ -module. As the action of  $U$  is unipotent,  $H_0(U, S)$  is non-zero. As in characteristic zero,  $SL(2)$ -theory implies that any weight of  $H_0(U, S)$  is antidominant. Therefore there exists a dominant weight  $\lambda$  and a non-zero  $B$ -morphism  $S \mapsto K(w_0\lambda)$ . By the universal property of the functor  $D$ , one gets a non-zero  $G$ -morphism  $S \rightarrow \nabla(\lambda) = DK_{w_0\lambda}$ . By Lemma 2.1,  $S$  is isomorphic to  $L(\lambda)$ . Q.E.D.

Therefore, the classification of simple  $G$ -modules is the same as in characteristic zero. As the space  $H^0(U, \nabla(\lambda))$  is one dimensional, the  $G$ -module  $\nabla(\lambda)$  is indecomposable. However, in general  $\nabla(\lambda)$  is reducible. A simple example is provided by the following remark: for any  $n \geq 1$ , the  $n$ -power map  $\Sigma_n : \nabla(\lambda) \rightarrow \nabla(n\lambda)$ ,  $f \mapsto f^n$  is a  $G$ -equivariant polynomial map and its image is a  $G$ -invariant cone. However, when  $n = p$  its image is a linear subspace, with the same dimension as  $\nabla(\lambda)$ . We can deduce that the module  $\nabla(p\lambda)$  is reducible for any  $\lambda \neq 0$ . For a more complicated example, note that the trivial module  $K$  is always a quotient of  $\nabla(2(p-1)\rho)$ . Indeed, with the definitions of Section 5,  $\nabla(2(p-1)\rho)$  can be identified with the space of  $\mathcal{O}_{G/B}^p$ -linear maps  $\sigma : \mathcal{O}_{G/B} \rightarrow \mathcal{O}_{G/B}^p$ . Therefore the map  $\sigma \mapsto \sigma(1)$  is a  $G$ -morphism from  $\nabla(2(p-1)\rho)$  to  $K$ . As  $G/B$  is Frobenius split, this map is not zero.

Next, Steinberg [St] showed how to reduce the computation of all simple modules to a finite problem, namely the determination of those whose highest weight is restricted. A dominant weight  $\lambda$  is called *re-*

stricted if  $\lambda(h_i) < p$  for any  $i \in I$ . The number of restricted weights is  $p^l$ . In order to introduce Steinberg's method, we first state a simple lemma.

**Lemma 2.3:** *Let  $\lambda$  be a restricted weight. Denote indistinctly by  $v_\lambda$  a non-zero highest weight vector of  $\Delta(\lambda)$  or of  $\nabla(\lambda)$ . We have  $\Delta(\lambda) = u(\mathfrak{u}^-).v_\lambda$ , and  $H^0(\mathfrak{u}, \nabla(\lambda)) = Kv_\lambda$ .*

*Proof:* By Lemma 2.1 (i), one gets  $\Delta(\lambda) = \mathcal{H}_{U^-}.v_\lambda$ . Moreover, we have:  $f_i^{(n)}u.v_\lambda = \sum_{a+b=n} Ad(f_i^{(a)})(u) f_i^{(b)}.v_\lambda$ , for any  $i \in I$ ,  $n \geq 0$  and  $u \in u(\mathfrak{u}^-)$ . As  $\lambda$  is restricted, we have  $f_i^{(b)}.v_\lambda = 0$  for any  $b \geq p$ , therefore  $u(\mathfrak{u}^-).v_\lambda$  is stable by  $\mathcal{H}_{U^-}$ . Hence  $\Delta(\lambda) = u(\mathfrak{u}^-).v_\lambda$ . The second point follows by duality. Q.E.D.

It follows from its definition that  $\mathbf{F}_p[G_{\mathbf{F}_p}]$  is a  $\mathbf{F}_p$ -form of the algebra  $K[G]$ , see Section 1. The Frobenius morphism<sup>2</sup>  $F : K[G] \rightarrow K[G]$  is the  $K$ -linear algebra morphism such that  $F(f) = f^p$  for any  $f \in \mathbf{F}_p[G_{\mathbf{F}_p}]$ . We will also denote by  $F : G \rightarrow G$  the corresponding group morphism. For  $G = SL(n)$ , the Frobenius morphism  $F$  is the map  $(a_{i,j}) \mapsto (a_{i,j}^p)$ . In general, the  $F$  is a group morphism which is pointwise bijective, but as a morphism of algebraic varieties, it is a finite morphism<sup>3</sup> of degree  $p^{\dim G}$ . Its image is the subring  $K[G]^p$ , which is stable by  $\mathcal{H}_G$ . Hence  $F$  induces an algebra morphism  $F : \mathcal{H}_G \rightarrow \mathcal{H}_G$ , and we have  $F(e_i^{(n)}) = F(f_i^{(n)}) = 0$  if  $n$  is not divisible by  $p$ , and  $F(e_i^{(n)}) = e_i^{(n/p)}$ ,  $F(f_i^{(n)}) = f_i^{(n/p)}$  otherwise. It follows easily that  $F(\mathfrak{g}) = 0$ . Therefore the  $\mathfrak{g}$ -module  $F_*M$  is trivial for any  $M \in Mod(G)$ . As  $F$  induces the multiplication by  $p$  on  $P$ , we have  $F_*L(\lambda) = L(p\lambda)$ .

Any non-negative integer  $n$  admits a finite  $p$ -adic decomposition:  $n = n_0 + p.n_1 + p^2.n_2 + \dots$ , where  $0 \leq n_k < p$  and almost all  $n_k$  are zero. Similarly, any dominant weight  $\lambda$  admits a  $p$ -adic decomposition  $\lambda = \lambda_0 + p.\lambda_1 + p^2.\lambda_2 + \dots$ , where all  $\lambda_k$  are restricted and almost all of them are zero.

**Theorem 2.4:** (Steinberg's tensor product Theorem) *Let  $\lambda = \lambda_0 + p.\lambda_1 + p^2.\lambda_2 + \dots$  as before. We have<sup>4</sup>:*

<sup>2</sup>The Frobenius morphism is also called the  $K$ -linear Frobenius morphism. It coincides with the absolute Frobenius morphism  $Fr$  on the subring  $\mathbf{F}_p[G_{\mathbf{F}_p}]$ .

<sup>3</sup>It is a purely characteristic  $p$  phenomenon that a finite morphism of degree  $> 1$  can be bijective. This arises in the presence of purely unseparable extensions.

<sup>4</sup>There is a similar statement for finite dimensional continuous simple representations of  $G_{\mathbb{C}}$ ; they are of the form  $L \otimes \sigma_*L'$ , where  $L, L'$  are holomorphic and where  $\sigma$  is the involution of  $Gal(\mathbb{R})$

$$L(\lambda) \simeq L(\lambda_0) \otimes F_*L(\lambda_1) \otimes F_*^2L(\lambda_1) \dots$$

*Proof:* It should be noted that  $\lambda_k = 0$  for  $k \gg 0$ , therefore  $F_*^k L(\lambda_k) = K$  for  $k \gg 0$  and the tensor product is indeed finite.

By induction, it is enough to prove  $L(\lambda) \simeq L(\lambda_0) \otimes F_*L(\lambda_1)$ , for any  $\lambda = \lambda_0 + p\lambda_1$  with  $\lambda_0$  restricted and  $\lambda_1 \in P^+$ . Let  $v_0^+, v_1^+$  be non-zero highest weight vectors of  $L(\lambda_0), F_*L(\lambda_1)$ . By Lemma 2.3, we have  $H^0(\mathfrak{u}, L(\lambda_0)) = K.v_0^+$ . Moreover  $\mathfrak{u}$  acts trivially on  $F_*L(\lambda_1)$ . Hence  $H^0(\mathfrak{u}, L(\lambda_0) \otimes F_*L(\lambda_1)) = K.v_0^+ \otimes F_*L(\lambda_1)$ , from which one deduces  $H^0(U, L(\lambda_0) \otimes F_*L(\lambda_1)) = K.v_0^+ \otimes v_1^+$ . Similarly, one proves  $H_0(U^-, L(\lambda_0) \otimes F_*L(\lambda_1)) = K.v_0^+ \otimes v_1^+$ . Therefore  $v_0^+ \otimes v_1^+$  generates  $L(\lambda_0) \otimes F_*L(\lambda_1)$  as a  $U^-$ -module and any non-zero  $U$ -submodule contains  $v_0^+ \otimes v_1^+$ . Hence the  $G$ -module  $L(\lambda_0) \otimes F_*L(\lambda_1)$  is simple. Its highest weight being  $\lambda$ , it is isomorphic to  $L(\lambda)$  by Theorem 2.2. Q.E.D.

Use again the notation  $F$  for the endomorphism of  $\mathbf{Z}[P]$  such that  $F e^\lambda = e^{p\lambda}$ . From Theorem 2.4, we deduce the following corollary.

**Corollary 2.5:** (Steinberg) *Let  $\lambda = \lambda_0 + p.\lambda_1 + p^2.\lambda_2 + \dots$  be the  $p$ -adic decomposition of a dominant weight  $\lambda$ . We have:*

$$\text{ch } L(\lambda) = \prod_{k \geq 0} F^k \text{ch } L(\lambda_k).$$

(Almost all terms of the previous product equals to 1 and the product is actually finite.) Set  $\rho = 1/2 \sum_{\alpha \in \Delta^+} \alpha$ . As usual,  $\rho$  is a weight because  $\rho(h_i) = 1$  for all  $i \in I$ . The next result of Steinberg is so remarkable that the module  $\Delta((p-1)\rho)$  is now called the *Steinberg module* and it deserves the special notation  $St$ .

**Theorem 2.6:** (Steinberg)

- (i) *As a  $u(\mathfrak{u}^-)$ -module,  $St$  is free of rank one.*
- (ii) *The  $G$ -module  $St$  is simple, and  $St \simeq \nabla((p-1)\rho)$ .*
- (iii)  $\text{ch } St = e^{(p-1)\rho} \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} \dots + e^{(1-p)\alpha})$ .

*Proof:* We claim:

- (i)  $u(\mathfrak{u}^-)$  contains a unique (up to scalar) vector  $X \neq 0$  of weight  $2(1-p)\rho$ , and
- (ii) any non-zero  $H$ -invariant left ideal  $I$  of  $u(\mathfrak{u}^-)$  contains  $X$ .

By Jacobson Theorem 1.2, we get:

$\text{ch } u(\mathfrak{u}^-) = \text{ch } \overline{S}u^- = \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} \dots + e^{(1-p)\alpha})$ , therefore any weight of  $u(\mathfrak{u}^-)$  is of the form  $\sum_{\alpha \in \Delta^+} m_\alpha \alpha$ , where  $-p < m_\alpha \leq 0$ . Thus its lowest weight, namely  $2(1-p)\rho$ , occurs with multiplicity 1, what shows the unicity of  $X$ . Let  $I$  be any non-zero  $H$ -invariant left ideal of  $u(\mathfrak{u}^-)$ . Let  $L : u(\mathfrak{u}^-) \rightarrow (\bigwedge^{(p-1)N} \mathfrak{u}^-)^{\otimes p-1} \simeq K(2(1-p)\rho)$

be the canonical map defined in Corollary 1.3. As the associated bilinear map is non-degenerated, we have  $L(I) \neq 0$ . In particular  $2(1-p)\rho$  is a weight of  $I$ . Therefore  $X \in I$  and the claim is proved.

It follows from Lemma 2.3 that  $L((p-1)\rho) = u(\mathfrak{u}^-).v^+$ , where  $v^+$  is a highest weight vector of  $L((p-1)\rho)$ . The lowest weight of  $L((p-1)\rho)$  is  $w_0(p-1)\rho = (1-p)\rho$ . Hence  $X.v^+$  is a non-zero lowest weight vector of  $L((p-1)\rho)$ . Therefore  $X$  does not belong to the annihilator  $I$  in  $u(\mathfrak{u}^-)$  of  $v^+$ . It follows from the previous claim that  $I = 0$ . Therefore, the  $u(\mathfrak{u}^-)$ -module  $L((p-1)\rho)$  is free of rank one and  $\Delta((p-1)\rho) = L((p-1)\rho)$ . By duality we also have  $\nabla((p-1)\rho) = L((p-1)\rho)$ , what proves assertions (i) and (ii). The last assertion follows from the fact that

$$\text{ch } L((p-1)\rho) = e^{(p-1)\rho} \text{ch } u(\mathfrak{u}^-). \quad \text{Q.E.D.}$$

### 3. Kempf's vanishing theorem and Weyl character formula.

In this section we will prove Kempf's vanishing Theorem [Ke], following the beautiful proof of Andersen [A3] and Haboush [Ha]. Its main corollary is the Weyl character formula for  $\nabla(\lambda)$ . We will also state Jantzen linkage Principle [J1][HJ] and Andersen strong linkage Principle [A4]. However, we will only sketch the proof of the linkage principles.

Note that  $\mathcal{O}_{G/B}^p \subset \mathcal{O}_{G/B}$  is a ring subsheaf. Therefore, any sheaf  $\mathcal{M}$  of  $\mathcal{O}_{G/B}$ -modules can be considered as a sheaf of  $\mathcal{O}_{G/B}^p$ -modules, hence we get a natural map  $\mathcal{O}_{G/B}^p \otimes_K \Gamma(G/B, \mathcal{M}) \rightarrow \mathcal{M}$ . Recall that  $St = \Gamma(G/B, \mathcal{L}((1-p)\rho))$ .

#### Lemma 3.1: (Andersen-Haboush formula [A3][Ha])

The natural map  $\theta : \mathcal{O}_{G/B}^p \otimes St \rightarrow \mathcal{L}((1-p)\rho)$  is an isomorphism.

*Proof:* The map  $\mathcal{O}_{G/B}^p \rightarrow \mathcal{L}((1-p)\rho)$ ,  $f \mapsto f.v^+$  (where  $v^+$  is a highest weight vector of  $St$ ) is clearly injective. By Lemma 2.3 we have:  $H^0(\mathfrak{u}, \mathcal{O}_{G/B}^p \otimes St) = \mathcal{O}_{G/B}^p \otimes k.v^+$ . Hence by  $\mathfrak{u}$ -equivariance,  $\theta$  is injective. Moreover, the source and the goal of  $\theta$  are locally free  $\mathcal{O}_{G/B}^p$ -sheaves of same rank, namely  $p^{\dim G/B}$ . Hence  $\theta$  is an isomorphism on some dense open subset. By  $G$ -equivariance,  $\theta$  is an isomorphism everywhere. Q.E.D.

#### Theorem 3.2: (Kempf's vanishing theorem [Ke]) We have:

$$H^k(G/B, \mathcal{L}(-\lambda)) = 0, \text{ for any } \lambda \in P^+ \text{ and } k > 0.$$

*Proof:* For any line bundle  $\mathcal{L}$  over  $G/B$ , denote by  $\mathcal{L}^p$  the image of the  $p$ -power map  $\Sigma_p : \mathcal{L} \rightarrow \mathcal{L}^{\otimes p}$ . Thus  $\mathcal{L}^p$  is an invertible  $\mathcal{O}_{G/B}^p$ -sheaf and  $\mathcal{L}^{\otimes p} = \mathcal{O}_{G/B} \otimes_{\mathcal{O}_{G/B}^p} \mathcal{L}^p$ . It follows Andersen-Haboush formula that

$\mathcal{L}(-p\lambda + (1-p)\rho) \simeq \mathcal{L}(-\lambda)^p \otimes St$ . As sheaves of abelian groups,  $\mathcal{L}(-\lambda)$  and  $\mathcal{L}(-\lambda)^p$  are isomorphic. Hence we get:

$$H^k(G/B, \mathcal{L}(-\lambda)) \neq 0 \Rightarrow H^k(G/B, \mathcal{L}(-p\lambda + (1-p)\rho)) \neq 0.$$

By induction, we get:

$$H^k(G/B, \mathcal{L}(-\lambda)) \neq 0 \Rightarrow H^k(G/B, \mathcal{L}_1^{\otimes q/p} \otimes \mathcal{L}_2^{\otimes q-1-(q/p)}) \neq 0,$$

for all power  $q$  of  $p$ , where  $\mathcal{L}_1 = \mathcal{L}(-p\lambda - \rho)$  and  $\mathcal{L}_2 = \mathcal{L}(-\rho)$ . Therefore, the proof follows from the following three assertions:

- (i)  $G/B$  is a projective variety,
- (ii)  $\mathcal{L}(-\mu - \rho)$  is ample for any  $\mu \in P^+$ ,
- (iii) For any two ample invertible sheaves  $\mathcal{L}_1, \mathcal{L}_2$  over a projective variety  $X$ , there are only finitely many  $m_1, m_2 \geq 0$  such that  $H^k(X, \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2}) \neq 0$  for some  $k > 0$ .

Assertion (iii) is a simple refinement of Serre's vanishing theorem whose proof is standard. Let us prove Assertion (i) and (ii).

First one claims that  $G/B$  is projective and  $\mathcal{L}((1-p)\rho)$  is very ample. Let  $v^+$  be the highest weight vector of  $St$ . First, consider the map  $\xi : G/B \rightarrow \mathbf{P}St, g \mapsto g.Kv^+$ . We observe that the differential at 1 of  $\xi$  can be identified with the map  $u^- \rightarrow St/Kv^+, u \mapsto u.v^+$ . By Theorem 2.6 (i), this map is injective. By  $G$ -invariance, one proves that  $\xi$  is an embedding. Moreover, by Lemma 2.1 and Theorem 2.6,  $Kv^+$  is the unique  $B$ -invariant line of  $\mathbf{P}St$ . Thus, by Lie's theorem, any closed  $B$ -invariant subset of  $\mathbf{P}St$  contains  $Kv^+$ . Therefore the image of  $\xi$  is closed and isomorphic to  $G/B$ . Thus the claim is proved.

In particular Assertion (i) holds. For any  $\mu \in P^+, \mathcal{L}(-\mu)$  has non-zero global sections. By  $G$ -invariance, it implies that  $\mathcal{L}(-\mu)$  is generated by its global sections. Therefore the sheaf  $\mathcal{L}(-\mu - \rho)^{\otimes p-1} \simeq \mathcal{L}((1-p)\mu) \otimes \mathcal{L}((1-p)\rho)$  is very ample, what proves Assertion (ii). Q.E.D.

*Remarks:* As  $G/B$  is projective, the functor  $D$  sends finite dimensional  $B$ -modules to finite dimensional  $G$ -modules (this fact can be proved more elementarily). Indeed, a refinement of Assertion (i) is true:  $\mathcal{L}(-\mu - \rho)$  is very ample whenever  $\mu$  is dominant. Moreover, the Kempf's vanishing theorem holds under the weaker hypothesis that  $\lambda + \rho$  is dominant. It should be noted that Kempf's theorem is a characteristic free generalization of Bott's vanishing theorem. However the part of Bott's theorem involving non-dominant weights fails in finite characteristics: there are weights  $\lambda$  such that  $H^k(G/B, \mathcal{L}(\lambda)) \neq 0$  for at least two values of  $k$ . See [A5] for a counterexample involving  $G = SL(3)$ . The original Kempf's proof is based on a case-by-case analysis.

**Corollary 3.3:** (Weyl character formula) *Let  $\lambda \in P^+$ . We have:*

$$\text{ch } \Delta(\lambda) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w) e^{w\rho}}.$$

*Proof:* By semi-continuity theorem, the character of the virtual  $G$ -module  $\sum_k (-1)^k H^k(G/B, \mathcal{L}(-\lambda))$  is independent of the characteristic of the field  $K$ , hence given by Weyl character formula. Hence, by Kempf's vanishing theorem, the character of  $\Delta(\lambda)$  is given by the Weyl character formula. Q.E.D.

As noted before, there are indecomposable  $G$ -modules which are not simple. Therefore the category  $Mod(G)$  of  $G$ -modules is not semisimple. As for any locally artinian category, we can decompose  $Mod(G)$  into blocks. We define now the notion of a *block* of a locally artinian abelian category  $\mathcal{M}$ . The *quiver*  $\Gamma$  of  $\mathcal{M}$  is the non-oriented graph whose vertices are the (isomorphism classes of) simple objects of  $\mathcal{M}$  and whose edges connect two simple objects  $L, L'$  whenever  $Ext_{\mathcal{M}}^1(L, L') \neq 0$  or  $Ext_{\mathcal{M}}^1(L', L) \neq 0$ . A block is a connected component of  $\Gamma$ . Any object  $M \in \mathcal{M}$  can be decomposed as  $\bigoplus_{b \in \pi_0(\Gamma)} M(b)$ , where any simple subquotient of  $M(b)$  belongs to the block  $b$ .

We will now give a more concrete description of the blocks of  $Mod(G)$ . By Theorem 2.2, the vertices of the quiver  $\Gamma_G$  of  $Mod(G)$  is identified with  $P^+$ . For any finite dimensional  $G$ -module  $M$ , set  $\tau M = \omega_* M^*$ , where  $\omega$  is the Cartan involution. The transformation  $\tau : M \mapsto \tau M$  is contravariant and  $\tau L(\lambda) \simeq L(\lambda)$  for any simple  $\lambda \in P^+$ . Hence the conditions  $Ext_G^1(L(\lambda), L(\lambda')) \neq 0$  and  $Ext_G^1(L(\lambda'), L(\lambda)) \neq 0$  are indeed equivalent, for any  $\lambda, \lambda' \in P^+$ . Let  $\lambda, \lambda' \in P^+$  be the two vertices of an edge of the quiver  $\Gamma_G$  and let

$$0 \rightarrow L(\lambda) \rightarrow Q \rightarrow L(\lambda') \rightarrow 0$$

be a non-split extension. Without loss of generality, we can assume  $\lambda \not\prec \lambda'$ . Therefore the natural  $B$ -morphism  $L(\lambda) \rightarrow L(\lambda)_{w_0\lambda}$  can be extended to a  $B$ -morphism  $Q \rightarrow L(\lambda)_{w_0\lambda}$ . Using the universal property of the induction functor, the natural embedding  $L(\lambda) \subset \nabla(\lambda)$  can be extended to a  $G$ -morphism  $Q \rightarrow \nabla(\lambda)$ . As  $Q$  is a non-split extension, this morphism is one-to-one. In particular, we have  $\lambda > \lambda'$ . Therefore the edges of  $\Gamma$  are exactly the pairs  $(\lambda, \lambda')$  such that:

- (i)  $L(\lambda')$  is a submodule of  $\Delta(\lambda)/L(\lambda)$  (what implies  $\lambda > \lambda'$ ), either
- (ii)  $L(\lambda)$  is a submodule of  $\Delta(\lambda')/L(\lambda')$  (what implies  $\lambda' > \lambda$ ).

By definition, the *affine Weyl group*  $W_{aff}$  is the group of affine transformations of  $P$  generated by  $W$  and the additional affine symmetry  $s_0 : \lambda \mapsto \lambda - (\lambda(h_0) - p)\alpha_0$ , where  $\alpha_0$  is the highest short root and  $h_0 = h_{\alpha_0}$  (thus  $h_0$  is the highest coroot).

**Theorem 3.4:** (Jantzen-Andersen linkage Principles [J1][A4])

(i) Let  $\lambda, \mu \in P^+$ . If  $L(\lambda)$  and  $L(\mu)$  are in the same block, then  $\lambda + \rho$  and  $\mu + \rho$  are conjugated by  $W_{aff}$ .

(ii) Let  $\lambda \in P^+$ ,  $\mu \in P$ . If  $L(-w_0\lambda)$  is a composition factor of  $H^*(G/B, \mathcal{L}(-\mu))$ , then  $\lambda + \rho$  and  $\mu + \rho$  are conjugated by  $W_{aff}$ .

The *linkage principle* (namely 3.4 (i)) has been conjectured by Verma and it has been proved by Jantzen [J1] for  $p \geq h$  and by Andersen [A4] in general. It follows from the definition of the edges of  $\Gamma_G$  that Assertion 3.4 (ii) implies Assertion 3.4 (i). Therefore, Assertion 3.4 (ii) which is due to Andersen [A4], is called the *strong linkage principle*. The sketch of proof given below is based on the Andersen's proof.

**Corollary 3.5:** *If  $(\lambda + \rho)(h_0) \leq p$ , then the module  $\nabla(\lambda)$  is simple.*

The simplest and standard proof of Corollary 3.5 is based the following fact. For any  $\lambda \in P^+$  with  $(\lambda + \rho)(h_0) \leq p$ , there are no dominant weight  $\mu < \lambda$  with  $\mu + \rho \in W_{aff}(\lambda + \rho)$ . Therefore  $L(\lambda)$  is the unique subquotient of  $\nabla(\lambda)$ , i.e.  $\nabla(\lambda)$  is simple.

However, we will sketch a more sophisticated proof in order to explain the idea of Andersen's proof of the linkage principle. First recall Demazure's trick in his "very simple proof" of Bott's theorem over a field of characteristic zero [De]. Therefore assume for a moment that  $K$  is a field of characteristic zero. Demazure's trick is the isomorphism:

$$H^k(G/B, \mathcal{L}(-\mu)) \simeq H^{k+1}(G/B, \mathcal{L}(-s_i(\mu + \rho) + \rho)),$$

for any  $k \geq 0$ ,  $i \in I$  and  $\mu \in P$  with  $\mu(h_i) \geq 0$ . Starting with a weight  $\lambda \in P^+$ , one gets by iterating an isomorphism:

$$H^k(G/B, \mathcal{L}(-\lambda)) \simeq H^{k+l(w)}(G/B, \mathcal{L}(-w(\lambda + \rho) + \rho)),$$

for any  $w \in W$ . However, by Serre's duality,  $H^{l(w)}(G/B, \mathcal{L}(-w_0(\lambda + \rho) + \rho)) \simeq \Delta(\lambda)$ , because  $w_0\rho = -\rho$  and  $\mathcal{L}(2\rho)$  is the sheaf of top forms of  $G/B$ . Thus for  $k = 0$  and  $w = w_0$ , one gets  $\nabla(\lambda) \simeq \Delta(\lambda)$  what amounts to the fact that the  $G$ -module  $\nabla(\lambda)$  is simple. Of course, there are simpler ways to prove this fact<sup>5</sup>.

Demazure's trick uses that the  $SL(2)$ -modules  $H^0(\mathbf{P}^1, \mathcal{L}(-n\rho))$  and  $H^0(\mathbf{P}^1, \mathcal{L}((n+2)\rho))$  are isomorphic for all  $n \geq 0$ . In characteristic  $p$ , it is true for all  $n < p$  and some other special values of  $n$ . Under the condition  $(\lambda + \rho)(h_0) \leq p$ , each step of Demazure's trick uses the previous isomorphism for some value of  $n < p$ . Therefore Demazure's proof applies to such a weight, and one gets a proof of Corollary 3.5.

Andersen's proof of the strong linkage principle is based on this idea. He uses a map  $H^k(G/B, \mathcal{L}(-\lambda)) \rightarrow H^{k+l(w)}(G/B, \mathcal{L}(-w(\lambda + \rho) +$

<sup>5</sup>In characteristic zero, the interest of Demazure's trick lies in the case  $k > 0$ : one gets Bott's vanishing theorem:  $H^k(G/B, \mathcal{L}(-\lambda)) \simeq H^{k+l(w_0)}(G/B, \mathcal{L}(-w_0(\lambda + \rho) + \rho)) = 0$ , because any cohomology in degree  $> \dim G/B$  vanishes.

$\rho$ )), and he determines by induction what are the possible composition factors of its kernel and cokernel.

**4. Good filtrations.** In this section, we will define the good filtrations. To my best knowledge, this notion has been invented by Jantzen and Humphreys (see [Hu]), and it has been used for the first time in Upadhyaya's work [U]. It will be useful to notice that any  $G$ -module has a canonical filtration: originally this idea comes from [Fr] but we will follow the approach of [M2], which is simpler and is better adapted. Using the canonical filtration, we will derive a simple proof of the characteristic  $p$  version of Peter-Weyl Theorem will be provided: this result is often attributed to Donkin (unpublished) and Koppinen [Ko]. At the end of the section, we will describe two different criteria [D1][M2] for the existence of a good filtration.

From now on, choose a one-to one linear map  $E : P \rightarrow \mathbf{R}$  such that  $E(\alpha_i) > 0$  for any  $i \in I$ ; the injectivity condition simply means that the real numbers  $E(\alpha_i)$  are  $\mathbf{Q}$ -linearly independent. By definition, we have  $E(\lambda) < E(\mu)$  whenever  $\lambda < \mu$ . Hence  $E$  induces a total ordering of  $P$  which extends the usual partial ordering  $<$ . For any non-zero dominant weight  $\lambda = \sum_{i \in I} x_i \alpha_i$ , it is well known that all coefficients  $x_i$  are positive. Hence the set  $E(P^+)$  is discrete, and we can uniquely write  $P^+ = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ , where  $E(\lambda_0) < E(\lambda_1) < E(\lambda_2) \dots$ .

For any  $B$ -module  $M$ , denote by  $M(k)$  its biggest  $B$ -submodule whose weights  $\mu$  all satisfy  $E(\mu) \leq E(\lambda_k)$ . By this way, we get a filtration  $M(0) \subset M(1) \subset \dots$  of  $M$ . The filtration  $(M(k))_{k \geq 0}$  of  $M$  is called the *canonical filtration* of  $M$ . However, one should be aware that the filtration depends effectively on  $E$ . As a matter of notation, we set  $h^0(M, \lambda) = \dim H^0(U, M)_\lambda$ .

**Lemma 4.1:** [M2] *Assume that  $M$  is a  $G$ -module.*

(i) *Each  $M(k)$  is indeed a  $G$ -submodule,*

(ii) *There is a canonical inclusion  $M(k)/M(k-1) \subset \nabla(\lambda_k)^{h^0(M, \lambda_k)}$ .*

*Proof:* Proof of Point (i): Use the PBW Decomposition 1.4:  $\mathcal{H}_G = \mathcal{H}_{U^-} \otimes \mathcal{H}_H \otimes \mathcal{H}_U$ . It follows that the  $G$ -submodule generated by  $M(k)$  is  $\mathcal{H}_{U^-} \cdot M(k)$ . However, any weight of  $\mathcal{H}_{U^-}$  is  $\leq 0$ . Thus  $\mathcal{H}_{U^-} \cdot M(k) \subset M(k)$ , hence  $M(k)$  is a  $G$ -module.

Proof of Point (ii): Clearly,  $\lambda_k$  is a highest weight of  $M(k)$ , with multiplicity  $h^0(M, \lambda_k)$ . Therefore the weight  $w_0 \lambda_k$  is a lowest weight of  $M(k)$  and the  $H$ -equivariant projection  $\pi : M(k) \rightarrow M(k)_{w_0 \lambda_k}$  is  $B$ -equivariant. By the universal property of the functor  $D$ ,  $\pi$  induces a morphism  $D\pi : M(k) \rightarrow D M(k)_{w_0 \lambda_k} \simeq \nabla(\lambda_k)^{h^0(M, \lambda_k)}$ , whose kernel is obviously  $M(k-1)$ . Q.E.D.

A filtration  $\mathcal{F}_0M \subset \mathcal{F}_1M \subset \mathcal{F}_2M \dots$  of a  $G$ -module  $M$  is called *good*<sup>6</sup> if for all  $k \geq 0$ ,  $\mathcal{F}_kM/\mathcal{F}_{k-1}M \simeq \nabla(\lambda_k)^{m_k}$ , for some  $\lambda_k \in P^+$  and some  $m_k \geq 0$ .

**Lemma 4.2:** *For any  $G$ -module  $M$ , we have:*

$$\text{ch } M \leq \sum_{\lambda \in P^+} h^0(M, \lambda) \text{ch } \nabla(\lambda).$$

Moreover, the following assertions are equivalent:

- (i)  $\dim M = \sum_{\lambda \in P^+} h^0(M, \lambda) \dim \nabla(\lambda)$ ,
- (ii)  $\text{ch } M = \sum_{\lambda \in P^+} h^0(M, \lambda) \text{ch } \nabla(\lambda)$ ,
- (iii) the canonical filtration is good.

*Proof:* From the previous lemma, we have:

$$\text{ch } M = \sum_{k \geq 0} \text{ch } M(k)/M(k-1) \leq \sum_{k \geq 0} h^0(M, \lambda_k) \text{ch } \nabla(\lambda_k).$$

Moreover, there is equality if and only if the canonical filtration is good. Q.E.D.

The following result is a generalization of Peter-weyl Theorem.

**Theorem 4.3:** (Donkin-Koppinen [Ko]) *There exist*

- (i) a  $G \times G$ -equivariant filtration  $\mathcal{F}_0K[G] \subset \mathcal{F}_1K[G] \dots$  of  $K[G]$
- (ii) a bijection  $\mathbf{Z}_{\geq 0} \rightarrow P^+, k \mapsto \mu_k$ ,

such that  $\mathcal{F}_kK[G]/\mathcal{F}_{k-1}K[G]$  is  $G \times G$ -isomorphic to  $\nabla(\mu_k) \otimes \nabla(-w_0\mu_k)$ , for all  $k \geq 0$ .

*Proof:* First we claim that  $H^0(U \times U^-, K[G]) \simeq \oplus_{\lambda \in P^+} K(\lambda) \otimes K(-\lambda)$  as an  $H \times H$ -module. Indeed  $\Omega = U.H.U^-$  is a dense open set in  $G$ , hence  $H^0(U \times U^-, K[G]) \subset H^0(U \times U^-, K[\Omega]) \simeq K[H] \simeq \oplus_{\lambda \in P} K(\lambda) \otimes K(-\lambda)$  as an  $H \times H$ -module. However the weight of any  $U$ -invariant vector in a  $G$ -module is dominant, hence we have  $H^0(U \times U^-, K[G]) \subset \oplus_{\lambda \in P^+} K(\lambda) \otimes K(-\lambda)$ . Conversely for  $\lambda \in P^+$ , set  $\phi_\lambda(g) = \langle \xi | g^{-1}.v^+ \rangle$ , where  $v^+$  is a highest weight vector of  $L(\lambda)$  and  $\xi$  is a lower weight vector of  $L(\lambda)^*$ . Then  $\phi_\lambda$  is a  $U \times U^-$ -invariant element of  $K[G]$  of weight  $(\lambda, -\lambda)$ . Therefore the claim is proved.

Now consider  $K[G]$  as  $G \times G$ -module, and let  $F : P \oplus P \rightarrow \mathbf{R}$  be an injective linear form as before. Denote by  $g_0 < g_1 < g_2 \dots$  the gaps of the canonical filtration of the  $G \times G$ -module  $K[G]$ , i.e. the integers  $g$  with  $K[G](g) \neq K[G](g-1)$ . Set  $\mathcal{F}_kK[G] = K[G](g_k)$  for all  $k \geq 0$ ; up to a change in the indices, the filtration  $\mathcal{F}_kK[G]$  is identical to the canonical filtration. It follows from the claim that any highest weight

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<sup>6</sup>The usual definition of good filtration requires that each subquotient is a dual of a Weyl module (i.e.  $m_k = 1$ ). However, our choice does not modify the notion of *module having a good filtration*.

in  $H^0(U \times U, \mathcal{F}_k K[G]/\mathcal{F}_{k-1} K[G])$  is of the form  $(\mu_k, -w_0 \mu_k)$  for some  $\mu_k \in P^+$  and has multiplicity one. Hence from Lemma 4.2, we get:

$$\dim \mathcal{F}_k K[G] \leq \sum_{j \leq k} (\dim \nabla(\mu_j))^2.$$

Clearly, the filtration  $\mathcal{F}$  could have been defined in characteristic zero as well. For  $\lambda \in P^+$ , denote by  $L_{\mathbf{C}}(\lambda)$  the simple  $G_{\mathbf{C}}$ -module with highest weight  $\lambda$ . By Peter-Weyl Theorem, we have  $\mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}] = \bigoplus_{j \leq k} L_{\mathbf{C}}(\mu_j) \otimes L_{\mathbf{C}}(-w_0 \mu_j)$ . By semi-continuity theorem<sup>7</sup>, we have  $\dim \mathcal{F}_k K[G] \geq \dim_{\mathbf{C}} \mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}]$ . Thus, we get:

$$\dim \mathcal{F}_k K[G] \geq \sum_{j \leq k} (\dim L_{\mathbf{C}}(\mu_j))^2.$$

By Weyl character formula, the previous inequalities are equalities, and the theorem follows from Lemma 4.2. Q.E.D.

Let  $M, N$  be two  $G$ -modules. Recall that  $Ext_G^1(M, N)$  is the classifying group of all extensions of  $G$ -modules:  $0 \rightarrow N \rightarrow X \rightarrow M \rightarrow 0$ .

**Lemma 4.4:** *Let  $M$  be a  $G$ -module and let  $\lambda \in P^+$ .*

(i) *if  $Ext_G^1(\Delta(\lambda), M) \neq 0$ , there is a weight  $\nu$  of  $M$  with  $\nu > \lambda$ .*

(ii) *if  $Ext_G^1(M, \nabla(\lambda)) \neq 0$ , there is a weight  $\nu$  of  $M$  with  $\nu > \lambda$ .*

*Proof:* Given one of its highest weight vector  $v_\lambda$ , the Weyl module  $\Delta(\lambda)$  satisfies the following universal property: for any module  $M$  and  $v \in H^0(U, M)_\lambda$ , there is a  $G$ -morphism  $\phi: \Delta(\lambda) \rightarrow M$  with  $\phi(v_\lambda) = v$ . Assume  $Ext_G^1(\Delta(\lambda), M) \neq 0$  and consider a non trivial extension of  $G$ -modules:  $0 \rightarrow M \rightarrow X \rightarrow \Delta(\lambda) \rightarrow 0$ . Let  $v \in X_\lambda$  be a lifting of  $v_\lambda$ . As the extension does not split,  $v$  is not  $U$ -invariant. Therefore the  $U$ -module generated by  $v$  contains a weight  $\nu > \lambda$ , what proves Assertion (i). The proof of Assertion (ii) is similar. Q.E.D.

The following Lemma is a very special case the Cline-Parshall-Scott-van der Kallen vanishing theorem (see the Appendix).

**Lemma 4.5:** [CPSV] *For any  $\lambda, \mu \in P^+$ , we have:*

$$Ext_G^1(\Delta(\lambda), \nabla(\mu)) = 0.$$

*Proof:* Both inequalities  $\mu > \lambda$  and  $\lambda > \mu$  cannot hold simultaneously. In any case, the claim follows from Assertion (i) or (ii) of the previous lemma. Q.E.D.

For  $M \in Mod(G)$  and  $\lambda \in P^+$ , set  $h^1(M, \lambda) = \dim Ext_G^1(\Delta(\lambda), M)$ .

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<sup>7</sup>The filtration  $\mathcal{F}$  is defined over  $\mathbf{Z}[G_{\mathbf{Z}}]$  as well. We have  $\mathbf{C} \otimes \mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}] = \mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}]$ , hence  $\mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}]$  is torsion free of rank  $\dim \mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}]$ . Moreover  $\mathbf{Z}[G_{\mathbf{Z}}]/\mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}]$  is torsion free, hence the map  $K \otimes \mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}] \rightarrow \mathcal{F}_k K[G]$  is one-to-one.

Thus we get  $\dim_K \mathcal{F}_k K[G] \geq \text{rk}_{\mathbf{Z}} \mathcal{F}_k \mathbf{Z}[G_{\mathbf{Z}}] = \dim_{\mathbf{C}} \mathcal{F}_k \mathbf{C}[G_{\mathbf{C}}]$ .

**Proposition 4.6:** *Let  $M$  be a  $G$ -module. We have:*

$$\begin{aligned} \text{ch } M &\leq \sum_{\lambda \in P^+} h^0(M, \lambda) \text{ch } \nabla(\lambda), \\ \text{ch } M &\geq \sum_{\lambda \in P^+} (h^0(M, \lambda) - h^1(M, \lambda)) \text{ch } \nabla(\lambda). \end{aligned}$$

*Proof:* The first inequality follows from Lemma 4.2. Prove the second inequality. The structural morphism  $M \rightarrow M \otimes K[G]$  is indeed a  $G$ -embedding  $M \subset K[G]^m$ , where  $m = \dim M$ . Using Theorem 4.3, there is a module with a good filtration  $X$  such that  $M \subset X$ : indeed a convenient choice is  $X = \mathcal{F}_k K[G]^m$  for  $k$  big enough. Then we get a short exact sequence:  $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$ . By Lemma 4.5, we have  $\text{Ext}_G^1(\Delta(\lambda), X) = 0$ , for any  $\lambda \in P^+$ . Therefore we get a four term exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Hom}_G(\Delta(\lambda), M) \rightarrow \text{Hom}_G(\Delta(\lambda), X) \\ \rightarrow \text{Hom}_G(\Delta(\lambda), N) \rightarrow \text{Ext}_G^1(\Delta(\lambda), M) \rightarrow 0 \end{aligned}$$

for any  $\lambda \in P^+$ . So we get:

$$(4.6.1) \quad h^0(X, \lambda) - h^0(N, \lambda) = h^0(M, \lambda) - h^1(M, \lambda).$$

Moreover by Lemma 4.2, we get:

$$(4.6.2) \quad \text{ch } X = \sum_{\lambda \in P^+} h^0(X, \lambda) \text{ch } \nabla(\lambda),$$

$$(4.6.3) \quad \text{ch } N \leq \sum_{\lambda \in P^+} h^0(N, \lambda) \text{ch } \nabla(\lambda).$$

As  $\text{ch } M = \text{ch } X - \text{ch } N$ , the relations (4.6.1 – 3) imply the required inequality. Q.E.D.

In the following statement, the equivalence of (ii) and (iii) is often called the Donkin criterion for good filtrations [D1].

**Corollary 4.7:** *For any  $G$ -module  $M$ , the following three properties are equivalent:*

- (i) *the canonical filtration of  $M$  is a good filtration,*
- (ii)  *$M$  has a good filtration,*
- (iii)  *$\text{Ext}_G^1(\Delta(\lambda), M) = 0$  for all  $\lambda \in P^+$ .*

*Moreover, if  $M$  has a good filtration, any direct summand of  $M$  has a good filtration.*

*Proof:* (i) $\Rightarrow$ (ii) is obvious, (ii) $\Rightarrow$ (iii) follows from Lemma 4.5, and (iii) $\Rightarrow$ (i) follows from Lemmas 4.2 and 4.5. Moreover, if  $M$  has a good filtration then  $\text{Ext}_G^1(\Delta(\lambda), N) = 0$  for any direct summand  $N$  and therefore  $N$  has a good filtration. Q.E.D.

We will state a criterion of different nature for the existence of a good filtration for a commutative  $G$ -algebra. This criterion is based on the notion of isotypical  $G$ -algebras. However, in Section 6 we will use also this notion for  $B$ -algebras and therefore we will give here the most general definition. Until the end of the section, we will no more

assume that the  $B$ -modules or  $G$ -modules are finite dimensional. Let  $\mu \in \mathbf{Q} \otimes_{\mathbf{Z}} P$ . A  $B$ -module  $M$  is called  $\mu$ -isotypical if  $\mu$  is the unique weight of  $H^0(U, M)$ . It follows from the definition that if  $\mu$  is not a weight, i.e. if  $\mu \in \mathbf{Q} \otimes_{\mathbf{Z}} P \setminus P$ , a  $\mu$ -isotypical module is zero. Assume now that  $\lambda \in P^+$  and that  $M$  is a  $\lambda$ -isotypical  $G$ -module  $M$ . Then set  $M^- = M_{w_0\lambda}$ . The  $H$ -equivariant projection  $M \rightarrow M^-$  is actually  $B$ -equivariant (here  $M^-$  is viewed as a  $B$ -module with a trivial  $U$ -action; this should not be confused with the natural embedding  $M_{w_0\lambda} \rightarrow M$  which is  $B^-$ -equivariant). It induces a natural map  $M \rightarrow DM^-$ . It is clear that this map is injective,  $H^0(U, M) = H^0(U, DM^-)$ , and  $DM^-$  is a direct sum of modules isomorphic to  $\nabla(\lambda)$ .

Let  $x \in \mathbf{Q} \otimes E(P)$  be a real number. By definition, a graded  $B$ -algebra  $A = \bigoplus_{n \geq 0} A_n$  is called *isotypical of slope  $x$*  if the  $B$ -module  $A_n$  is a  $n\mu$ -isotypical for all  $n \geq 0$ , where  $\mu \in \mathbf{Q} \otimes_{\mathbf{Z}} P$  is uniquely defined by  $E(\mu) = x$ . It will be convenient to extend this definition for any real number  $x$ , by requiring that an isotypical of slope  $x$  graded  $B$ -algebra  $A$  is zero if  $x \notin \mathbf{Q} \otimes E(P)$ . Assume now that  $x = E(\lambda)$  for some  $\lambda \in P^+$  and that  $A$  is a graded  $G$ -algebra which is isotypical of slope  $x$ . Then set  $A^- = \bigoplus_{n \geq 0} (A_n)_{nw_0\lambda}$ . As before  $A^-$  is viewed as a  $B$ -module with a trivial  $U$ -action and the natural projection  $A \rightarrow A^-$  is a morphism of  $B$ -algebras (this should not be confused with the natural  $B^-$ -equivariant embedding  $A^- \rightarrow A$ ). By the universal property of the induction functor  $D$ , there is a natural morphism  $DA^- \otimes DA^- \rightarrow D(A^- \otimes A^-)$ . Therefore  $DA^-$  carries a natural structure of  $G$ -algebra. As the map  $A \rightarrow DA^-$  is a one-to-one morphism of  $G$ -algebras, we will consider  $A$  as a subalgebra of  $DA^-$ . We have  $H^0(U, A) = H^0(U, DA^-)$ , and therefore  $DA^-$  is  $x$ -isotypical as well. Moreover  $DA^-$  is commutative (respectively associative, unitary, reduced, without zero divisors) whenever  $A^-$  is commutative (respectively, whenever it satisfies the same property). Therefore  $DA^-$  is commutative, ... if and only if  $A$  is commutative, ...

We will see later (see the proof of Theorem 6.2) that any graded  $G$ -algebra has a filtration whose any subquotient is an  $x$ -isotypical algebra for some  $x$ . Therefore the next proposition is a good filtration criterion for any commutative associative  $G$ -algebra.

**Proposition 4.8:** [M2] *Let  $\lambda \in P^+$  and set  $x = E(\lambda)$ . Let  $A$  be a commutative associative graded  $G$ -algebra which is isotypical of slope  $x$ . If  $A_1$  is not a direct sum of  $\nabla(\lambda)$ , there is an element  $f \in DA^-$  such that  $f \notin A$  but  $f^p \in A$ .*

*Proof:* Without loss of generality, we can assume  $A_0 = K$  and  $A$  is unitary. In what follows, we will only consider unitary subalgebras of

$DA^-$ , what allows us to use the terminology of the algebraic geometry. By definition,  $DA_1^-$  is a direct sum of  $\nabla(\lambda)$  and the hypothesis is indeed equivalent to  $A_1 \neq DA_1^-$ . Therefore  $DA_1^-$  contains a submodule  $M$  isomorphic to  $\nabla(\lambda)$  such that  $M$  is not contained in  $A_1$ . Let  $y \in M_{w_0\lambda}$  be a non-zero vector and let  $Y$  be the subalgebra of  $A^-$  generated by  $y$ . Note that  $DY$  is a graded subalgebra of  $DA^-$  and set  $A' = DY \cap A$ . We have  $Y = A'^-$ ,  $A_1' \neq M = DA_1'$ , and we will prove that there is  $f \in DA'^- \setminus A'$  with  $f^p \in A'$ . Therefore we can assume that  $A' = A$ .

Indeed we will prove the following assertion:

(\*) for any  $x \in DA_1^-$ ,  $x^{mp^n}$  belongs to  $DA$  for all  $n, m \gg 0$ .

However this is enough: if  $x \in DA_1^- \setminus A_1$  satisfies (\*), then  $f = x^N$  satisfies the conclusion of the proposition for some  $N \geq 1$ .

Assume first that  $A^-$  is not isomorphic to the polynomial ring  $K[y]$ . As  $A^-$  is a graded algebra, we have  $A^- = K[y]/(y^N)$  for some  $N > 1$ . Thus  $A_n = 0$  for any  $n \geq N$ , and any  $x \in DA_1^-$  satisfies (\*). The proposition is proved in this case. Therefore we can assume from now on that  $A^-$  is isomorphic to  $K[y]$ . Thus, the algebra  $DA^-$  is isomorphic to  $\bigoplus_{n \geq 0} \Gamma(G/B, \mathcal{L}^{\otimes n})$ , where  $\mathcal{L} = \mathcal{L}(w_0\lambda)$ . In particular the algebra  $DA^-$  is an integrally closed domain.

First we claim that the algebras  $A, DA^-$  are finitely generated and the morphism  $\text{Spec } DA^- \rightarrow \text{Spec } A$  is finite and bijective. Let  $\mathcal{E} \subset A$  be the subalgebra generated by  $\bigoplus_w M_{w\lambda}$  and let  $\mathcal{E}^+$  be its maximal homogenous ideal. As  $\mathcal{L}$  is  $G$ -equivariant and generated by its global sections, the algebra  $DA^-$  is finitely generated<sup>8</sup>. The spectrum of  $DA^-$  can be identified with the subset  $G.v^+ \cup \{0\}$  of  $M^*$ , where  $v^+$  is any highest weight vector of  $M^* \simeq \Delta(-w_0\lambda)$ . By Bruhat decomposition, for any  $g \in G$ , we can find  $w \in W$  such that  $\langle wy|g.v^+ \rangle \neq 0$  (here  $\langle | \rangle$  denotes the pairing between  $M$  and  $M^*$ ). As  $wy \in \mathcal{E}^+$ , the radical of the ideal  $\mathcal{E}^+.DA^-$  is the maximal homogenous ideal of  $DA^-$  and therefore  $DA^-$  is a finitely generated  $\mathcal{E}$ -module. As  $M_{w\lambda} \subset A_1$  for any  $w \in W$ , we have  $\mathcal{E} \subset A$ . Therefore  $DA^-$  is finitely generated and the extension  $A \subset DA^-$  is finite. In particular  $\text{Spec } DA^- \rightarrow \text{Spec } A$  is onto. Let  $\bar{v}^+$  be a highest weight vector of  $\bar{A}_1^*$ . The stabilizers in  $G$  of  $K.\bar{v}^+$  and  $K.v^+$  are the same parabolic subgroups. Hence the morphism  $\text{Spec } DA^- \rightarrow \text{Spec } A$  is bijective and the claim is proved.

Let  $\bar{A}$  be the integral closure of  $A$ . We have  $A \subset \bar{A} \subset DA^-$  and  $\bar{A}$  is a graded subalgebra of  $DA^-$ . Note by  $\sigma : \text{Spec } DA^- \rightarrow \text{Spec } \bar{A}$  and by  $\pi : \text{Spec } \bar{A} \rightarrow \text{Spec } A$  the corresponding morphisms. It follows from

<sup>8</sup>Using a less elementary approach, one can prove that  $DA^-$  is generated by  $M$ . However, we will not use this fact.

the previous claim that the morphisms  $\pi$  and  $\sigma$  are finite and bijective. Moreover  $\pi$  is an isomorphism on some non-empty open subset. By  $G$ -invariance,  $\pi$  is an isomorphism outside  $\{0\}$ . Therefore we have:

$$(i) \quad A_n = \overline{A}_n \text{ for any } n \gg 0.$$

Let  $K_A, K_{DA^-}$  be the fraction fields of  $A$  and  $DA^-$ . The extension  $K_A \subset K_{DA^-}$  is purely inseparable, because its separability degree is the cardinal of the fibers of  $\sigma$ . Thus there is a power  $q$  of  $p$  such that  $K_{DA^-}^q \subset K_A$ . Hence we get:

$$(ii) \quad (DA^-)^q \subset \overline{A}.$$

It is clear that Assertion (\*) follows from Assertions (i) and (ii). Therefore Proposition 4.8 is proved. Q.E.D.

**5. Canonical Frobenius splittings.** The notion of Frobenius splittings is due to Metha, Ramanan and Ramanathan [MR][RR]. Somehow, their beautiful idea originated in Andersen-Haboush formula 3.1. Here we will not follow their original paper, based on Cartier operators [C]. Instead we will follow another approach, based on the characteristic  $p$  version of the change of variables formula of [M1]: this approach is very elementary and more intuitive. The notion of canonical Frobenius splittings comes from [M2].

The basic change of variables formula:

$$\int_X \phi dx_1 \wedge \dots \wedge dx_n = \int_X \phi \det\left(\frac{\partial x_i}{\partial y_j}\right) dy_1 \wedge \dots \wedge dy_n$$

is valid in the context of the Differential Geometry. Here  $X$  is a  $n$ -dimensional oriented manifold,  $(x_1, \dots, x_n)$  and  $(y_1, \dots, y_n)$  are two systems of parameters and  $\phi$  is any test function. As the integral is somehow the inverse of the derivation, we can write this formula as follows:

$$\left(\frac{\partial^n}{\partial x_1 \dots \partial x_n}\right)^{0-1} = \left(\frac{\partial^n}{\partial y_1 \dots \partial y_n}\right)^{0-1} \circ [\det\left(\frac{\partial y_i}{\partial x_j}\right)]^{0-1}.$$

Of course, this is only a formal identity. The same equality holds in characteristic  $p$ , where 0 is changed to  $p$ : this is why we wrote  $0 - 1$  instead of  $-1$ . Before stating the theorem, we need to comment the notations. It is usual to write differential operators under the form  $\sum_{\alpha} f_{\alpha} \partial^{\alpha}$ , where the  $f_{\alpha}$  are functions and  $\partial^{\alpha}$  are partial derivatives. However, it will be more convenient here to write functions on the right side of partial derivatives. For example, the differential operator  $d/dx \circ f$  should be understood as the operator sending an arbitrary function  $\phi$  to  $d/dx(f\phi)$ : with the usual notation, this differential operator is denoted by  $f' + f d/dx$ .

**Theorem 5.1:** ([M1]) *Let  $X$  be a smooth algebraic variety over  $K$  of dimension  $n$ , and let  $(x_1, \dots, x_n)$  and  $(y_1, \dots, y_n)$  be two systems of parameters on some open subset  $U$ . Then as differential operators, we have:*

$$\left(\frac{\partial^n}{\partial x_1 \dots \partial x_n}\right)^{p-1} = \left(\frac{\partial^n}{\partial y_1 \dots \partial y_n}\right)^{p-1} \circ [\det\left(\frac{\partial y_i}{\partial x_j}\right)]^{p-1}.$$

See the proof in [M1]. Let  $X$  be an affine variety and set  $A = K[X]$ . Note that  $A^p$  is a subring of  $A$ . A  $F$ -map is an  $A^p$ -linear map  $\sigma : A \rightarrow A^p$ . Such a map is called a *Frobenius splitting* if  $\sigma(1) = 1$ , i.e. if  $\sigma$  is a splitting of the embedding  $A^p \subset A$ . The localization relative to a non-zero element  $f \in A$  is the same as the localization relative to  $f^p$ . Hence any  $F$ -map  $\sigma$  commutes with localizations and it defines a morphism of  $\mathcal{O}_X^p$ -sheaves  $\sigma : \mathcal{O}_X \rightarrow \mathcal{O}_X^p$ . Therefore the notion of  $F$ -maps and Frobenius splittings can be defined for any reduced scheme, not only for an affine variety<sup>9</sup>. By definition, a *Frobenius split scheme* is a reduced scheme  $X$  which admits a Frobenius splitting. For any reduced scheme  $X$  denote by  $\mathcal{FM}_X$  the sheaf of  $F$ -maps, i.e. the sheaf of  $\mathcal{O}_X^p$ -linear maps from  $\mathcal{O}_X$  to  $\mathcal{O}_X^p$ . Also set  $FM(X) = \Gamma(X, \mathcal{FM}_X)$ . We will consider  $\mathcal{FM}_X$  as a sheaf of  $\mathcal{O}_X$ -modules (by multiplication at the source).

Let us compute the sheaf  $\mathcal{FM}_X$  for a smooth variety  $X$  of dimension  $n$ . Let  $x_1, \dots, x_n$  be a system of parameters at a point  $x \in X$ . We have  $\left(\frac{\partial}{\partial x_i}\right)^m x_i^n = n(n-1) \dots (n-m+1)x_i^{n-m}$ . Thus we get:

(i)  $\left(\frac{\partial}{\partial x_i}\right)^p = 0,$

(ii) A function  $f \in \mathcal{O}_{X,x}$  belongs to  $\mathcal{O}_{X,x}^p$  if and only if  $\frac{\partial}{\partial x_i} f = 0, \forall i$ .

Hence the image of the differential operator  $\left(\frac{\partial^n}{\partial x_1 \dots \partial x_n}\right)^{p-1}$  consist of  $p$ -power only. It follows easily that any  $F$ -map  $\sigma : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}^p$  is a differential operator of the form  $\left(\frac{\partial^n}{\partial x_1 \dots \partial x_n}\right)^{p-1} \circ \phi$ , where  $\phi \in \mathcal{O}_{X,x}$ . Therefore  $\mathcal{FM}_X$  is invertible. Moreover the previous theorem gives a formula for the transition functions of the line bundle define by  $\mathcal{FM}_X$ . Thus we get:

**Theorem 5.2:** (Metha-Ramanathan formula [MR]) *For a smooth variety  $X$ , we have  $\mathcal{FM}_X \simeq \mathcal{K}_X^{\otimes(1-p)}$ , where  $\mathcal{K}_X$  is the sheaf of top differential forms on  $X$ .*

*Examples:*

Example (a): let  $X = \text{Spec } A$  be a smooth affine variety. The  $A^p$ -module  $A/A^p$  is projective. Thus the exact sequence of  $A^p$ -modules

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<sup>9</sup>In the original framework [MR], the notion of Frobenius splittings is defined for any scheme, including non-reduced: the definition given here is different from the original one, but it is essentially equivalent whenever the scheme is reduced (moreover a Frobenius split scheme is automatically reduced, see[Ra]). The present approach is intuitive and more adapted to the notion of canonical Frobenius splittings.

$$0 \rightarrow A^p \rightarrow A \rightarrow A/A^p \rightarrow 0$$

splits. Therefore  $X$  is Frobenius split.

Example (b): let  $X$  be a toric variety of dimension  $n$ . By definition there exists a torus  $T$  acting on  $X$  such that  $X$  contains an open orbit isomorphic to  $T$ . There is a basis  $x_1, \dots, x_n$  of the Lie algebra of  $T$  such that  $x_i^p = x_i$  for all  $i$ . Set  $\sigma = \prod_{1 \leq i \leq n} (1 - x_i^{p-1})$ . It is clear that  $\sigma(K(T)) = K(T)^p$ , where  $K(T)$  is the field of rational functions of  $T$ . If  $X$  is normal, we have  $\mathcal{O}_X^p = \mathcal{O}_X \cap K(T)^p$ . Therefore, if  $X$  is normal,  $\sigma$  is a Frobenius splitting of  $X$ . For example, the projective space  $\mathbf{P}^n$  is Frobenius split. Let us give a concrete example: write  $\mathbf{P}^1 = \text{Spec } K[z] \cup \text{Spec } K[z^{-1}]$ , let  $T = K^*$  be the one dimensional torus which acts on  $\mathbf{P}^1$  by  $t : z \mapsto tz$  for any  $t \in K^*$ . Then the corresponding Frobenius splitting  $\sigma$  is uniquely defined by  $\sigma(z^n) = 0$  if  $n$  is not divisible by  $p$ , and  $\sigma(z^n) = z^n$  otherwise.

Example (c): we will determine when a complete smooth curve  $X$  of genus  $g$  is Frobenius split. By the previous example,  $X$  is Frobenius split if  $g = 0$ . If  $g \geq 2$ , we have  $FM(X) = 0$  by Theorem 5.2 and therefore  $X$  is not Frobenius split. If  $g = 1$ , the space  $\mathfrak{t}$  of vector fields on  $X$  has dimension 1, and we need to consider two cases. If  $X$  is not supersingular, there is  $x \in \mathfrak{t}$  with  $x^p = x$  and the differential operator  $1 - x^{p-1}$  is a Frobenius splitting. If  $X$  is supersingular, we have  $x^p = 0$  for any  $x \in \mathfrak{t}$ , and it is easy to prove that  $X$  is not Frobenius split.

Let  $X$  be a variety, and let  $\mathcal{L}$  be an invertible sheaf. Denote by  $\mathcal{L}^p$  the image of  $p$ -power map  $\Sigma_p : \mathcal{L} \rightarrow \mathcal{L}^{\otimes p}$ . As  $X$  is reduced, the map  $\Sigma_p : \mathcal{L} \rightarrow \mathcal{L}^p$  is an isomorphism of sheaves of abelian groups, and we have  $\mathcal{L}^{\otimes p} = \mathcal{O}_X \otimes_{\mathcal{O}_X^p} \mathcal{L}^p$ . Hence any  $F$ -map  $\sigma$  induces a map  $\sigma_{\mathcal{L}} : \mathcal{L}^{\otimes p} \rightarrow \mathcal{L}^p$ , namely  $\sigma_{\mathcal{L}} = \sigma \otimes 1_{\mathcal{L}^p}$ . We will also denote by  $\sigma_{\mathcal{L}}^{1/p} : \mathcal{L}^{\otimes p} \rightarrow \mathcal{L}$  the map  $\Sigma_p^{-1} \circ \sigma_{\mathcal{L}}$ . Therefore  $\sigma_{\mathcal{L}}$  is an  $\mathcal{O}_X^p$ -morphism and we have  $\sigma_{\mathcal{L}}^{1/p}(f^p x) = f \sigma_{\mathcal{L}}^{1/p}(x)$  for all  $f \in \mathcal{O}_X, x \in \mathcal{L}^{\otimes p}$ .

For a  $B$ -variety  $X$ , there are usually no non-trivial  $B$ -invariant  $F$ -maps. For example by Theorem 5.2,  $FM(G/B) \simeq \nabla(2(p-1)\rho)$  and any  $B$ -invariant  $F$ -map on  $G/B$  is zero. Therefore we consider a weaker form of  $B$ -invariance, namely the notion of canonical  $F$ -maps. From this viewpoint the terminology "almost  $B$ -invariant  $F$ -maps" could have been better. However the terminology "canonical" is explained by Proposition 5.5: e.g. it is easy to show that  $G/B$  admits a *unique* canonical Frobenius splitting. A  $F$ -map  $\sigma$  is called *canonical* if and only if  $\sigma$  is  $H$ -invariant and  $Ad(e_i^{(n)})(\sigma) = 0$  for any  $i \in I$  and any  $n \geq p$ . Recall that the adjoint action  $Ad$  is defined in the context of Hopf algebras, i.e.  $Ad(e_i^{(n)})(\sigma) = \sum_{a+b=n} (-1)^a e_i^{(a)} \circ \sigma \circ e_i^{(b)}$ . One should not confuse

Ad with the commutator: e.g. there are canonical  $F$ -maps  $\sigma$  such that  $[\sigma, e_i^{(p+1)}] \neq 0$ . However, we have:

**Lemma 5.3:** (Commutation relations [M2]) *Let  $X$  be a  $B$ -variety and let  $\sigma \in FM(X)$  be an  $H$ -invariant  $F$ -map. Then  $\sigma$  is canonical if and only if we have:  $e_i^{(pn)} \circ \sigma_{\mathcal{L}} = \sigma_{\mathcal{L}} \circ e_i^{(pn)}$ , for any  $i \in I$ , any  $n > 0$  and any  $B$ -equivariant invertible sheaf  $\mathcal{L}$ .*

*Proof:* First assume that  $\sigma$  is canonical, and let  $\mathcal{L}$  be any  $B$ -equivariant invertible sheaf. As any section of  $\mathcal{L}^p$  is a  $p$ -power, we have  $e_i \mathcal{L}^p = 0$ . Moreover we have  $e_i^{(m)} = e_i^{(pa)} e_i^{(b)}$ , if  $m = pa + b$  where  $0 \leq b < p$ . So we have:  $e_i^{(m)} \circ \sigma_{\mathcal{L}} = 0$  for any  $m$  not divisible by  $p$ , and the hypothesis  $Ad(e_i^{(pn)})(\sigma) = 0$  can be restated as:

$$\sum_{a+b=n} (-1)^a e_i^{(pa)} \circ \sigma_{\mathcal{L}} \circ e_i^{(pb)} = 0, \forall n > 0.$$

We will prove now the commutation relation  $e_i^{(pn)} \circ \sigma_{\mathcal{L}} = \sigma_{\mathcal{L}} \circ e_i^{(pn)}$  by induction over  $n > 0$ . Consider first the case where  $n$  is a power of  $p$ . Then the previous relation can be written as:

$$e_i^{(pn)} \circ \sigma_{\mathcal{L}} - \sigma_{\mathcal{L}} \circ e_i^{(pn)} + \sum_{0 < a < n} (-1)^a e_i^{(pa)} \circ \sigma_{\mathcal{L}} \circ e_i^{(p(n-a))} = 0.$$

By the induction hypothesis, we have:  $e_i^{(pa)} \circ \sigma_{\mathcal{L}} \circ e_i^{(pb)} = e_i^{(pa)} \circ e_i^{(pb)} \circ \sigma_{\mathcal{L}} = \binom{n}{a} e_i^{(pn)} \circ \sigma_{\mathcal{L}} = 0$ , for any  $a, b > 0$  with  $a + b = n$ ; indeed as  $n$  is a power of  $p$ ,  $\binom{n}{a} = 0$  modulo  $p$  for all  $a \neq 0, n$ . Therefore  $e_i^{(pn)} \circ \sigma_{\mathcal{L}} = \sigma_{\mathcal{L}} \circ e_i^{(pn)}$ . When  $n$  is not a power of  $p$ , we can find  $a$ , with  $0 < a < n$  such that  $\binom{n}{a} \neq 0$  modulo  $p$ , hence by induction we get:

$$\begin{aligned} e_i^{(pn)} \circ \sigma_{\mathcal{L}} &= \binom{n}{a}^{-1} e_i^{(pa)} \circ e_i^{(p(n-a))} \circ \sigma_{\mathcal{L}} \\ &= \binom{n}{a}^{-1} \sigma_{\mathcal{L}} \circ e_i^{(pa)} \circ e_i^{(p(n-a))} \\ &= \sigma_{\mathcal{L}} \circ e_i^{(pn)}. \end{aligned}$$

Therefore the commutation relations are proved by induction.

Conversely, assume that the commutation relations hold. In particular they hold for the trivial line bundle  $\mathcal{O}_X$ , for which we have  $\sigma_{\mathcal{L}} = \sigma$ . By the same proof as before, one proves  $Ad(e_i^{(n)})(\sigma) = 0$  whenever  $n > 1$  is a power of  $p$ . For any  $n \geq p$ , there are two integers  $a, b$  with  $a + b = n$ ,  $b > 1$ ,  $b$  is a power of  $p$  and  $\binom{n}{b} \neq 0$  modulo  $p$ . Hence  $Ad(e_i^{(n)})(\sigma) = \binom{n}{b}^{-1} Ad(e_i^{(a)}) \circ Ad(e_i^{(b)})(\sigma) = 0$ . Q.E.D.

Let us explain why a canonical  $F$ -map is "almost  $B$ -invariant":

**Proposition 5.4:** (M2]) *Let  $X$  be a  $B$ -variety, let  $\mathcal{L}$  be a  $B$ -equivariant invertible sheaf and let  $\sigma$  be a canonical  $F$ -map. Then the induced map  $\sigma_{\mathcal{L}}^{1/p} : \Gamma(X, \mathcal{L}^{\otimes p}) \rightarrow \Gamma(X, \mathcal{L})$  sends  $B$ -modules to  $B$ -modules.*

*Proof:* By definition, the map  $\sigma_{\mathcal{L}}^{1/p} : \Gamma(X, \mathcal{L}^{\otimes p}) \rightarrow \Gamma(X, \mathcal{L})$  is the composition of  $\sigma_{\mathcal{L}} : \Gamma(X, \mathcal{L}^{\otimes p}) \rightarrow \Gamma(X, \mathcal{L}^p)$  and of the inverse of  $\Sigma_p : \Gamma(X, \mathcal{L}) \rightarrow \Gamma(X, \mathcal{L}^p)$ . We have  $e_i^{(pn)} \circ \Sigma_p = \Sigma_p \circ e_i^{(n)}$ , for any  $n \geq 0$  and any  $i \in I$ . From the commutation relations 5.3, we get  $e_i^{(n)} \circ \sigma_{\mathcal{L}}^{1/p} = \sigma_{\mathcal{L}}^{1/p} \circ e_i^{(pn)}$ . It follows that  $\sigma_{\mathcal{L}}^{1/p}$  sends  $U$ -modules to  $U$ -modules. Moreover by  $H$ -invariance of  $\sigma$ ,  $\sigma_{\mathcal{L}}^{1/p}$  sends  $H$ -modules to  $H$ -modules. Hence  $\sigma_{\mathcal{L}}$  sends  $B$ -modules to  $B$ -modules. Q.E.D.

*Remark:* If  $X$  is a  $G$ -variety, and if  $\sigma$  is canonical, it follows from  $SL(2)$ -theory that we have  $Ad(f_i^{(n)})(\sigma) = 0$  for all  $i \in I$  and all  $n \geq p$ . Hence  $\sigma_{\mathcal{L}}$  sends also  $G$ -modules to  $G$ -modules.

For any  $B$ -variety  $X$ , denote by  $CFM(X)$  the set of canonical  $F$ -maps of  $X$ .

**Proposition 5.5:** (M2] *Let  $X$  a  $B$ -variety and set  $Y = G \times_B X$ .*

*If  $X$  has a canonical Frobenius splitting, then  $Y$  also admits a canonical Frobenius splitting.*

*More precisely, there is a natural map  $I_X : CFM(X) \rightarrow CFM(Y)$  which sends canonical Frobenius splittings of  $X$  to those of  $Y$ .*

*Proof:* Start with some notations used in the proof. Set  $\omega = (p-1)\rho$ ,  $St = \nabla(\omega)$  and let  $v^-$  be a non-zero lowest weight vector of the Steinberg module  $St$ . Denote by  $\mathfrak{u}$  is the Lie algebra of  $U$ , by  $u(\mathfrak{u})$  its restricted enveloping algebra and by  $\mathcal{H}_B$  the hyperalgebra of  $B$ . Let  $u$  be a non-zero element of the one dimensional vector space  $u(\mathfrak{u})_{2\omega}$  (see the proof of Theorem 2.6) and set  $v^+ = u.v^-$ . Then  $v^+$  is a non-zero highest weight vector of  $St$ . The choices of the elements  $u, v^-$  and of an isomorphism  $St \simeq \Gamma(G/B, \mathcal{L}(-\omega))$  are unique only up to a scalar. We will not care too much about this, but indeed the map  $I_X$  defined below is independent of these choices.

There is a natural identification  $\tau : CFM(X) \simeq Hom_B(St, FM(X) \otimes K(-\omega))$ . Indeed  $St$  is the  $\mathcal{H}_B$ -module generated by one vector of weight  $-\omega$ , namely  $v^-$ , and defined only by the relations  $e_i^{(n)}.v^- = 0$  for any  $i \in I$  and any  $n \geq p$ . Therefore any  $\sigma \in CFM(X)$  induces the  $B$ -equivariant map  $\tau(\sigma) : St \rightarrow FM(X) \otimes K(-\omega)$  which is uniquely defined by  $\tau(\sigma)(v^-) = \sigma \otimes 1$ .

Let  $X_1$  be the variety homeomorphic to  $X$  whose structure sheaf is  $\mathcal{O}_X^p$  and set  $Y_1 = G \times_B X_1$ . Thus  $Y_1$  is homeomorphic to  $Y$  and  $\mathcal{O}_{Y_1} = \mathcal{O}_{G/B} \otimes_{\mathcal{O}_{G/B}}^p \mathcal{O}_X^p$ . Denote by  $i : X_1 \rightarrow Y_1$  and by  $\pi : Y_1 \rightarrow G/B$  the natural maps. The functor  $i^*$  is an equivalence from the category of  $G$ -equivariant coherent sheaves on  $Y_1$  to the category of  $B$ -equivariant coherent sheaves on  $X_1$ . Hence let  $\mathcal{D}$  be its inverse. Set

$\mathcal{L} = \pi^* \mathcal{L}(-\omega)$ . Since  $\pi(X_1) = B/B$ , the sheaf  $i^* \mathcal{L}$  is  $B$ -isomorphic to  $\mathcal{O}_X^p \otimes K(-\omega)$ . Thus we have  $i^* \mathcal{H}om_{\mathcal{O}_{Y_1}}(\mathcal{O}_Y, \mathcal{L}) = \mathcal{H}om_{\mathcal{O}_{X_1}}(\mathcal{O}_X, i^* \mathcal{L}) = \mathcal{F}M_X \otimes K(-\omega)$ , and therefore  $\mathcal{H}om_{\mathcal{O}_{Y_1}}(\mathcal{O}_Y, \mathcal{L}) = \mathcal{D}(\mathcal{F}M_X \otimes K(-\omega))$ . Note that  $\Gamma(Y_1, \mathcal{D}\mathcal{M}) = D\Gamma(X_1, \mathcal{M})$  for any  $B$ -equivariant coherent sheaf  $\mathcal{M}$  on  $X_1$ . Hence, we have:

$$\mathcal{H}om_{\mathcal{O}_{Y_1}}(\mathcal{O}_Y, \mathcal{L}) = D(\mathcal{F}M(X) \otimes K(-\omega)).$$

Therefore, for any  $\sigma \in CFM(X)$ , the  $B$ -equivariant map  $\tau(\sigma) : St \rightarrow FM(X) \otimes K(-\omega)$  induces a  $G$ -equivariant map  $D\tau(\sigma) : St \rightarrow \mathcal{H}om_{\mathcal{O}_{Y_1}}(\mathcal{O}_Y, \mathcal{L})$ . Set  $\Theta_\sigma = D\tau(\sigma)(v^-)$ . By definition,  $\Theta_\sigma : \mathcal{O}_Y \rightarrow \mathcal{L}$  is a morphism of  $\mathcal{O}_{Y_1}$ -modules, we have  $Ad(e_i^{(n)})(\Theta_\sigma) = 0$  for all  $i \in I$  and all  $n \geq p$  and  $\Theta_\sigma$  has weight  $-\omega$ . It follows from Lemma 5.3 that  $\sigma(1)$  is a  $B$ -invariant function on  $X_1$ . Therefore it extends to a  $G$ -invariant function  $\overline{\sigma(1)}$  on  $Y_1$  and one easily check that  $\overline{\sigma(1)}$  belongs to  $\Gamma(Y, \mathcal{O}_Y^p)$ . One easily shows that  $\Theta_\sigma(1) = \overline{\sigma(1)}v^-$  (here  $v^-$  is viewed as a global section of  $\mathcal{L}$ ).

Next we define a  $\mathcal{O}_Y^p$ -linear map  $\Theta : \mathcal{L} \rightarrow \mathcal{O}_Y^p$ . Using Andersen-Haboush formula 3.1, we have  $\mathcal{L} = \mathcal{O}_Y^p \otimes St$ . Hence  $u.\mathcal{L} = \mathcal{O}_Y^p \otimes Kv^+$ , and we define  $\Theta$  by the requirement  $u.s = \Theta(s) \otimes v^+$  for any section  $s$  of  $\mathcal{L}$ . As  $u$  is a highest weight vector of  $u(\mathfrak{u})$ ,  $u$  is  $Ad(U)$ -invariant and therefore the map  $\Theta$  is  $U$ -equivariant, of weight  $\omega$ . Moreover  $\Theta(v^-) = 1$ .

For any  $\sigma \in CFM(X)$ , set  $I_X(\sigma) = \Theta \circ \Theta_\sigma$ . By definition,  $I_X(\sigma)$  is a  $F$ -map of  $Y$  and it follows from the previous remarks that  $I_X(\sigma)$  is canonical. Therefore,  $I_X$  is a well-defined map from  $CFM(X)$  to  $CFM(Y)$ . Moreover, we have  $I_X(\sigma)(1) = \overline{\sigma(1)}$ . Therefore,  $I_X(\sigma)$  is a Frobenius splitting of  $Y$  whenever  $\sigma$  is a Frobenius splitting of  $X$ . Q.E.D.

Let  $X$  be a variety, let  $Z$  be a subvariety of  $X$ , and let  $\mathcal{I}_Z$  be the its defining ideal. By assumption, subvarities are reduced, thus  $\mathcal{I}_Z^p = \mathcal{I}_Z \cap \mathcal{O}_X^p$ . Following [MR], we say that a  $F$ -map  $\sigma$  is *compatible* with  $Z$  if  $\sigma(\mathcal{I}_Z) \subset \mathcal{I}_Z^p$ . Therefore, a compatible  $F$ -map  $\sigma$  induces a  $F$ -map of  $Z$ .

**Lemma 5.6:** *Let  $X$  be a  $B$ -variety and let  $Z$  be a  $B$ -invariant subvariety. Let  $\sigma \in CFM(X)$  be compatible with  $Z$ . Then  $I_X(\sigma)$  is compatible with the subvariety  $\overline{BwB} \times_B Z$  for any  $w \in W$ . Moreover its restriction to  $X$  is  $\sigma$ .*

*Proof:* First prove that  $I_X(\sigma)$  is compatible with  $X$ . With the notations of the previous proof, this follows easily from the following two observations:

- (i) the map  $\Theta_\sigma : \mathcal{O}_{G \times_B X} \rightarrow \mathcal{L}$  is a lifting of the map  $\sigma : \mathcal{O}_X \rightarrow$

$$\mathcal{O}_X^p \simeq i^* \mathcal{L}.$$

(ii) the ideal  $\mathcal{I}_X$  defining  $X$  in  $G \times_B X$  is  $U$ -invariant and therefore it is invariant under the element  $u \in \mathfrak{u}(\mathfrak{u})$ .

Next prove the lemma by induction on  $w$ . Let  $\mathcal{I}_w$  be the ideal defining  $\overline{BwB} \times_B Z$ . For  $w = 1$ , the assertion is already proved. For  $w \neq 1$ , write  $w = s_i v$ , where  $s_i$  is a simple reflexion and  $v \leq w$ , and let  $\mathcal{H}_i$  be the subalgebra of  $\mathcal{H}_G$  spanned by all  $f_i^{(n)}$ . By induction, we can assume that  $I_X(\sigma)(\mathcal{I}_v) \subset \mathcal{I}_v^p$ . However, it is clear that  $\mathcal{I}_w$  is the biggest subsheaf of  $\mathcal{I}_v$  stable by  $\mathcal{H}_i$ . Hence it follows from the commutation relations 5.3 that  $I_X(\sigma)(\mathcal{I}_w) \subset \mathcal{I}_w^p$ . Thus  $I_X(\sigma)$  is compatible with  $\overline{BwB} \times_B Z$ . Q.E.D.

Indeed the map  $I_X$  defined in Proposition 5.5 establishes a bijection between the canonical Frobenius splittings of  $X$  and the canonical Frobenius splittings of  $G \times_B X$  which are compatible with  $X$ . In particular, for any parabolic group  $P \supset B$ ,  $\sigma$  induces a canonical Frobenius splitting of  $P \times_B X$ . Therefore, starting with the trivial Frobenius splitting of the point  $B/B$ , we get by induction:

**Corollary 5.7:** *Any Demazure variety admits a unique canonical Frobenius splitting which is compatible to any Demazure subvariety.*

**6. Tensor products of good filtrations.** In this section,  $B$ -modules and  $G$ -modules are of arbitrary dimension. As in Section 4, let  $E : P \rightarrow \mathbf{R}$  be an injective additive map with  $E(\alpha_i) > 0$  for all  $i \in I$ . We will use a different convention for the canonical filtrations defined in Section 4. For any  $B$ -module  $M$  and  $x \in \mathbf{R}$ , we denote by  $\mathcal{F}_x M$  the biggest  $B$ -submodule whose weights  $\mu$  all satisfy  $E(\mu) \leq x$  and set  $\mathcal{F}_x^- M = \cup_{y < x} \mathcal{F}_y M$ . For any  $B$ -module  $M$ ,  $\mathcal{F}_x M / \mathcal{F}_x^- M$  is either zero (if  $x \notin E(P)$ ) or it is  $\mu$ -isotypical (if  $x = E(\mu)$ , where  $\mu \in P$ ): therefore any  $B$ -module admits a filtration (in a generalized sense) whose subquotients are isotypical.

By definition, a  $\mathcal{H}_G - B$ -module is a  $B$ -module  $M$  endowed with a compatible  $\mathcal{H}_G$ -structure: the compatibility means that the  $\mathcal{H}_B$ -structure comes from the  $B$ -action. Denote by  $M^{int}$  the subspace of all vectors  $m \in M$  such that  $\dim \mathcal{H}_G \cdot m < \infty$ . It is easy to prove that  $M^{int}$  is the biggest submodule on which the  $\mathcal{H}_G$ -action integrates to a  $G$ -action. For  $\mu \in P$ , denote by  $I(\mu)$  the injective envelope of the  $B$ -module  $K(\mu)$ : indeed, we have  $I(\mu) \simeq K[B/H] \otimes K(\mu)$ . If we identify  $B/H$  with the open  $B$ -orbit of  $G/B$ , we have  $I(\mu) \simeq H^0(B/H, \mathcal{L}(w_0 \mu))$ , hence  $I(\mu)$  is a  $\mathcal{H}_G - B$ -module. Indeed its structure of  $\mathcal{H}_G - B$ -module is unique, as it is proved in Assertion (ii) of the next lemma.

**Lemma 6.1:** [M2] *Let  $\mu \in P$ ,  $\lambda \in P^+$  and  $x \in \mathbf{R}$ .*

(i) *For any injective  $B$ -module  $M$ ,  $\mathcal{F}_x M / \mathcal{F}_x^- M$  is again injective.*

(ii) *Let  $M, N$  be two  $\mu$ -isotypical  $\mathcal{H}_G - B$ -modules. Any  $B$ -morphism  $L : M \rightarrow N$  is  $\mathcal{H}_G$ -equivariant. In particular any  $\mu$ -isotypical  $B$ -module admits at most one structure of  $\mathcal{H}_G - B$ -module.*

(iii) *Let  $M$  be a  $\mathcal{H}_G - B$ -module. Then  $\mathcal{F}_x M$  and  $\mathcal{F}_x^- M$  are  $\mathcal{H}_G - B$ -submodules.*

(iv) *Let  $\overline{M}$  be a  $\lambda$ -isotypical injective  $B$ -module (therefore  $M$  is a  $\mathcal{H}_G - B$ -module) and let  $M \subset \overline{M}^{int}$  be a  $G$ -submodule. Then there is a canonical embedding of  $G$ -modules  $DM^- \subset \overline{M}^{int}$ .*

*Proof:* Proof of Assertion (i): we have  $\mathcal{F}_x I(\mu) = I(\mu)$  if  $E(\mu) \leq x$  and  $\mathcal{F}_x I(\mu) = 0$  otherwise. Any injective  $B$ -module is a direct sum of  $I(\mu)$ 's. Therefore for any injective  $B$ -module  $M$ ,  $\mathcal{F}_x M / \mathcal{F}_x^- M$  is again injective.

Proof of Assertion (ii): Assume that  $L$  is not  $\mathcal{H}_G$ -invariant. There is some  $i \in I$  and  $n > 0$  such that  $Ad(f_i^{(n)})(L) \neq 0$ . Moreover we can assume  $Ad(f_i^{(m)})(L) = 0$  for any positive integer  $m < n$ . Set  $L' = Ad(f_i^{(n)})(L)$  and let  $k > 0$ . The fact that  $Ad(e_j^{(k)})(L') = 0$  is obvious for any  $j \neq i$  and follows from  $SL(2)$ -computation for  $j = i$ . Hence  $L'$  is  $U$ -invariant of weight  $-n\alpha_i$ , and  $L'(M)$  is a  $B$ -submodule whose weights  $\nu$  all satisfy  $E(\nu) \leq x - nE(\alpha_i) < x$ . Therefore  $L' = 0$  what means that  $L$  is  $\mathcal{H}_G$ -invariant. It follows also that any  $\mu$ -isotypical  $B$ -module admits at most one structure of  $\mathcal{H}_G - B$ -module.

The proof of Assertion (iii) is the same as the proof of Lemma 4.1.

Proof of Assertion (iv): The  $B$ -module  $\overline{M}$  is a direct sum of  $I(\lambda)$ , therefore  $\overline{M} \simeq I(\lambda) \otimes C$  for some vector space  $C$ . It follows from Assertion (ii) that  $\overline{M} \simeq I(\lambda) \otimes C$  as  $\mathcal{H}_G - B$ -modules, and therefore  $\overline{M}^{int} \simeq \nabla(\lambda) \otimes C$  as  $G$ -modules. We have  $M_{w_0\lambda} \subset \overline{M}_{w_0\lambda}^{int}$ , and therefore we have  $DM^- \subset D(\overline{M}^{int})^- = \overline{M}^{int}$ . Q.E.D.

**Theorem 6.2:** (M2) *Let  $X$  be a  $G$ -variety which admits a canonical Frobenius splitting. For any  $G$ -equivariant line bundle  $\mathcal{L}$ , the  $G$ -module  $\Gamma(X, \mathcal{L})$  has a good filtration.*

*Proof:* First, we claim that we can assume the following additional hypothesis:  $X$  contains a  $B$ -invariant dense open subset  $\Omega$  such that  $\Omega \simeq B \times_H Y$  for some  $H$ -variety  $Y$ . Indeed set  $X' = G/B \times X$  and  $\mathcal{L}' = \mathcal{O}_{G/B} \otimes \mathcal{L}$ . We have  $X' \simeq G \times_B X$ , and  $\Omega = Bw_0B \times_B X$  is a dense open subset of  $X'$  which is isomorphic to  $B \times_H w_0X$ . Moreover, by Proposition 5.5,  $X'$  admits a canonical Frobenius splitting. Also, we

have  $\Gamma(X', \mathcal{L}') = \Gamma(X, \mathcal{L})$ . Therefore using  $X'$  and  $\mathcal{L}'$  instead of  $X$  and  $\mathcal{L}$ , we can assume that the additional hypothesis holds.

For any  $n \geq 0$ , set  $\bar{\mathcal{A}}_n = \Gamma(\Omega, \mathcal{L}^{\otimes n})$  and  $\mathcal{A}_n = \Gamma(X, \mathcal{L}^{\otimes n})$ . Also set  $\bar{\mathcal{A}} = \bigoplus_{n \geq 0} \bar{\mathcal{A}}_n$  and  $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ . Let  $\lambda \in P^+$  and set  $x = E(\lambda)$ . It is clear that  $\bar{\mathcal{A}}_x = \bigoplus_{n \geq 0} \mathcal{F}_{nx} \bar{\mathcal{A}}_n$  and  $\mathcal{A}_x = \bigoplus_{n \geq 0} \mathcal{F}_{nx} \mathcal{A}_n$  are subalgebras of  $\bar{\mathcal{A}}$ . Moreover  $\bar{\mathcal{I}}_x = \bigoplus_{n \geq 0} \mathcal{F}_{nx}^- \bar{\mathcal{A}}_n$  (respectively  $\mathcal{I}_x = \bigoplus_{n \geq 0} \mathcal{F}_{nx}^- \mathcal{A}_n$ ) is an ideal of  $\bar{\mathcal{A}}_x$  (respectively: an ideal of  $\mathcal{A}_x$ ). Thus set  $\bar{A} = \bar{\mathcal{A}}_x / \bar{\mathcal{I}}_x$  and  $A = \mathcal{A}_x / \mathcal{I}_x$ . By Lemma 6.1,  $\bar{\mathcal{I}}_x$  and  $\bar{\mathcal{A}}_x$  are  $\mathcal{H}_G - B$ -submodules of  $\bar{\mathcal{A}}$ , and therefore  $\bar{A}$  is an  $x$ -isotypical  $\mathcal{H}_G - B$ -algebra. Similarly, by Lemma 4.1,  $A$  is an  $x$ -isotypical  $G$ -algebra. As the natural map  $A \rightarrow \bar{A}$  is obviously one-to-one, we will consider  $A$  as a subalgebra of  $\bar{A}$ . The additional hypothesis  $\Omega \simeq B \times_H Y$  implies that the  $B$ -module  $\Gamma(\Omega, \mathcal{L}^{\otimes n})$  is injective for any  $n \geq 0$ . Therefore, by Lemma 6.1, there is the following series of inclusions of graded commutative algebras:

$$A \subset D A^- \subset \bar{A}^{int} \subset \bar{A}.$$

Now let  $\sigma$  be a canonical Frobenius splitting of  $X$ . Define a map  $\Theta : \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}}$  as follows: for  $f \in \bar{\mathcal{A}}_n$ , set  $\Theta(f) = \sigma_{\mathcal{L}^{\otimes n/p}}^{1/p}(f)$  if  $n$  is divisible by  $p$  and  $\Theta(f) = 0$  otherwise. By Proposition 5.4,  $\Theta$  sends  $B$ -modules to  $B$ -modules. As  $\Theta$  is  $H$ -invariant, we have  $\Theta(\bar{\mathcal{A}}_x) \subset \bar{\mathcal{A}}_x$  and  $\Theta(\bar{\mathcal{I}}_x) \subset \bar{\mathcal{I}}_x$ . Moreover by construction we have  $\Theta(\mathcal{A}) \subset \mathcal{A}$ . Hence  $\Theta$  induces a map  $\theta : \bar{A} \rightarrow \bar{A}$  such that:

- (i)  $\theta(f^p) = f$ , for all  $f \in \bar{A}$ ,
- (ii)  $\theta(A) \subset A$ .

It follows that any  $f \in D A^-$  such that  $f^p \in A$  belongs to  $A$ . Therefore, by Proposition 4.8,  $A_1$  is a direct sum of  $\nabla(\lambda)$ .

As  $x \in E(P^+)$  is arbitrary,  $\bar{A}_1 = \mathcal{F}_x \Gamma(X, \mathcal{L}) / \mathcal{F}_x^- \Gamma(X, \mathcal{L})$  can be any subquotient of the canonical filtration of  $\Gamma(X, \mathcal{L})$ . Therefore  $\Gamma(X, \mathcal{L})$  has a good filtration. Q.E.D.

There is a refinement of the previous theorem which is useful in some applications which are not involved here (see [M2]). With the same notations, let  $\mathcal{I} \subset \mathcal{O}_X$  be an ideal defining a  $G$ -subscheme  $Z \subset X$  which is compatibly split for some canonical Frobenius splitting  $\sigma$  of  $X$ . Then  $\Gamma(X, \mathcal{I} \otimes \mathcal{L})$  has a good filtration. Its proof is the same.

The following Corollary 6.3 has been first proved in [Wa] for  $G = SL(n)$  or  $p \gg 0$ . Then both corollaries 6.3 and 6.4 have been proved, by a very long case-by-case analysis, in [D2] for all groups, except  $E_7$  and  $E_8$  and characteristic 2. The general proof based on Frobenius splitting comes from [M2]. A nice account of it can be found in [Kn][vdK].

**Corollary 6.3:** *Let  $M, N$  two  $G$ -modules which admit a good fil-*

tration. Then  $M \otimes N$  has a good filtration.

*Proof* It is enough to prove that  $\nabla(\lambda) \otimes \nabla(\mu)$  has a good filtration, for any  $\lambda, \mu \in P^+$ . We have  $\nabla(\lambda) \otimes \nabla(\mu) = \Gamma(G/B \times G/B, \mathcal{L})$ , where  $\mathcal{L} = \mathcal{L}(w_0\lambda) \otimes \mathcal{L}(w_0\mu)$ . Moreover  $G/B \times G/B \simeq G \times_B (G \times_B B/B)$ , hence by Proposition 5.5,  $G/B \times G/B$  admits a canonical Frobenius splitting. Thus the proof follows from the previous theorem. Q.E.D.

**Corollary 6.4:** *Let  $G'$  be a Levi subgroup of  $G$  and let  $M$  be a  $G$ -module which admits a good filtration. Then the  $G'$ -module  $M|_{G'}$  has a good filtration.*

*Proof:* It is enough to prove that  $\nabla(\lambda)|_{G'}$  has a good filtration, for any  $\lambda \in P^+$ . By Proposition 5.5,  $G/B$  admits a canonical Frobenius splitting  $\sigma$  and by definition  $\sigma$  is canonical relative to  $G'$ . Thus the proof follows from Theorem 6.2. Q.E.D.

**7. Tilting modules.** The notion of tilting objects has been first introduced by Brenner and Butler [BB1][BB2] and then it has been generalized by Ringel [Ri] for the context of modules over a quasi-hereditary algebra. However its first appearance for algebraic groups is due to Donkin [D3]. Let  $M$  be a  $G$ -module. A *Weyl filtration* of  $M$  is the dual of a good filtration. The  $G$ -module  $M$  is called *tilting* if  $M$  has both a good filtration and a Weyl filtration. Usually, these two filtrations are distinct.

**Lemma 7.1:** *Let  $M$  be a  $G$ -module.*

(i) *There exists an exact sequence  $0 \rightarrow M \rightarrow G(M) \rightarrow W(M) \rightarrow 0$  such that  $G(M)$  has a good filtration and  $W(M)$  has a filtration by Weyl modules.*

(ii) *Moreover, we can assume that the highest weights of  $M$  and  $G(M)$  are the same and  $M_\lambda = G(M)_\lambda$  for any highest weight  $\lambda$  of  $M$ .*

*Proof:* Choose a linear form  $E : P \rightarrow \mathbf{R}$  as in Section 4 and write accordingly  $P^+ = \{\lambda_k | k \geq 0\}$ , where  $E(\lambda_0) < E(\lambda_1) < E(\lambda_2) \dots$ . There is an integer  $n > 0$  such that  $E(\lambda) \leq E(\lambda_n)$  for any dominant weight  $\lambda$  of  $M$ . We define a sequence of modules  $M = G_n(M) \subset G_{n-1}(M) \subset \dots \subset G_0(M)$  by decreasing induction as follows. Assume that  $G_r(M)$  is already defined. Then set  $m_{r-1} = \dim \text{Ext}_G^1(\Delta(\lambda_{r-1}), G_r(M))$ , and define  $G_{r-1}(M)$  by choosing any exact sequence:

$$0 \rightarrow G_r(M) \rightarrow G_{r-1}(M) \rightarrow \Delta(\lambda_{r-1})^{m_{r-1}} \rightarrow 0,$$

such that, the connecting homomorphism:

$$\delta : \text{Hom}_G(\Delta(\lambda_{r-1}), \Delta(\lambda_{r-1})^{m_{r-1}}) \rightarrow \text{Ext}_G^1(\Delta(\lambda_{r-1}), G_r(M))$$

is an isomorphism.

We claim  $Ext_G^1(\Delta(\lambda_s), G_r(M)) = 0$ , for all  $s \geq r$ . First consider the case  $r = n$ : by Lemma 4.4,  $Ext_G^1(\Delta(\lambda_s), G_n(M)) = 0$  for any  $s \geq n$ . Then the proof runs by decreasing induction over  $r$ : this follows from  $Ext_G^1(\Delta(\lambda_s), \Delta(\lambda_r)) = 0$  for any  $s \geq r$  (by Lemma 4.4) and the fact that the connecting homomorphism  $\delta$  is bijective.

Set  $G(M) = G_0(M)$ . By the previous claim  $Ext_G^1(\Delta(\lambda), G(M)) = 0$  for any  $\lambda \in P^+$ . Therefore by Corollary 4.7, the  $G$ -module  $G(M) = G_0(M)$  has a good filtration. By its definition, the  $G$ -module  $W(M) = M/G(M)$  has a Weyl filtration, hence Assertion (i) is proved. Moreover, it follows by induction from Lemma 4.4 that for any weight  $\nu$  of  $W(M)$ , we have  $\nu < \mu$  for some weight  $\mu$  of  $M$ . Hence Assertion (ii) holds. Q.E.D.

**Theorem 7.2:** (Donkin [D3])

(i) For any  $\lambda \in P^+$ , there is a unique indecomposable tilting module  $T(\lambda)$  such that  $\dim T(\lambda)_\lambda = 1$  and any weight of  $T(\lambda)$  is in the convex hull of  $W.\lambda$ .

(ii)  $T(\lambda) \simeq T(\mu)$  if and only if  $\lambda = \mu$ ,

(iii) Any indecomposable tilting module is isomorphic to some  $T(\lambda)$ .

For any  $G$ -module  $M$ , denote by  $T(M)$  the image of the composite map:  $H^0(U, M) \rightarrow M \rightarrow H_0(U^-, M)$ . As  $T(M)$  is an  $H$ -module, we will denote by  $T_\lambda(M)$  its weight spaces. Set  $t(M, \lambda) = \dim T_\lambda(M)$  for any  $\lambda \in P^+$ .

**Lemma 7.3:** ([MP1]) Let  $M$  be a tilting module. Then

$$M \simeq \bigoplus_{\lambda \in P^+} T(\lambda)^{t(M, \lambda)}.$$

*Proofs:* It will be more convenient to prove Theorem 7.2 and Lemma 7.3 together. By Lemma 7.1, there is an exact sequence:

$$0 \rightarrow \Delta(\lambda) \rightarrow G(\Delta(\lambda)) \rightarrow W(\Delta(\lambda)) \rightarrow 0;$$

such that  $G(\Delta(\lambda))$  has a good filtration,  $W(\Delta(\lambda))$  has a Weyl filtration and any weight of  $W(\Delta(\lambda))$  is  $< \lambda$ . Thus  $G(\Delta(\lambda))$  contains a unique indecomposable direct summand  $T(\lambda)$  with  $\dim T(\lambda)_\lambda = 1$ . Note that  $G(\Delta(\lambda))$  has also a Weyl filtration. Hence  $G(\Delta(\lambda))$  is tilting and by Corollary 4.7 its direct summand  $T(\lambda)$  is tilting. Thus the existence of  $T(\lambda)$  is proved: at this stage of the proof we found for any  $\lambda \in P^+$  one indecomposable tilting module  $T(\lambda)$  satisfying the requirements of point (i) of Theorem 7.2. These modules satisfy Assertion (ii) by definition.

Make further remarks about the module  $T(\lambda)$ . The last quotient of its canonical filtration is  $\nabla(\lambda)$ , hence there is an exact sequence:

$$0 \rightarrow X \rightarrow T(\lambda) \rightarrow \nabla(\lambda) \rightarrow 0,$$

where  $X$  has a good filtration. Similarly, we can find an exact sequence:

$$0 \rightarrow \Delta(\lambda) \rightarrow T(\lambda) \rightarrow Y \rightarrow 0,$$

where  $Y$  has a Weyl filtration.

We will now prove together the remaining statements. Let  $M$  be an indecomposable tilting module. For any highest weight  $\mu$  of  $M$ , we have  $M_\mu \simeq T_\mu(M)$ . Therefore  $T(M) \neq 0$ . Let  $\lambda$  be any weight of  $T(M)$ , and choose  $v \in H^0(U, M)_\lambda$  whose image in  $T(M)$  is not zero. There is a  $B^-$ -equivariant morphism  $L : M \rightarrow K(\lambda)$  with  $L(v) = 1$ , where  $K(\lambda)$  is the one dimensional  $B^-$ -module with weight  $\lambda$ . Denote by the same symbol  $v_\lambda$  some highest weight vector of  $\nabla(\lambda), \Delta(\lambda)$  and  $T(\lambda)$ . By the universal property of Weyl modules, there is a map  $\psi_1 : \Delta(\lambda) \rightarrow M$  sending  $v_\lambda$  to  $v$ . Similarly, there is a map  $\psi_2 : M \rightarrow \nabla(\lambda)$  such that  $\psi_2(m) = L(m)v_\lambda$  for any  $m \in M_\lambda$ . Since  $M$  has a good filtration, we have  $Ext_G^1(T(\lambda)/\Delta(\lambda), M) = 0$  (by Corollary 4.7). Thus, the map  $\psi_1$  can be extended to a map  $\phi_1 : T(\lambda) \rightarrow M$ . In the same way, the map  $\psi_2$  can be lifted to a map  $\phi_2 : M \rightarrow T(\lambda)$ . So we get the following commutative diagram:

$$\begin{array}{ccccc}
 & & T(\lambda) & & \\
 & & \phi_2 \nearrow \downarrow & & \\
 \Delta(\lambda) & \longrightarrow & M & \longrightarrow & \nabla(\lambda) \\
 \downarrow & \nearrow \phi_1 & & & \\
 & & T(\lambda) & & 
 \end{array}$$

By definition, we have  $\phi_2 \circ \phi_1(v_\lambda) = v_\lambda$ . Therefore,  $\phi_2 \circ \phi_1$  is a non-nilpotent endomorphism of the indecomposable module  $T(\lambda)$ . By Fitting's Lemma,  $\phi_2 \circ \phi_1$  is invertible. Thus,  $T(\lambda)$  is a direct factor of  $M$  and so we have  $M \simeq T(\lambda)$ . Therefore Assertion (iii) of the theorem is proved.

If  $\mu$  is another weight of  $T(M)$ , we get  $T(\mu) \simeq M \simeq T(\lambda)$ , and therefore  $\lambda = \mu$ . Thus  $\lambda$  is the unique weight of  $T(T(\lambda))$  and its multiplicity is 1. Therefore Lemma 7.3 is proved when  $M$  is indecomposable, and the general case follows. Q.E.D.

**8. The functor  $T^\Gamma$ .** In view of the next section, we recall some simple facts about the modular representation theory of an abstract group  $\Gamma$ . Let  $Mod_f(\Gamma)$  be the category of finite dimensional  $K\Gamma$ -modules. For  $M \in Mod_f(\Gamma)$ , there are natural maps  $H^0(\Gamma, M) \rightarrow M$  and  $M \rightarrow H_0(\Gamma, M)$ . We denote by  $T^\Gamma(M)$  the image of the composite map  $H^0(\Gamma, M) \rightarrow H_0(\Gamma, M)$ . (When  $\Gamma = G$  and  $M$  is a rational representation of  $G$ , we have  $T^G(M) = T_0(M)$ .) Similarly for  $M, N \in Mod_f(\Gamma)$ , set  $T^\Gamma(M, N) = T^\Gamma(Hom(M, N))$ . For  $M, N \in Mod_f(\Gamma)$ , the natural

map  $H^0(\Gamma, M) \otimes H^0(\Gamma, N) \rightarrow H^0(\Gamma, M \otimes N)$  induces a natural map  $T^\Gamma(M) \otimes T^\Gamma(N) \rightarrow T^\Gamma(M \otimes N)$ . This elementary property implies that the functor  $T^\Gamma(-, -)$  is composable: given  $L, M, N \in \text{Mod}_f(\Gamma)$ , there is a natural map  $T^\Gamma(L, M) \times T^\Gamma(M, N) \rightarrow T^\Gamma(L, N)$ . Therefore  $T^\Gamma(M, M)$  is an algebra.

**Lemma 8.1:** [Be] *Let  $M, N$  be two indecomposable  $\Gamma$ -modules. If  $M$  and  $N$  are isomorphic and their dimensions are not divisible by  $p$ , we have  $T^\Gamma(M, N) = K$ . Otherwise we have  $T^\Gamma(M, N) = 0$ .*

*Proof:* The dual of the  $\Gamma$ -module  $\text{Hom}(M, N)$  is  $\text{Hom}(N, M)$ , and the duality pairing is given by  $x, y \in \text{Hom}(M, N) \times \text{Hom}(N, M) \mapsto \text{Tr } yx$ , where  $\text{Tr}$  denotes the trace. Therefore the image of any  $x \in \text{Hom}_\Gamma(M, N)$  in  $T^\Gamma(M, N)$  is zero if and only if  $\text{Tr } yx = 0$  for any  $y \in \text{Hom}_\Gamma(N, M)$ . As  $M$  is indecomposable, any  $\Gamma$ -equivariant endomorphism  $z$  of  $M$  has a unique eigenvalue, say  $\lambda(z)$ . Thus we get  $\text{Tr } yx = \lambda(yx) \dim M$ . Therefore we have  $T^\Gamma(M, N) = 0$  if  $M$  and  $N$  are not isomorphic or if  $\dim M$  is divisible by  $p$ .

Furthermore the nilpotent radical of the  $K$ -algebra  $\text{End}_\Gamma(M)$  is the kernel of  $\lambda$ , and we have  $\lambda(xy) = \lambda(x)\lambda(y)$ , for any  $x, y \in \text{End}_\Gamma(M)$ . Thus  $T^\Gamma(M, M) = \text{End}_K(M)/\text{Ker}(\lambda) \simeq K$  if the dimension of  $M$  is not divisible by  $p$ . Q.E.D.

For any positive integer  $n$ , denote by  $M_n(K)$  the  $K$ -algebra of  $n$  by  $n$  matrices.

**Corollary 8.2:** *Let  $M \in \text{Mod}_f(\Gamma)$  and let  $M = \bigoplus P^{m_P}$  be its decomposition into indecomposable  $\Gamma$ -modules. Then  $T^\Gamma(M, M)$  is a semisimple algebra. More precisely, we have  $T^\Gamma(M, M) = \bigoplus_P M_{m_P}(K)$ , where  $P$  runs over all indecomposable modules with  $\dim P \neq 0$  modulo  $p$ .*

Let  $\mathcal{C}$  be a full subcategory of  $\text{Mod}_f(\Gamma)$ . Define a new category  $\mathcal{C}_T$  by the following requirements.

- (i) The objects of  $\mathcal{C}_T$  are the objects of  $\mathcal{C}$ .
  - (ii) For any two objects  $M$  and  $N$ , set  $\text{Hom}_{\mathcal{C}_T}(M, N) = T^\Gamma(M, N)$ .
- By definition,  $\mathcal{C}$  is called a *Karoubi category* if it is additive and subtractive, i.e. for any  $\Gamma$ -modules  $L, M, N$  with  $L \simeq M \oplus N$ , we have:

$$(L \in \mathcal{C}) \text{ if and only if } (M, N \in \mathcal{C}).$$

**Lemma 8.3:** *Let  $\mathcal{C}$  be a full Karoubi subcategory of  $\text{Mod}_f(\Gamma)$ . Then  $\mathcal{C}_T$  is abelian and semisimple. Moreover its simple objects are the indecomposable modules of  $\mathcal{C}$  whose dimension is not divisible by  $p$ .*

The lemma follows from the previous corollary. Assume now that  $\mathcal{C}$  is stable by tensor product. Then the category  $\mathcal{C}_T$  is endowed with a tensor product and  $K_0(\mathcal{C}_T)$  is a ring.

**Lemma 8.4:** [GM2] *Let  $\mathcal{C} \subset \text{Mod}_f(\Gamma)$  be a full Karoubi subcategory, which is stable by tensor product. Then the ring  $K_0(\mathcal{C}_T)$  is reduced.*

*Proof:* In order to prove the lemma, we can assume that  $\mathcal{C} = \text{Mod}_f(\Gamma)$ . Therefore, we can assume that  $\mathcal{C}$  is stable by duality. By Lemma 8.3, any  $x \in K_0(\mathcal{C}_T)$  can be uniquely written as  $\sum_{S \in \mathcal{S}} m_S [S]$ , where  $\mathcal{S}$  is the set of isomorphism classes of indecomposable  $K\Gamma$ -modules of dimension not divisible by  $p$ . Let  $I(x) = m_K$  be the multiplicity (as a direct summand) of the trivial representation in  $x$ . When  $y$  represents an effective  $K\Gamma$ -module  $M$ , we have  $I(y) = \dim T^\Gamma(M)$ . Hence by Lemma 8.1, we have  $I(x.x^*) = \sum_{S \in \mathcal{S}} m_S^2$ . Thus we have  $I(x.x^*) > 0$  for any  $x \neq 0$ .

Let  $x \in K_0(\mathcal{C}_T)$  be non-zero and set  $z = x.x^*$ . As  $I(z) > 0$ , we get  $z \neq 0$ , hence  $I(z.z^*) > 0$ , hence  $z.z^* \neq 0$ . Therefore  $x^2.(x^*)^2 \neq 0$  and  $x^2 \neq 0$ . Thus the ring  $K_0(\mathcal{C}_T)$  is reduced. Q.E.D.

Less formally, the compatibility of the functors  $T^\Gamma$  and  $\otimes$  can be stated as follows:

**Lemma 8.5:** [Be][GM2] *Let  $M, N \in \text{Mod}_f(\Gamma)$ . Assume that  $M$  is indecomposable and its dimension is divisible by  $p$ . Then the dimension of any direct summand of  $M \otimes N$  is divisible by  $p$ .*

*Proof:* By lemma 8.1, the ring  $T^\Gamma(M, M)$  is zero, i.e.  $1_M = 0_M$  in  $T^\Gamma(M, M)$ . Hence  $1_{M \otimes N} = 1_M \otimes 1_N = 0$ , i.e.  $T^\Gamma(M \otimes N, M \otimes N) = 0$ . Hence by Corollary 8.2, the dimension of any direct summand of  $M \otimes N$  is divisible by  $p$ .

*Remark:* The functor  $T$  can be defined for any Hopf algebra  $\mathcal{H}$  over an arbitrary field  $K$ , and it is interesting when  $\mathcal{H}$  admits non-zero representations of zero 'dimension', where the 'dimension' is defined in the setting of Hopf algebras. In characteristic zero, this occurs only for non-commutative Hopf algebras. Here is an example which has not been investigated. Let  $\mathfrak{g}$  be a Lie super-algebra over  $\mathbb{C}$ . The *superdimension* of  $\mathfrak{g}$ -module  $M$  is  $\dim M_0 - \dim M_1$ . The analog of Lemma 8.5 can be stated as follows: for any indecomposable  $\mathfrak{g}$ -module  $M$  of superdimension 0, any summand of  $M \otimes N$  has superdimension 0, for any other  $\mathfrak{g}$ -module  $N$ . It turns out that for a classical Lie superalgebra  $\mathfrak{g}$ , the simple representations  $L$  of superdimension 0 are the most atypical (so a priori the most complicated). Therefore, it seems interesting to investigate similar questions for Lie superalgebras in order to understand these very atypical representations. For other characteristic zero examples see e.g. [A6][GM1][AP].

## 9. The Verlinde's formula and the modular Verlinde's for-

**mula.** Most of the analogies of this section are borrowed from [GM1]. Let  $G_{\mathbf{C}}$  be a connected simply connected algebraic group over  $\mathbf{C}$ , let  $\mathfrak{g}_{\mathbf{C}}$  be its Lie algebra and let  $\Delta$  be its root system. Normalize the invariant bilinear form  $\kappa$  of  $\mathfrak{g}_{\mathbf{C}}$  by the following requirement:  $\kappa(h_{\alpha}, h_{\alpha}) = 2$  for any short root  $\alpha$  (i.e. for any root if  $\mathfrak{g}_{\mathbf{C}}$  belongs to the  $ADE$  series). Set  $\mathcal{O} = \mathbf{C}[[t]]$ , let  $\mathbf{K} = \mathbf{C}((t))$  be its quotient field and let  $m = t \cdot \mathbf{C}[[t]]$  be its maximal ideal. The loop algebra  $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$  is the central extension of  $\mathfrak{g}_{\mathbf{C}} \otimes \mathbf{K}$  defined by the 2-cocycle  $(x_1 \otimes f_1, x_2 \otimes f_2) \mapsto \kappa(x_1, x_2) \text{Res}_0 f_1 df_2$ , for any  $x_i \in \mathfrak{g}_{\mathbf{C}}$ ,  $f_i \in \mathbf{K}$  (where 0 denotes the closed point of  $\text{Spec } \mathcal{O}$ ). The central element of  $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$  is denoted by  $c$  and set  $\mathcal{L}^+ = \mathfrak{g}_{\mathbf{C}} \otimes m$ . A  $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ -module  $M$  is called:

- smooth*<sup>10</sup> if  $\mathcal{L}^+$  acts locally nilpotently,
- integrable* if  $x \otimes f$  acts locally nilpotently, for any ad-nilpotent  $x \in \mathfrak{g}_{\mathbf{C}}$  and any  $f \in \mathbf{K}$ ,
- of level  $l$*  if  $c$  acts as the scalar  $l$ .

For a non-negative integer  $l$ , let  $\mathcal{O}_l^{\text{int}}$  be the category of level  $l$  smooth integrable  $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ -modules of finite length. This notation should not be confused with the valuation ring  $\mathcal{O}$ . Denote by  $h'_0$  the highest short coroot, i.e.  $h'_0 = h_{\beta}$  where  $\beta$  is the highest root. For  $\mathfrak{g}_{\mathbf{C}}$  of type  $ADE$ ,  $h'_0 = h_0$  is the highest coroot. However for  $\mathfrak{g}_{\mathbf{C}}$  of type  $BCFG$ ,  $h'_0 \neq h_0$ : for example  $h_0 = h_1 + 2h_2$  and  $h'_0 = h_1 + h_2$  if  $\mathfrak{g}_{\mathbf{C}}$  is of type  $B_2$ . Set  $P_l^+ = \{\lambda \in P^+ \mid \lambda(h'_0) \leq l\}$ . Note that  $h'_0 = \sum_{1 \leq i \leq l} m_i h_i$ , where all  $m_i$  are positive integers, hence  $P_l^+$  is finite. First recall the main results about the category  $\mathcal{O}_l^{\text{int}}$ .

**Theorem 9.1:** (Kac) [Ka]

- (i) Let  $M \in \mathcal{O}_l^{\text{int}}$  be a simple  $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ -module. Then  $H^0(\mathcal{L}^+, M)$  is a simple finite dimensional  $\mathfrak{g}_{\mathbf{C}}$ -module, and its highest weight  $\lambda = \lambda(M)$  belongs to  $P_l^+$ .
- (ii) The map  $M \mapsto \lambda(M)$  is a bijection from simple modules of  $\mathcal{O}_l^{\text{int}}$  to  $P_l^+$ .

**Theorem 9.2:** (Deodhar, Gabber, Kac [DGK]) The category  $\mathcal{O}_l^{\text{int}}$  is semisimple.

We will now introduce more notations. For  $\lambda \in P^+$ , denote by  $L_{\mathbf{C}}(\lambda)$  the simple  $\mathfrak{g}_{\mathbf{C}}$ -module whose highest weight is  $\lambda$ . For  $\lambda \in P_l^+$ , denote by  $L_l(\lambda)$  the simple  $\mathcal{L}(\mathfrak{g}_{\mathbf{C}})$ -module such that  $H^0(\mathcal{L}^+, L_l(\lambda)) \simeq L_{\mathbf{C}}(\lambda)$ . Also for any  $\lambda, \mu, \nu \in P^+$ , denote by  $K_{\lambda\mu}^{\nu}$  the classical ten-

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<sup>10</sup>the terminology "smooth" is used in [KL] by analogy with representation theory of  $p$ -adic groups. However, the terminology "level" has no connection.

tensor product multiplicities which are defined by:  $L_C(\lambda) \otimes L_C(\mu) \simeq \bigoplus_{\nu \in P^+} L_C(\nu) K_{\lambda\mu}^\nu$ . They are explicitly determined by Kostant's formula.

Following [MS], we would like to define some modified tensor product multiplicities  $V_{\lambda\mu}^{\nu}$  which are defined for all  $\lambda, \mu, \nu \in P_l^+$ . However, for any  $M, N \in \mathcal{O}_l^{int}$ , their tensor product  $M \otimes N$  has level  $2l$  and the multiplicities of its simple components are usually infinite. Therefore the ordinary tensor product cannot be used to define  $V_{\lambda\mu}^{\nu}$ . Indeed, Moore and Seiberg [MS] underlined the possibility to define a new tensor product  $\hat{\otimes}$  in a such way that

(i)  $M \hat{\otimes} N \in \mathcal{O}_l^{int}$ ,

(ii) the tensor product multiplicities  $V_{\lambda\mu}^{\nu}$  are given by Verlinde's formula [V].

More precisely, in the axiomatic of Moore and Seiberg, we have

$$L_l(\lambda) \hat{\otimes} L_l(\mu) = \bigoplus_{\nu \in P_l^+} L_l(\nu) V_{\lambda\mu}^{\nu},$$

where the constant  $V_{\lambda\mu}^{\nu}$  are given by the formula

$$V_{\lambda\mu}^{\nu} = \sum_{w \in W'_{aff}} \epsilon(w) K_{\lambda\mu}^{w(\nu+\rho)-\rho}.$$

The meaning of symbols used in the previous formula is now explained. The integer  $h'$  is the dual Coxeter number  $\rho(h'_0) + 1$ . The group  $W'_{aff}$  is the dual affine Weyl group, generated by the linear reflexions  $s_i \in W$  and by the additional affine reflexion  $s'_0$  relative to the hyperplane  $\lambda(h'_0) = l + h'$ . The multiplicities  $K_{\lambda\mu}^\nu$  have been previously defined for  $\nu \in P^+$  and we set  $K_{\lambda\mu}^\nu = 0$  if  $\nu \notin P^+$ .

By definition, the *fusion ring* is  $K_0(\mathcal{O}_l^{int})$ , where the product is induced by the modified tensor product  $\hat{\otimes}$ . It follows from Kac's theorem 9.1 that the fusion ring is the free  $\mathbf{Z}$ -module with a basis  $[L_l(\lambda)]$  indexed by  $\lambda \in P_l^+$  and the algebra structure constants are  $V_{\lambda\mu}^{\nu}$ . In addition, Seiberg and Moore described a series of axioms satisfied by the tensor category  $(\mathcal{O}_l^{int}, \hat{\otimes})$ . We will now explain their combinatorial consequences for the structure constants  $V_{\lambda\mu}^{\nu}$ . First, the existence of  $\hat{\otimes}$  implies that  $[L_l(\lambda)] \cdot [L_l(\lambda)]$  is an effective representation, hence we have:

(AX1)  $V_{\lambda\mu}^{\nu} \geq 0$ .

The next axiom is the fact that  $\hat{\otimes}$  is somehow commutative and associative. The associativity axiom implies that the fusion ring is associative, or equivalently the combinatorial identity holds:

(AX2)  $\sum_{\nu \in P_l^+} V_{\lambda\mu}^{\nu} V_{\nu\pi}^{\sigma} = \sum_{\nu \in P_l^+} V_{\lambda\nu}^{\sigma} V_{\mu\pi}^{\nu}$ , for any  $\lambda, \mu, \pi, \sigma \in P_l^+$ .

Another axiom is the existence of duals in  $\mathcal{O}_l^{int}$ , and the dual of  $L_l(\lambda)$  is  $L_l(-w_0(\lambda))$ . Hence:

(AX3) the map  $\lambda, \mu, \nu \mapsto V_{\lambda\mu}^{\prime-w_0(\nu)}$  is a symmetric function into the three arguments.

(AX4) the fusion ring is reduced.

Following [MS], the space  $Hom_{\mathcal{O}_i^{int}}(L_l(\nu), L_l(\lambda) \hat{\otimes} L_l(\mu))$  should be described as a space of fields, which are uniquely determined by their residue values in  $Hom_{G_C}(L(\nu), L(\lambda) \otimes L(\mu))$ . The injectivity of the residue map implies the inequality:

$$(AX5) V_{\mu,\lambda}^{\nu} \leq K_{\lambda\mu}^{\nu}.$$

In what follows, we will explain how to define a category  $\mathcal{P}$  in the setting of modular representations of the Chevalley group  $G$ , following [GM1][GM2]. This category satisfies the previous axioms (AX1 – 5) of  $\mathcal{O}_{p-h}^{int}$ , the main differences are as follows:

- (i) the category  $\mathcal{P}$  is  $K$ -linear, where  $K$  is a field of characteristic  $p$  (instead of being  $\mathbb{C}$ -linear),
- (ii) Morphisms are modified (instead of being the usual one),
- (iii) The tensor product is the ordinary tensor product (instead of being modified),
- (iv) The structure constants of the tensor product involves the affine Weyl group (instead of the dual affine Weyl group).

Assume that  $p \geq h$ , where  $h = \rho(h_0) + 1$ . Define the category  $\mathcal{P}$  by  $\mathcal{P} = \mathcal{T}_T$  (see Section 8), where  $\mathcal{T}$  is the category of tilting modules. By Corollary 4.7, the category  $\mathcal{T}$  is a Karoubi category, hence by Lemma 8.3,  $\mathcal{P}$  is a semisimple abelian category. In order to explain why the category  $\mathcal{P}$  satisfies the previous axioms, we first state a lemma, whose proof will be postponed until the next section. Set  $C^0 = \{\lambda \in P^+ | (\lambda + \rho)(h_0) < p\}$ .

**Lemma 9.3:** ([GM1][GM2]) *Assume  $p \geq h$ . Let  $\lambda \in P^+$ .*

- (i) *If  $\lambda \in C^0$ , then  $T(\lambda) = \nabla(\lambda) = \Delta(\lambda)$  and its dimension is not divisible by  $p$ .*
- (ii) *Otherwise, the dimension of  $T(\lambda)$  is divisible by  $p$ .*

It follows from lemmas 8.3 and 9.3 that  $([\nabla(\lambda)])_{\lambda \in C^0}$  is a basis of the  $\mathbb{Z}$ -module  $K_0(\mathcal{P})$ . For a group  $G$  of type  $ADE$ , we have  $C^0 = P_{p-h}^+$ , and the index set of the bases of  $K_0(\mathcal{P})$  and of the fusion ring  $K_0(\mathcal{O}_1^{int})$  are the same. It follows from Corollary 6.3 that  $\mathcal{P}$  is a tensor category. By definition, the tensor product multiplicity  $V_{\lambda\mu}^{\nu}$  are the structure constants of the ring  $K_0(\mathcal{P})$ , where  $\lambda, \mu, \nu \in C^0$ .

Now we will check that the multiplicities  $V_{\lambda\mu}^{\nu}$  satisfy the previous axioms. By definition we have  $\nabla(\lambda) \otimes \nabla(\mu) = \bigoplus_{\nu \in C^0} \nabla(\nu) V_{\lambda\mu}^{\nu} \oplus T$ , where  $T$  is a sum of indecomposable modules of dimension divisible by  $p$ . Hence by definition, we get  $V_{\lambda\mu}^{\nu} \geq 0$  (AX1). Similarly, the associativity of the constant structure  $V_{\lambda\mu}^{\nu}$  comes from the fact that the tensor product of  $\mathcal{P}$

is truly associative (see (AX2)). We have  $V_{\lambda\mu}^\nu = \dim T^G(\nabla(\nu), \nabla(\lambda) \otimes \nabla(\mu))$ . However, we have  $T(\nu)^* = T(-w_0(\nu))$ , hence we get  $V_{\lambda\mu}^{-w_0(\nu)} = \dim T^G(\nabla(\nu) \otimes \nabla(\lambda) \otimes \nabla(\mu))$ , and  $V_{\lambda\mu}^{-w_0(\nu)}$  is symmetric into the three arguments (AX3). The fact that the ring  $K_0(\mathcal{P})$  is reduced comes from Lemma 8.4, therefore  $K_0(\mathcal{P})$  satisfies (AX4). To check the last axiom (AX5), we prove:

**Lemma 9.4:** *For any  $\lambda, \mu, \nu \in C^0$ , we have  $V_{\lambda\mu}^\nu \leq K_{\lambda\mu}^\nu$ .*

*Proof:* It follows from Lemma 9.3 that  $T(\lambda) = \nabla(\lambda)$ ,  $T(\mu) = \nabla(\mu)$  and  $T(\nu) = \nabla(\nu)$ . Set  $M = \nabla(\lambda) \otimes \nabla(\mu)$ . It follows from the definition of  $T^G$  that  $V_{\lambda\mu}^\nu \leq \dim \text{Hom}_G(\Delta(\nu), M)$ , thus we get  $V_{\lambda\mu}^\nu \leq h^0(M, \nu)$ . By Corollary 6.3,  $M$  has a good filtration. By Lemma 4.2, one gets  $\text{ch } M = \sum_{\pi \in P^+} h^0(M, \pi) \text{ch } \nabla(\pi)$ . In particular, the numbers  $h^0(M, \pi)$  are independent of the characteristic. Thus  $h^0(M, \nu) = K_{\lambda\mu}^\nu$ , and the lemma is proved. Q.E.D.

We still have to compute the number  $V_{\lambda\mu}^\nu$  to show the complete similarity between the category  $\mathcal{P}$  and the fusion category  $\mathcal{O}_{p-h}^{\text{int}}$ . Indeed we have:

**Theorem 9.5:** (modular Verlinde’s formula [GM1][GM2]):

*Assume  $p \geq h$ . For any  $\lambda, \mu, \nu \in C^0$ , we have:*

$$V_{\lambda\mu}^\nu = \sum_{w \in W_{\text{aff}}} \epsilon(w) K_{\lambda\mu}^{w(\nu+\rho)-\rho};$$

The previous theorem was proved in [GM1][GM2]. However, we will give a simpler proof in the next section. Indeed the hypothesis  $p \geq h$  is useless: however if  $p < h$ , the set  $C^0$  is empty.

Also, it is easy to realize the ring  $K_0(\mathcal{P})$  as a quotient of  $K_0(G)$ . For any additive category  $\mathcal{A}$ , denote by  $K'_0(\mathcal{A})$  the group generated by the symbols  $([M])_{M \in \mathcal{A}}$  submitted to the relations  $[M] + [N] - [L]$  whenever  $L \simeq M \oplus N$ . It follows from Donkin’s theorem 4.2 that  $K'_0(\mathcal{T}) = \mathbf{Z}[P]^W = K_0(G_{\mathbf{C}})$ , where  $G_{\mathbf{C}}$  is the simply connected complex group of same type as  $G$ . Using the natural morphism  $K'_0(\mathcal{T}) \rightarrow K_0(\mathcal{P})$ , we get the epimorphism  $K_0(G_{\mathbf{C}}) \rightarrow K_0(\mathcal{P})$ . As  $K_0(\mathcal{P})$  is reduced, it is enough to determine the spectrum of  $\mathbf{C} \otimes K_0(\mathcal{P})$ , as a subset of  $\text{Spec}(\mathbf{C} \otimes K_0(G_{\mathbf{C}}))$ . Recall that  $\text{Spec}(\mathbf{C} \otimes K_0(G_{\mathbf{C}}))$  is exactly the set of conjugacy classes of semisimple elements of  $G_{\mathbf{C}}$ . Denote by  $Z^{1/p}$  the set of all regular semisimple conjugacy classes  $[g]$  in  $G_{\mathbf{C}}$  such that  $g^p$  is central.

**Theorem 9.6:** ([GM1][GM2]) *Assume  $p \geq h$ . Then we have:*

$$\text{Spec}(\mathbf{C} \otimes K_0(\mathcal{P})) = Z^{1/p}.$$

*Remark:* The remarkable paper of Moore and Seiberg [MS] describes the axioms of the tensor product category  $\mathcal{O}_i^{int}$ . However, their papers did not provide a rigorous mathematical proof of the existence of a tensor product, and at that time there were no rigorous mathematical approach of it. Kazhdan and Lusztig [KL] introduced a tensor product on a category of representation of affine Lie algebras. It was very surprising that these authors use a category of negative level representation instead of  $\mathcal{O}_i^{int}$  (see also [HL] for a vertex algebra approach). Using the approach of Kazhdan and Lusztig, Finkelberg [F] provided a tensor product on  $\mathcal{O}_i^{int}$ , and he gave another approach of the Verlinde's formula for quantum groups. See [Sor] for a recent account of Verlinde's formula in the context of algebraic geometry.

**10. Proof of the modular Verlinde's formula.** The proof given here is a bit simpler than the original proof (and a bit more precise). Indeed in [GM1][GM2], we were not aware of Jantzen's lemma 10.2, which allows some simplifications.

A weight  $\lambda$  is called *p-singular* if  $(\lambda + \rho)(h_\alpha) = 0$  modulo  $p$  for some root  $\alpha$ . Otherwise,  $\lambda$  is called *p-regular*. A  $G$ -module  $M$  is called *regular* (respectively *singular*) if the highest weight of any simple subquotient of  $M$  is *p-regular* (respectively *p-singular*). By the linkage principle 3.4, a block of  $Mod(G)$  contains only regular modules or only singular modules. Thus any  $G$ -module  $M$  can be decomposed as  $M = M_{reg} \oplus M_{sing}$ , where  $M_{reg}$  is regular and  $M_{sing}$  is singular. Accordingly, we can split the category  $Mod(G)$  as  $Mod_{reg}(G) \oplus Mod_{sing}(G)$ .

**Lemma 10.1:** *Assume  $p \geq h$ . For any singular module  $M$ , we have:*  

$$\dim M = 0 \text{ modulo } p.$$

*Proof:* Let  $\lambda \in P^+$  be a *p-singular* weight. By Weyl dimension formula, we have  $\dim \nabla(\lambda) = \prod_{\alpha \in \Delta^+} (\lambda + \rho)(h_\alpha) / \prod_{\alpha \in \Delta^+} \rho(h_\alpha)$ . We have  $\rho(h_\alpha) \leq h - 1$  for any root  $\alpha \in \Delta^+$ , hence the denominator is not divisible by  $p$ , but the numerator is divisible by  $p$ , thus  $\dim \nabla(\lambda)$  is divisible by  $p$ . By Linkage Principle 3.4, the character of  $M$  is a linear combination of  $\text{ch } \nabla(\lambda)$  where  $\lambda$  is *p-singular*. Therefore its dimension is divisible by  $p$ . Q.E.D.

Set  $P_{\mathbf{R}} = \mathbf{R} \otimes P$ , set  $P_{\mathbf{R}}^{reg} = \{\lambda \in P_{\mathbf{R}} | (\lambda + \rho)(h_\alpha) \notin p\mathbf{Z}, \forall \alpha \in \Delta^+\}$  and let  $P_{\mathbf{R}}^{sub}$  be the set of all  $\lambda \in P_{\mathbf{R}}$  such that  $(\lambda + \rho)(h_\alpha) \in p\mathbf{Z}$  for exactly one  $\alpha \in \Delta^+$ . The connected components of  $P_{\mathbf{R}}^{reg}$  are called the *alcoves*, those of  $P_{\mathbf{R}}^{sub}$  are called the *walls*. The *fundamental alcove* is the alcove  $C_{fund} = \{\lambda \in P_{\mathbf{R}} | (\lambda + \rho)(h_0) < p \text{ and } (\lambda + \rho)(h_i) > 0, \forall i \in I\}$ . Therefore  $C^0 = C_{fund} \cap P$ . The *fundamental walls* are the

walls contained in  $\overline{C_{fund}}$ . Thus there are  $l + 1$  fundamental walls, namely  $(F_i)_{i \in I \cup \{0\}}$ , which are defined as follows:

$$F_0 = \{\lambda \in P_{\mathbf{R}} \mid (\lambda + \rho)(h_0) = p \text{ and } (\lambda + \rho)(h_i) > 0, \forall i \in I\}.$$

$$F_i = \{\lambda \in P_{\mathbf{R}} \mid (\lambda + \rho)(h_i) = 0, (\lambda + \rho)(h_0) < p$$

$$\text{and } (\lambda + \rho)(h_j) > 0, \forall j \in I, j \neq i\},$$

for any  $i \in I$ . Any alcove is conjugated under  $W_{aff}$  to  $C_{fund}$ . Any wall  $F$  is conjugated to exactly one  $F_i$ : in such case, we say that  $i$  is the *type* of  $F$ .

Let  $\lambda \in P_{\mathbf{R}}^{reg}$ , let  $i \in I \cup \{0\}$ , and let  $C$  be the unique alcove containing  $\lambda$ . There is a unique wall  $F$  of type  $i$  with  $F \subset \overline{C}$ . We set  $s_i * \lambda = s_F \lambda$ , where  $s_F$  is the affine reflexion relative to the affine hyperplane containing  $F$ . There is another definition of  $s_i * \lambda$ : write  $\lambda = w(\mu + \rho) - \rho$ , where  $w \in W_{aff}$  and  $\mu \in C_{fund}$ . Then  $s_i * \lambda = w s_i(\mu + \rho) - \rho$ . Hence the group generated by the operators  $s_i$  is isomorphic to  $W_{aff}$ . This action of  $W_{aff}$ , noted  $*$ , is often called the right action of  $W_{aff}$  on the regular weights.

**Lemma 10.2:** (Jantzen [J3]) *Assume  $p \geq h$ . Then any alcove and any wall contain an integral weight.*

*Proof:* For an alcove, the proof is obvious. For a wall, the proof is given in [J3], claim 6.3 (1).

Set  $\overline{C}^0 = \overline{C_{fund}} \cap P$ . For  $\lambda \in \overline{C}^0$  and  $M \in Mod(G)$ , denote by  $P_\lambda M$  be the biggest submodule of  $M$  such that the highest weight of any simple subquotient of  $M$  is in  $W_{aff} * \lambda$ . It follows from Theorem 3.4 that  $M = \bigoplus_{\lambda \in \overline{C}^0} P_\lambda M$ , therefore the functors  $M \mapsto P_\lambda M$  are exact. For  $\lambda, \mu \in \overline{C}^0$ , the *translation functor*  $T_\lambda^\mu : Mod(G) \rightarrow Mod(G)$  is defined as follows (see [J3], ch. 7). Let  $\nu$  be the unique dominant weight in the  $W$ -orbit of  $\mu - \lambda$  and let  $S$  be any indecomposable module such that  $\dim S_\nu = 1$  and all its weights are  $\leq \nu$ . For any  $G$ -module  $M$ , set  $T_\lambda^\mu M = P_\mu(S \otimes P_\lambda M)$ . It follows from [J3] (remark 1 of section 7.6) that  $T_\lambda^\mu M$  does not depend on the choice of  $S$ , and it will be convenient to use  $S = T(\nu)$ . Therefore it follows that  $T_\lambda^\mu M$  is a direct summand of  $T(\nu) \otimes M$ .

For any simple reflexion  $s$  of  $W_{aff}$ , denote by  $T_s : Mod_{reg}(G) \rightarrow Mod_{reg}(G)$  the *reflexion functor* invented by Jantzen (sometimes these functors are called wall crossing functors). Roughly speaking, the functor  $T_s$  is defined as follows. First assume  $p \geq h$  (otherwise  $Mod_{reg}(G) = 0$ ). By Jantzen's lemma 10.2, there is an integral weight  $\mu$  in the unique fundamental wall fixed by  $s$ . Then

$$T_s = \bigoplus_{\lambda \in C^0} T_\mu^\lambda \circ T_\lambda^\mu.$$

For any  $M \in \text{Mod}_{\text{reg}}(G)$ ,  $T_s M$  is a direct summand of a tensor product of  $M$  by tilting modules. Therefore, by Corollary 6.3,  $T_s M$  has a good filtration (respectively has a Weyl filtration, is tilting) if  $M$  has a good filtration (respectively has a Weyl filtration, is tilting). Moreover,  $T_s M$  is a direct summand in a direct sum of type  $X \otimes Y$ , where  $X \in \text{Mod}_{\text{sing}}(G)$ . Therefore it follows from Lemma 8.5 and Lemma 10.1 that the dimension of any direct summand of  $T_s M$  is divisible by  $p$ .

**Corollary 10.3:** [J3] *Assume  $p \geq h$ . Let  $\lambda \in P^+$  be  $p$ -regular and let  $s$  be a simple reflexion of  $W_{\text{aff}}$ . Then  $T_s \Delta(\lambda)$  is indecomposable. Moreover:*

- (i) *If  $s * \lambda \notin P^+$ , we have  $T_s \Delta(\lambda) = 0$ .*
- (ii) *If  $s * \lambda \in P^+$  and  $s * \lambda > \lambda$ , there is a non-split extension*  

$$0 \rightarrow \Delta(s * \lambda) \rightarrow T_s \Delta(\lambda) \rightarrow \Delta(\lambda) \rightarrow 0.$$
- (iii) *If  $s * \lambda \in P^+$  and  $s * \lambda < \lambda$ , there is a non-split extension*  

$$0 \rightarrow \Delta(\lambda) \rightarrow T_s \Delta(\lambda) \rightarrow \Delta(s * \lambda) \rightarrow 0.$$

*Proof:* Assertion (i) is clear, therefore we can assume that  $s * \lambda \in P^+$ . The existence of the extension follows from the fact that  $T_s \Delta(\lambda)$  has a Weyl filtration and  $\text{ch } T_s \Delta(\lambda) = \text{ch } \Delta(\lambda) + \text{ch } \Delta(s * \lambda)$ . Moreover,  $T_s \Delta(\lambda)$  is indecomposable because any direct summand has a Weyl filtration and its dimension is divisible by  $p$ . Therefore the extensions are non-split. Q.E.D.

For  $\lambda, \mu \in P^+$ , denote by  $[T(\lambda) : \nabla(\mu)]$  the multiplicity of  $\nabla(\mu)$  in a good filtration of  $T(\lambda)$ . By Lemma 4.2, we have  $[T(\lambda) : \nabla(\mu)] = h^0(T(\lambda), \mu)$ . We extend this notation by requiring  $[T(\lambda) : \nabla(\mu)] = 0$  if  $\mu \notin P^+$ .

**Lemma 10.4:** ([GM1][GM2]) *Assume  $p \geq h$ . Let  $\lambda \in P^+$ .*

(i) *If  $\lambda$  belongs to  $C^0$ , then  $T(\lambda) = \nabla(\lambda) = \Delta(\lambda)$  and its dimension is not divisible by  $p$ .*

(ii) *Otherwise,  $\dim T(\lambda)$  is divisible by  $p$ , and we have:*

$$\sum_{w \in W_{\text{aff}}} \epsilon(w) [T(\lambda) : \nabla(w(\lambda + \rho) - \rho)] = 0.$$

*Proof:* *Proof of (i):* If  $\lambda \in C^0$ , then  $\nabla(\lambda)$  is simple and isomorphic to  $\Delta(\lambda)$  by Corollary 3.5. Thus  $T(\lambda) = \nabla(\lambda)$  and its dimension is  $\prod_{\alpha \in \Delta^+} (\lambda + \rho)(h_\alpha) / \prod_{\alpha \in \Delta^+} \rho(h_\alpha)$ , which is not divisible by  $p$ .

*Proof of (ii):* If  $\lambda$  is  $p$ -singular,  $\dim T(\lambda)$  is divisible by  $p$  by Lemma 10.1. Moreover, there is a reflexion  $s \in W_{\text{aff}}$  such that  $s(\lambda + \rho) - \rho = \lambda$ . Hence, we have:

$\epsilon(w) [T(\lambda) : \nabla(w(\lambda + \rho) - \rho)] + \epsilon(ws) [T(\lambda) : \nabla(ws(\lambda + \rho) - \rho)] = 0$ ,  
 for any  $w \in W_{\text{aff}}$ . Therefore we get:

$$\sum_{w \in W_{\text{aff}}} \epsilon(w) [T(\lambda) : \nabla(w(\lambda + \rho) - \rho)] = 0.$$

Henceforth, we can assume that  $\lambda$  is regular. Write  $\lambda = w(\mu + \rho) - \rho$ , where  $\mu \in C^0$ . By induction, we can assume that Assertion (ii) holds for the weight  $x(\lambda + \rho) - \rho$ , for any  $x \neq 1$  with  $l(x) < l(w)$ . Write  $w = vs$ , where  $s$  is a simple reflexion,  $l(v) = l(w) - 1$  and  $v(\lambda + \rho) - \rho \in P^+$  and set  $\lambda' = v(\lambda + \rho) - \rho$ . Note that  $\lambda$  is a highest weight of the tilting module  $T_s T(\lambda')$  (e.g. this follows from Corollary 10.3). Therefore  $T(\lambda)$  is a direct summand of  $T_s T(\lambda')$ . Hence its dimension is divisible by  $p$ . Moreover, the other indecomposable summands are some tilting modules  $T(x(\lambda + \rho) - \rho)$ , with  $l(x) < l(w)$ . Moreover  $T(\mu)$  is not a direct summand because its dimension is not divisible by  $p$ . Hence Assertion (ii) follows by induction. Q.E.D.

*Remark:* In their paper [AP], Andersen and Paradowski found a refinement of the previous Lemma. Indeed they prove that if  $\nu$  is  $p$ -singular and  $\lambda$  is  $p$ -regular, then  $T_\nu^\lambda T(\nu)$  is an indecomposable tilting module (there is a similar statement for quantum groups in [A6]). The previous proof (or the one of [GM2]) only implies that  $T(\mu)$  never occurs as a direct summand in  $T_\nu^\mu T(\nu)$ , whenever  $\mu \in C^0$ .

The following statement is an immediate consequence of Lemma 10.4:

**Corollary 10.5:** *Let  $M$  be a tilting module and let  $\lambda \in C^0$ . Then*

$$[M : T(\lambda)] = \sum_{w \in W_{aff}} \epsilon(w) [M : \nabla((w(\lambda + \rho) - \rho))].$$

In what follows, we prove the statements of Section 9, whose proofs were postponed.

*Proof of Lemma 9.3:* See Lemma 10.4 (i).

*Proof of Theorems 9.5:* For  $\lambda, \mu, \nu \in C^0$ , we have:

$$\begin{aligned} V_{\lambda\mu}^\nu &= [T(\lambda) \otimes T(\mu) : T(\nu)]. \\ &= \sum_{w \in W_{aff}} \epsilon(w) [T(\lambda) \otimes T(\mu) : \Delta((w(\nu + \rho) - \rho))] \\ &= \sum_{w \in W_{aff}} \epsilon(w) K_{\lambda\mu}^{w(\nu + \rho) - \rho} \qquad \text{Q.E.D.} \end{aligned}$$

*Proof of Theorems 9.6:* Let  $I_1$  be the kernel of the morphism  $K_0(G_C) \rightarrow K_0(\mathcal{P})$ . Let  $I_2$  be the subgroup of  $K_0(G)$  generated by all  $\text{ch } \Delta(\lambda)$ , where  $\lambda$  is  $p$ -singular and all  $\text{ch } \Delta(\lambda) + \text{ch } \Delta(s(\lambda + \rho) - \rho)$ , where  $\lambda$  is  $p$ -regular and  $s$  is a reflexion with  $s(\lambda + \rho) - \rho \in P^+$ . Let  $I_3$  be the ideal of all  $f \in K_0(G)$  such that  $f(g) = 0$  for any  $g \in Z^{1/p}$ . By Lemma 8.3,  $I_1$  is spanned by all  $\text{ch } T(\lambda)$ , where  $\dim T(\lambda) = 0$  modulo  $p$ . Therefore, it follows from Lemma 10.4 (ii) that  $I_1 \subset I_2$ .

We claim that  $I_2 \subset I_3$ . Let  $\lambda \in P^+$ , let  $s \in W_{aff}$  be an affine reflexion such that  $s(\lambda + \rho) - \rho \in P^+$  and let  $g \in Z^{1/p}$ . For  $\mu \in P$ , set  $\delta_\mu(g) = \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho)}(g)$ . By Weyl's denominator formula, we have  $\delta_0(g) = e^{-\rho}(g) \prod_{\alpha \in \Delta^+} (1 - e^\alpha(g))$ . By regularity of  $g$ , we have  $\delta_0(g) \neq 0$ . Let  $\bar{s}$  be the linear reflexion associated with  $s$ . For any weight  $\mu$ ,  $\bar{s}\mu - s\mu \in pQ$ . As  $g^p$  is central, we have  $e^{\bar{s}\mu}(g) = e^{s\mu}(g)$ . Therefore  $\delta_{s\lambda}(g) = \delta_{\bar{s}\lambda}(g)$ . However  $\bar{s}$  belongs to  $W$  and  $\epsilon(\bar{s}) = -1$ . Hence  $\delta_{\bar{s}\lambda}(g) = -\delta_\lambda(g)$  and  $\delta_{s\lambda}(g) + \delta_\lambda(g) = 0$ . Therefore the evaluation of  $\text{ch } \Delta(\lambda) + \text{ch } \Delta(s(\lambda + \rho) - \rho)$  at  $g$  is  $\delta_0(g)^{-1}(\delta_{s\lambda}(g) + \delta_\lambda(g)) = 0$ , and  $I_2 \subset I_3$ .

Moreover  $K_0(G)/I_1$  is a free abelian group of rank  $\text{Card}(C^0)$  and by Galois theory  $K_0(G)/I_3$  has rank  $\text{Card } Z^{1/p}$ . By a case by case analysis, one checks that  $C^0$  and  $Z^{1/p}$  have the same cardinality. Therefore  $I_1 = I_2 = I_3$ , what proves Theorem 9.6. Q.E.D.

**11. Tilting modules and commutant algebras.** Let  $M$  be a tilting  $G$ -module and let  $C$  be its commutant<sup>11</sup>. For  $\lambda \in P^+$ , the  $C$ -module  $H^0(U, M)_\lambda$  will be denoted  $Sp(\lambda)$  and it will be called the *Specht module*. This terminology is not standard: usually a Specht module is only a representation of the symmetric group  $S_n$ . We will see in Section 14 why this notion generalizes the classical notion of a Specht module. Also set  $S(\lambda) = T_\lambda(M)$ . The natural map  $H^0(U, M)_\lambda \rightarrow T_\lambda(M)$  is a morphism of  $C$ -modules  $Sp(\lambda) \rightarrow S(\lambda)$ . Let  $\mathcal{K}(\lambda)$  be its kernel and let  $\mathcal{R}$  be the radical of  $C$ . Denote by  $[M : T(\lambda)]$  the multiplicity of  $T(\lambda)$  as a direct summand in  $M$ .

**Lemma 11.1:** *Let  $\lambda \in P^+$  such that  $S(\lambda) \neq 0$ .*

(i) *The  $C$ -module  $S(\lambda)$  is simple and its dimension is  $[M : T(\lambda)]$ .*

(ii) *We have  $\mathcal{K}(\lambda) = \mathcal{R}.Sp(\lambda)$ .*

(iii) *In particular, any  $v \in Sp(\lambda) \setminus \mathcal{K}(\lambda)$  generates the  $C$ -module  $Sp(\lambda)$  and  $S(\lambda)$  is the unique simple quotient of  $Sp(\lambda)$ .*

*Proof:* It follows from Lemma 7.3 that the  $C$ -module  $S(\lambda)$  is simple whenever it is not zero. Thus Assertion (i) is obvious. Set  $m = \dim S(\lambda)$ . By Lemma 7.3, we have  $M = T(\lambda)^m \oplus M'$ , where  $M'$  has no summand isomorphic to  $T(\lambda)$ . Therefore  $\dim S(\lambda) = [M : T(\lambda)]$ . Let  $x \in \mathcal{K}(\lambda)$ . By assumption, its image in  $T_\lambda(M)$  is zero, thus  $x$  belongs to  $M'$ . There

<sup>11</sup>In principle, one should consider  $M$  as a right  $C$ -module. However, we have  $M^* \simeq M^\omega$ , where  $\omega : G \rightarrow G$  is the Cartan involution. Hence the algebras  $C$  and  $C^{opp}$  are isomorphic. Thus the distinction between right  $C$ -modules and left  $C$ -modules is not important, and we will simply speak of  $C$ -modules.

is a unique  $G$ -equivariant map  $\pi : \Delta(\lambda) \rightarrow M'$  such that  $\pi(v_\lambda) = x$ , where  $v_\lambda$  is a highest weight vector of  $\Delta(\lambda)$ . As  $T(\lambda)/\Delta(\lambda)$  has a Weyl filtration and  $M'$  has a good filtration, we have  $Ext_G^1(T(\lambda)/\Delta(\lambda), M') = 0$  (this follows from Corollary 4.7). Therefore we can extend  $\pi$  to a map  $\pi' : T(\lambda) \rightarrow M'$ . As  $m > 0$ , we can write  $M = T(\lambda) \oplus M''$ , where  $M'' = T(\lambda)^{m-1} \oplus M'$ . Thus we can define a  $G$ -equivariant map  $\pi'' : M \rightarrow M$  by requiring that its restriction to  $T(\lambda)$  is  $\pi'$  and its restriction to  $M''$  is zero. It follows from its definition that  $\pi''$  is in the radical of  $C$ . Moreover,  $x$  is in the image of  $\pi''$ . Thus  $\mathcal{K}(\lambda) \subset \mathcal{R}.Sp(\lambda)$ , what proves Assertion (ii). Then Assertion (iii) follows. Q.E.D.

**Proposition 11.2:** *Let  $(M_j)_{1 \leq j \leq k}$  be  $k$  tilting modules whose commutant algebras are denoted  $(C_j)_{1 \leq j \leq k}$ . Let  $\lambda \in P^+$  such that  $\dim T(\lambda) \neq 0$  modulo  $p$ . Then  $T_\lambda(M_1 \otimes M_2 \otimes \dots \otimes M_k)$  is semisimple as a  $C_1 \otimes C_2 \otimes \dots \otimes C_k$ -module.*

*Proof:* Set  $\overline{C}_j = T^G(M_j, M_j)$ . By Lemma 8.3, the algebra  $\overline{C}_j$  is a semisimple quotient of  $C_j$ . It follows from Lemmas 7.3 and 8.3 that the functor  $T_\lambda$  factors through the category  $\mathcal{P}$ . Hence the action of  $C_j$  over  $T_\lambda(M_1 \otimes M_2 \otimes \dots \otimes M_n)$  factors through  $\overline{C}_j$ , and the proposition follows. Q.E.D.

Let  $M$  be a tilting module whose commutant is  $C$ . For the next proposition, it will be convenient to extend the definition of Specht modules  $Sp(\mu)$  to all weights  $\mu$  by setting  $Sp(\mu) = 0$  if  $\mu \notin P^+$ . We will extend the definition of the symbol  $[T(\lambda) : \nabla(\mu)]$  by requiring that  $[T(\lambda) : \nabla(\mu)] = 0$  if  $\lambda \notin P^+$  or if  $\mu \notin P^+$ . The image of a  $C$ -module  $X$  in  $K_0(C)$  will be denoted by  $[X]$ .

**Proposition 11.3:** *In  $K_0(C)$  the following equalities hold:*

- (i) For  $\lambda \in P^+$ , we have  $[Sp(\lambda)] = \sum_{w \in W} \epsilon(w) [M_{\lambda + \rho - w\rho}]$ .
- (ii) For  $\lambda \in C^0$ , we have  $[S(\lambda)] = \sum_{w \in W_{aff}} \epsilon(w) [Sp(w(\lambda + \rho) - \rho)]$ .
- (iii) For  $\lambda \in C^0$ , we have  $[S(\lambda)] = \sum_{w \in W_{aff}} \epsilon(w) [M_{w(\lambda + \rho) - \rho}]$ .

*Proof:* There is a semisimple associative subalgebra  $S$  of  $C$  such that  $C = S \oplus \mathcal{R}$  and we have  $K_0(C) = K_0(S)$ . Therefore, it is enough to establish these equalities in  $K_0(S)$ . We have  $M = \bigoplus_{\mu \in P^+} T(\mu) \otimes S(\mu)$  as a  $G \times S$ -modules. By Lemma 7.3, the  $S$ -module  $Sp(\lambda)$  is isomorphic to  $\bigoplus_{\mu \in P^+} S(\mu)^{h^0(T(\mu), \lambda)}$  for any  $\lambda \in P^+$ . By Lemma 4.2, we have  $h^0(T(\mu), \lambda) = [T(\mu) : \nabla(\lambda)]$ , hence we obtain Donkin's formula:

$$(*) \quad [Sp(\lambda)] = \sum_{\mu \in P^+} [T(\mu) : \nabla(\lambda)] [S(\mu)],$$

compare with [D3]. From Weyl character formula we deduce that

$$\begin{aligned}
 [T(\mu) : \nabla(\lambda)] &= \sum_{w \in W} \epsilon(w) \dim T(\mu)_{\lambda + \rho - w\rho}. \text{ Hence one gets:} \\
 [Sp(\lambda)] &= \sum_{\mu \in P^+} \sum_{w \in W} \epsilon(w) \dim T(\mu)_{\lambda + \rho - w\rho} [S(\mu)] \\
 &= \sum_{w \in W} \epsilon(w) [M_{\lambda + \rho - w\rho}],
 \end{aligned}$$

what proves Assertion (i).

From now on, assume that  $\lambda \in C^0$ . It follows from the linkage principle (Theorem 3.4) that  $[T(w(\lambda + \rho) - \rho) : \nabla(\mu)] = 0$  unless  $\mu + \rho$  and  $\lambda + \rho$  are  $W_{aff}$ -conjugated. Thus by Lemma 10.4, we have:

$$\begin{aligned}
 \sum_{w \in W_{aff}} \epsilon(w) [T(\mu) : \nabla(w(\lambda + \rho) - \rho)] &= \delta_{\lambda\mu}. \text{ Using } (*), \text{ one gets:} \\
 \sum_{w \in W_{aff}} \epsilon(w) [Sp(w(\lambda + \rho) - \rho)] &= \sum_{w \in W_{aff}} \sum_{\mu \in P^+} \epsilon(w) [T(\mu) : \nabla(w(\lambda + \rho) - \rho)] [S(\mu)] \\
 &= \sum_{\mu \in P^+} \delta_{\lambda\mu} [S(\mu)] \\
 &= [S(\lambda)].
 \end{aligned}$$

Therefore Assertion (ii) follows. From the first two assertions one gets:

$$[S(\lambda)] = \sum_{x \in W_{aff}^+} \sum_{y \in W} \epsilon(x)\epsilon(y) [M_{x(\lambda + \rho) - y\rho}], \text{ where } W_{aff}^+ \text{ is the set of all } x \in W_{aff} \text{ such that } x(\lambda + \rho) - \rho \in P^+. \text{ By } W\text{-invariance, the } C\text{-module } M_{x(\lambda + \rho) - y\rho} \text{ is isomorphic to } M_{y^{-1}x(\lambda + \rho) - \rho}. \text{ Any element } w \in W_{aff} \text{ can be uniquely written as } w = y^{-1}x, \text{ where } x \in W_{aff}^+ \text{ and } y \in W. \text{ Hence Assertion (iii) follows. Q.E.D.}$$

Let  $M, N$  be two tilting modules, let  $\lambda \in P^+$  and set  $C = \text{End}_G(M)$ . Note that  $C$  acts over  $M \otimes N^{\otimes n}$  (any  $u \in C$  acts as  $u \otimes 1$ ). Therefore  $T_\lambda(M \otimes N^{\otimes n})$  is a  $C$ -module and we can define the  $K_0(C)$ -valued formal series  $\chi^\lambda(z)$  by:  $\chi^\lambda(z) = \sum_{n \geq 0} [T_\lambda(M \otimes N^{\otimes n})] z^n$ . Recall that  $Z^{1/p}$  is the set of regular conjugacy classes of  $G_C$  such that  $g^p$  is central. Since  $\text{ch } N$  can be identified with a central function of  $G_C$ , let  $Z(N)$  the set of its values over  $Z^{1/p}$ , i.e.  $Z(N) = \{x \in C \mid \exists g \in Z^{1/p} : x = \text{ch } N(g)\}$ .

**Theorem 11.4:** *Let  $\lambda \in C^0$ . Then the generating series  $\chi^\lambda(z)$  is a rational function, and its poles are simple. More precisely, we have  $\chi^\lambda(z) = \sum_{x \in Z(N)} \frac{a_x}{1 - xz}$ , for some  $a_x \in C \otimes K_0(C)$ .*

*Proof.* Let  $\text{Hom}(K_0(\mathcal{P}), C)$  be the space of additive maps from  $K_0(\mathcal{P})$  to  $C$ . For  $g \in \text{Spec}(C \otimes K_0(\mathcal{P}))$  denote by  $e_g : K_0(\mathcal{P}) \rightarrow C$  the corresponding character. The hypothesis  $C^0 \neq \emptyset$  implies  $p \geq h$ . Thus Theorem 9.6 identifies  $\text{Spec}(C \otimes K_0(\mathcal{P}))$  with the finite set  $Z^{1/p}$  and by Lemma 8.4 the ring  $K_0(\mathcal{P})$  is reduced. Therefore  $(e_g)_{g \in Z^{1/p}}$  is a basis of

$Hom(K_0(\mathcal{P}), \mathbf{C})$ . For  $\nu \in C^0$ , the functor  $T_\nu : L \in \mathcal{T} \mapsto T_\nu(L)$  factors through  $\mathcal{P}$  and therefore it induces an additive map  $t_\nu : K_0(\mathcal{P}) \rightarrow \mathbf{C}$  defined by  $t_\nu([L]) = \dim T_\nu(L)$  for any tilting module  $L$ . Therefore, we have  $t_\nu = \sum_{g \in Z^{1/p}} b_\nu^g e_g$ , for some  $b_\nu^g \in \mathbf{C}$ .

Using that  $\mathcal{P}$  is semisimple and Proposition 11.2, we get:

$$T_\lambda(M \otimes N^{\otimes n}) \simeq \bigoplus_{\mu, \nu \in C^0} T_\mu(M) \otimes T_\nu(N^{\otimes n}) V_{\mu\nu}^\lambda.$$

Therefore we get:

$$\begin{aligned} \chi^\lambda(z) &= \sum_{n \geq 0} \sum_{\mu, \nu \in C^0} V_{\mu\nu}^\lambda t_\nu(N^{\otimes n}) z^n [T_\mu(M)] \\ &= \sum_{n \geq 0} \sum_{\mu, \nu \in C^0} \sum_{g \in Z^{1/p}} V_{\mu\nu}^\lambda b_\nu^g e_g(N^{\otimes n}) z^n [T_\mu(M)] \\ &= \sum_{\mu, \nu \in C^0} \sum_{g \in Z^{1/p}} V_{\mu\nu}^\lambda b_\nu^g \frac{[T_\mu(M)]}{1 - e_g(N)z}. \end{aligned}$$

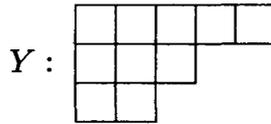
Since  $e_g(N) = \text{ch } N(g)$ , the theorem is proved. Q.E.D.

**12. Application of tilting modules to representation theory of  $GL(V)$ .** For simplicity, the results of the previous sections have been stated for simple algebraic groups only. However, it will be more convenient to work with  $GL(V)$  instead of  $SL(V)$ , and all statements of the previous sections can be easily adapted to reductive groups. We will start with a few definition involving Young tableaux and polynomial weights.

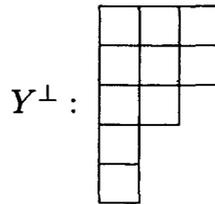
Let  $V$  be a vector space of dimension  $n$  with basis  $x_1, \dots, x_n$ . The Cartan subgroup  $H$  of  $GL(V)$  is the subgroup of diagonal matrices and its Borel subgroup  $B$  is the subgroup of upper triangular matrices. Denote by  $\epsilon_i$  the weight of  $x_i$ . Therefore we have  $\epsilon_i(h) = z_i$  for a diagonal matrix  $h \in H$  with diagonal entries  $z_1, z_2, \dots$ . Any weight  $\mu$  of  $H$  can be additively written  $\mu = \sum_{i \geq 1} m_i \epsilon_i$ , where all  $m_i$  are integers. Moreover a weight  $\mu$  is dominant if and only if  $m_1 \geq m_2 \geq m_3 \dots$ . A weight  $\mu$  is called *polynomial* if and only if  $m_i \geq 0$  for any  $i \geq 1$ . Its *degree* is  $\sum_{i \geq 0} m_i$ . A rational representation  $\phi$  of  $GL(V)$  is called *polynomial* if and only if all its weights are polynomial: this is equivalent to Green's definition [G], namely that the matrix coefficients of  $\phi(g)$  are polynomial functions into the entries  $g_{i,j}$  of the matrix. Note that for a general rational representation  $\phi$ , its matrix coefficients are element of  $K[g_{i,j}, \det g^{-1}]$ . Hence any representation of  $GL(V)$  can be written as  $M \otimes L$ , where  $M$  is polynomial and  $L$  is one dimensional. Therefore, any statement about polynomial representations of  $GL(V)$  can be easily extended to all rational representations and we will restrict ourself to polynomial representations.

A Young diagram  $Y$  is a finite sequence of integers  $(m_1, m_2 \dots)$  with  $m_1 \geq m_2 \dots \geq 0$ . A Young diagram is often represented by a set

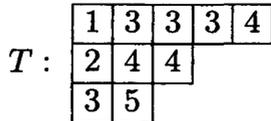
of boxes in the plane, with  $m_1$  boxes on the first line,  $m_2$  boxes on the second line and so on. For example, the graphic representation of the Young diagram  $Y = (5, 3, 2)$  is:



Let  $Y$  be a Young diagram. Its *degree*  $\deg Y$  is the total number of boxes. Its *height*  $\text{ht } Y$  is the number of boxes on the first column. We denote by  $Y^\perp$  its mirror image through the main diagonal. In the previous example,  $\deg Y = 10$ ,  $\text{ht } Y = 3$  and  $Y^\perp$  is the Young diagram:



A *tableau*  $T$  of shape  $Y$  is a labeling of the boxes of  $Y$  by the integers  $1, 2, \dots, n$ . Its *weight*  $w(T)$  is  $\sum_{i \geq 1} n_i \epsilon_i$ , where  $n_i$  is the number of occurrences of the index  $i$  in  $T$ . As usual, a tableau is called *semi-standard* if the filling is non decreasing from left to right and increasing from top to bottom. For example the tableau:



is semi-standard of shape  $Y$  and weight  $\epsilon_1 + \epsilon_2 + 4\epsilon_3 + 3\epsilon_4 + \epsilon_5$ .

There is a bijective correspondence  $\lambda \mapsto Y(\lambda)$  between dominant polynomial weights of  $GL(V)$  and Young tableaux of height  $\leq n$ . Indeed to each dominant polynomial weight  $\lambda = \sum_{1 \leq i \leq n} m_i \epsilon_i$  one associates the Young diagram  $Y(\lambda) = (m_1, m_2, \dots)$ . Let  $Y \mapsto \lambda(Y)$  be its inverse. In what follows, we will often use  $Y$  to denote the dominant weight  $\lambda(Y)$ . We will use various groups  $GL(V)$ . Therefore, we will denote by  $L_V(Y)$  the simple  $GL(V)$ -module with highest weight  $\lambda(Y)$ . It will be convenient to set  $L_V(Y) = 0$  if  $\text{ht } Y > n$ . We will use a similar notation and convention for the Weyl module  $\Delta_V(Y)$ , the dual of the Weyl module  $\nabla_V(Y)$  and the tilting module  $T_V(Y)$  with highest weight  $\lambda(Y)$ . It is easy to prove that  $V \mapsto L_V(Y)$  can be realized by a polynomial functor of degree  $\deg Y$ .

Let  $M$  be an auxiliary vector space of dimension  $m$ . Set  $G = GL(M)$ . The functor  $T_\lambda$  defined on  $\text{Mod}(G)$  (see Section 7) will be denoted by  $T_Y^M$ . There could be no confusion with the functor  $T^G$  defined in Section 8: indeed  $T^G = T_\emptyset^V$ . Set  $\mathbf{M} = \wedge(V \otimes M)$  and denote by  $\phi_V$  the action of  $GL(V)$  on  $\mathbf{M}$ .

**Lemma 12.1:** *The  $G$ -module  $\mathbf{M}$  is tilting.*

*Proof:* For any  $k \geq 0$ , the  $G$ -module  $\bigwedge^k M$  is tilting: indeed it is irreducible and satisfies Weyl character formula. Therefore  $\bigwedge^k M$  is both a Weyl module and a dual of a Weyl module. The  $G$ -module  $\mathbf{M}$  is isomorphic to  $(\bigwedge M)^{\otimes n}$ . Thus by Corollary 6.3,  $\mathbf{M}$  is tilting. Q.E.D.

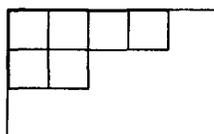
**Theorem 12.2:** (Donkin [D3]) *The algebra  $\text{End}_G(\mathbf{M})$  is generated by  $\phi_V(\text{GL}(V))$ .*

The reference for this theorem is [D3], Proposition 3.11. Indeed, we obtain a dual statement by exchanging  $V$  and  $M$ . However, it should be noted that usually  $\mathbf{M}$  is not tilting as a  $\text{GL}(V) \times G$ -module. In Howe's terminology,  $(\text{GL}(V), G)$  is a dual pair in  $\text{GL}(\mathbf{M})$ . Indeed, for fields of characteristic zero, this duality is due to Howe [Ho]. In this setting, Howe showed that the  $\text{GL}(V) \times G$ -module  $\mathbf{M}$  is isomorphic to  $\bigoplus_Y L_V(Y) \otimes L_M(Y^\perp)$ , where  $Y$  runs over all Young diagrams contained in the  $n \times m$  rectangle (i.e. such that  $\text{ht } Y \leq n$  and  $\text{ht } Y^\perp \leq m$ ). With our conventions, this restriction is not necessary, since  $L_V(Y) \otimes L_M(Y^\perp) = 0$  if  $Y$  is not contained in the  $n \times m$  rectangle. In what follows, we will always consider  $\mathbf{M}$  as a  $G$ -module. Therefore the associated Specht modules  $Sp(Y^\perp)$  (defined in Section 11) are  $\text{GL}(V)$ -modules. We will see that in our context Specht modules are simply Weyl modules, as it is proved by the next lemma (the second assertion comes from [MP1]):

**Lemma 12.3:** *Let  $Y$  be a Young diagram contained in the  $n \times m$  rectangle. As  $\text{GL}(V)$ -modules, we have:*

$$Sp(Y^\perp) = \Delta_V(Y) \text{ and } T_{Y^\perp}^M(\mathbf{M}) = L_V(Y).$$

*Proof:* Denote by  $(y_i)_{1 \leq i \leq m}$  a basis of  $M$ , and set  $z_{i,j} = x_i \otimes y_j$ . Also let  $b_{i,j}$  be the box of the  $n \times m$  rectangle located at the intersection of the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column. Denote by  $U_V$  (respectively  $U_V^-$ ) the subgroup of upper (respectively lower) unipotent triangular matrices of  $\text{GL}(V)$  and denote by  $U_M, U_M^-$  the corresponding subgroups of  $G$ . We can assume  $Y \neq \emptyset$  and we place the Young diagram  $Y$  inside the rectangle in a such way that it contains the upper left box  $b_{1,1}$ . For example:



Let  $Z$  (respectively  $Z^-$ ) be the subspace of  $V \otimes M$  generated by all  $z_{i,j}$  with  $b_{i,j} \in Y$  (respectively with  $b_{i,j} \notin Y$ ). Note that  $Z$  is a  $d$ -

dimensional  $U_V \times U_M$ -submodule of  $V \otimes M$  and  $Z^-$  is a  $(nm - d)$ -dimensional  $U_V^- \times U_M^-$ -submodule of  $V \otimes M$ , where  $d = \deg Y$ . Choose non-zero vectors  $v \in \wedge^d Z$  and  $v^- \in \wedge^{nm-d} Z^-$ .

We claim that  $T_{Y^\perp}^M(\mathbf{M})$  contains a non-zero  $U_V$ -invariant vector of weight  $Y$ . The vector  $v$  is  $U_V \times U_M$ -invariant of weight  $(Y, Y^\perp)$ . Hence it defines a  $U_V$ -invariant element  $\bar{v} \in T_{Y^\perp}^M(\mathbf{M})$ . Since  $v^-$  is  $U_M^-$ -invariant and  $v \wedge v^-$  is a non-zero vector of the trivial  $U_M^-$ -module  $\wedge^{nm}(V \otimes M)$ , we have  $\bar{v} \neq 0$ . Hence  $T_{Y^\perp}^M(\mathbf{M})$  contains a non-zero  $U_V$ -invariant vector of weight  $Y$ , namely  $\bar{v}$ .

By Lemma 11.1 (i), the  $GL(V)$ -module  $T_{Y^\perp}^M(\mathbf{M})$  is simple. Moreover it contains a  $U_V$ -invariant vector of weight  $Y$ . By the classification of simple  $GL(V)$ -modules (Theorem 2.2), we get  $T_{Y^\perp}^M(\mathbf{M}) = L_V(Y)$ . As the  $G$ -module  $\mathbf{M}$  has a good filtration, it follows from Lemma 4.2 (ii) that the character of the  $GL(V)$ -module  $Sp(Y^\perp)$  is independent of the characteristic. By the previous result of Howe, we get  $\text{ch } Sp(Y^\perp) = \text{ch } \Delta_V(Y)$ . Moreover by Lemma 11.1 (iii), the  $GL(V)$ -module  $Sp(Y^\perp)$  is generated by its highest weight vector  $v$ . Therefore  $Sp(Y^\perp) = \Delta_V(Y)$ . Q.E.D.

**Corollary 12.3:** ([MP1]) *Let  $Y$  be a Young diagram contained in the  $n \times m$  rectangle, and let  $\mu = \sum_{1 \leq k \leq n} k_i \epsilon_i$  be a polynomial weight. We have:*

$$\dim L_V(Y)_\mu = [\wedge^{k_1} M \otimes \wedge^{k_2} M \otimes \dots : T_M(Y^\perp)].$$

*Proof:* The  $\mu$ -weight space of  $\mathbf{M}$  is the subspace  $\wedge^{k_1} M \otimes \wedge^{k_2} M \otimes \dots$ . Therefore the corollary follows from Lemma 12.2 and Lemma 7.3.

We will now determine when  $Y^\perp$  is in the fundamental alcove of  $G$ . Note that this last condition depends on  $m$ . To explain this condition, we need to introduce the definition of  $m$ -special Young diagrams. For a Young diagram  $Y$ , denote by  $c_i(Y)$  the number of boxes on the  $i^{\text{th}}$ -column. For any integer  $m < p$ , say that the Young diagram  $Y$  is  $m$ -special if the number of columns is  $\leq m$  and if  $c_1(Y) - c_m(Y) \leq p - m$ . The following easy lemma is proved in [MP1].

**Lemma 12.4:** ([MP1]) *Assume  $m < p$ . The map  $Y \mapsto Y^\perp$  is a bijection from the set of  $m$ -special Young diagram to the fundamental alcove  $C^0$  of  $G$ .*

Therefore, we can use Corollary 12.3 and the modular Verlinde's formula 9.5 to compute the character of  $L_V(Y)$  whenever  $Y^\perp$  is in the fundamental alcove of  $G$ . Indeed the character formula can be computed in a combinatorial way. To explain the underlying combinatorics, we need to define the notion of a  $m$ -semi-standard tableau. For a tableau

$T$ , denote by  $T[i]$  the subset of boxes with labels  $\leq i$ . Therefore, when  $T$  is semi-standard,  $T[i]$  is again a Young diagram. By definition, a semi-standard tableau  $T$  is *m-semi-standard* if all Young diagrams  $T[i]$  are *m-special*. Set  $N(Y, \mu)$  the number of *m-semi-standard* tableaux of shape  $Y$  and weight  $\mu$ . The details of the proof of the following theorem are given in [MP1], Theorem 4.3:

**Theorem 12.4:** ([MP1]) *Let  $Y$  be a  $m$ -special Young diagram for some  $m < p$ . Any weight of  $L_V(Y)$  is polynomial and for any polynomial weight  $\mu$  we have:*

$$\dim L_V(Y)_\mu = N(Y, \mu).$$

As an example of character formula, note that for any fundamental weight  $\omega_i$ ,  $Y(m\omega_i)$  is *m-special*. Therefore, by using Steinberg tensor product Theorem 2.4, one can deduce a character formula for the  $GL(V)$ -module  $L_V(N\omega_i)$  for any  $n, N, i$  and any characteristic  $p$ . See [MP1], Theorem 5.3.

There is a general conjecture, due to Lusztig [Lu1], [Lu2] about the character of a simple rational  $GL(V)$ -module. The experts believe that this conjecture holds for  $p \geq n$  (see e.g. the introduction of [So]) and it has been proved for  $p \gg n$  by Andersen, Jantzen and Soergel [AJS]. In contrast, the previous character formula applies only to some peculiar highest weights, but they hold for  $V$  of any dimension  $n$  and are therefore outside the validity domain of Lusztig's Conjecture. This conjecture does not seem adapted to the investigation of simple polynomial functors. Using Weyl's polarizations, the simple polynomial functor  $V \mapsto L_V(Y)$  is entirely determined by the  $GL(n)$ -module  $L_{K^n}(Y)$ , where  $n = \deg Y$ . Therefore, Lusztig's Conjecture only applies to polynomial functors of degree  $\leq p$ . In contrast, there are *m-special* Young diagrams of arbitrary degree.

Let  $V = \bigoplus_{1 \leq j \leq k} V_j$  be any decomposition of  $V$ . Therefore  $\prod_{1 \leq j \leq k} GL(V_j)$  is a subgroup of  $GL(V)$ . The following theorem has been proved independently (by very different methods) in [BKS] and in [MP1]. The proof given below is simpler.

**Theorem 12.6:** ([BKS] [MP1]) *Let  $Y$  be a  $m$ -special Young diagram for some  $m < p$ . As a  $\prod_{1 \leq j \leq k} GL(V_j)$ -module,  $L_V(Y)$  is semisimple.*

*Proof:* Set  $\mathbf{M}_j = \bigwedge(V_j \otimes M)$ . By Theorem 12.1,  $GL(V_j)$  generates the commutant algebra  $C_j = \text{End}_G(\mathbf{M}_j)$ . By Lemmas 10.4 and 12.4, the dimension of the  $G$ -module  $T_M(Y^\perp)$  is not divisible by  $p$ . Therefore by Proposition 11.2, the  $\prod_{1 \leq j \leq k} GL(V_j)$ -module  $T_{Y^\perp}^M(\bigotimes_{1 \leq j \leq k} \mathbf{M}_j)$  is semisimple. Therefore the theorem follows from the fact that  $\mathbf{M} \simeq$

$\otimes_{1 \leq j \leq k} \mathbf{M}_j$  and from the isomorphism  $T_{Y^\perp}^M(\mathbf{M}) \simeq L_V(Y)$  proved in Lemma 12.3. Q.E.D.

**13. An easy example: the fusion ring of  $SL(p - 1)$ .** As before,  $V$  is a vector space of arbitrary dimension. In this section, the auxiliary vector space  $M$  is  $K^{p-1}$  and we will investigate the fusion ring for  $G = SL(p - 1)$ , which is very simple. Then we recover some well-known multiplicity one results. We denote by  $\omega_i^M$  the fundamental weights of  $G$ , and set  $\omega_0^M = 0$ . For an integer  $n$ , it will be convenient to set  $\omega_n^M = \omega_{\bar{n}}^M$  where  $\bar{n}$  is its residue modulo  $p - 1$ .

**Lemma 13.1:** *For  $G = SL(p - 1)$ , the only weights in  $C^0$  are  $\omega_0^M, \omega_1^M, \dots, \omega_{p-2}^M$ . In the category  $\mathcal{P}$ , we have:*

$$T(\omega_i^M) \otimes T(\omega_j^M) = T(\omega_{i+j}^M).$$

This Lemma follows from easy computation. As the tensor product multiplicities are 1, it follows that for any  $(p - 1)$ -special Young diagram  $Y$  the  $GL(V)$ -module  $L_V(Y)$  is multiplicity free. By this way, we recover a result of Doty. We denote by  $\omega_i$  the fundamental weights of  $GL(V)$ .

**Theorem 13.2:** (Doty [Do]) *Let  $\lambda \in P^+$  be of the form  $\lambda = a\omega_i + b\omega_{i+1}$  with  $a + b = p - 1$ . Then any weight of  $GL(V)$ -module  $L_V(\lambda)$  has multiplicity one.*

*Proof:* The Young diagram  $Y(\lambda)$  is  $(p - 1)$ -special. Therefore by Corollary 12.3 and Lemma 13.1, any any weight of the  $GL(V)$ -module  $L_V(\lambda)$  has multiplicity one. Q.E.D.

Doty's proof is based on the fact that the module  $L_V(\lambda)$  is a quotient of  $S^N V$ , where  $N = ai + b(i + 1)$ . Indeed with the notations of Section 1, we have  $L_V(\lambda) \simeq \overline{S}^N V$ . For  $\lambda \in \mathbf{C} \otimes P$ , denote by  $L_{\mathbf{C}}(\lambda)$  be the simple highest weight  $\mathfrak{gl}(n, \mathbf{C})$ -module with highest weight  $\lambda$  (when  $\lambda$  is not dominant, this module is infinite dimensional). We can also recover the following characteristic zero result:

**Theorem 13.3:** (Benckart-Britten-Lemire [BBL]) *Let  $\lambda \in \mathbf{C} \otimes P$  be a weight of the form  $\lambda = a\omega_i + b\omega_{i+1}$  where  $a, b \in \mathbf{C}$  and  $a + b = -1$ . Then any weight of  $L_{\mathbf{C}}(\lambda)$  has multiplicity one.*

*Proof:* Let  $\Lambda$  be the set of all weights of the form  $a\omega_i + b\omega_{i+1}$  with  $a + b = -1$ , and  $\Lambda_{\mathbf{Z}}$  those which are integral, i.e. such that  $a, b$  are integers. As  $\Lambda_{\mathbf{Z}}$  is Zariski dense in  $\Lambda$ , it is enough to prove the assertion for the weights in  $\Lambda_{\mathbf{Z}}$ . Then the result follows easily from the previous theorem and by semi-continuity principle. Q.E.D.

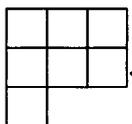
The last proof is based on the idea of [MP2]. See also [MP2] for more complicated examples of combinatorial weight multiplicity formulas in the category  $\mathcal{O}$  of  $\mathfrak{gl}(n, \mathbf{C})$ , which can be deduced from Theorem 12.4. Indeed the combinatorics is based on semi-infinite Young diagrams.

**14. Application of tilting modules to the symmetric group  $S_n$ .** In this section, we will investigate the representation of the symmetric group  $S_n$ . Let  $M$  be an auxiliary vector space of dimension  $m$  and set  $G = GL(M)$ . Denote by  $\phi$  the natural action of  $S_n$  on  $M^{\otimes n}$ . By Lemma 12.1, the  $G$ -module  $M^{\otimes n}$  is tilting. As in Section 12, we will use that  $\phi(KS_n)$  is the commutant of the  $G$ -module  $M^{\otimes n}$ . This fact has been proved by Weyl in characteristic zero and it has been extended to finite characteristics by de Concini and Procesi. See [dP] for the proof of the next result:

**Theorem 14.1:** (de Concini and Procesi [dP]) *We have:*

$$\text{End}_G(M^{\otimes n}) = \phi(KS_n).$$

We can also describe a Young diagram  $Y$  by a finite sequence  $(m_1^{a_1}, m_2^{a_2}, \dots)$  by the following rule:  $m_1, m_2, \dots$  are the various lengths of the non-empty lines of  $Y$  and  $a_k$  is the number of lines of  $Y$  of length  $m_k$ . Therefore  $m_1, m_2, \dots$  are distinct positive integers. We do not require that the sequence  $(m_1, m_2, \dots)$  is ordered, therefore  $(m_1^{a_1}, m_2^{a_2}, \dots)$  is defined up to permutation. For example, the Young diagram defined by the sequence  $(3^2, 1^1)$  is:



A Young diagram is called  $p$ -regular if  $a_k < p$  for any  $k$ . It is clear that  $Y$  is  $p$ -regular if and only if the weight  $\lambda(Y^\perp)$  is restricted. This usual terminology conflicts with the notion of  $p$ -regular weights. An element  $g \in S_n$  is  $p$ -regular if its order is not divisible by  $p$ .

**Lemma 14.2:** *There is a natural bijection between the  $p$ -regular conjugacy classes of  $S_n$  and the  $p$ -regular Young diagrams of degree  $n$ .*

*Proof:* To any conjugacy class  $[g]$  of  $S_n$ , one associates a finite sequence  $(m_1^{a_1}, m_2^{a_2}, \dots)$  by the following rule:  $m_1, m_2, \dots$  are the various lengths of the cycles of  $g$  and  $a_k$  is the number of cycles of  $[g]$  of length  $m_k$ . Therefore, there is a bijection between:

- (i) all the  $p$ -regular conjugacy classes of  $S_n$ , and
- (ii) all the sequences  $(m_1^{a_1}, m_2^{a_2}, \dots)$  of degree  $n$ , with no parts  $m_k$  divisible by  $p$ .

Any integer  $a \geq 1$  admits a  $p$ -adic expansion  $a = \sum_{r \geq 0} b_r p^r$ , where  $0 \leq b_r < p$  for any  $r \geq 0$ . For any one term sequence  $m^a$ , set  $\psi(m^a) = (m^{b_0}, (pm)^{b_1}, (p^2m)^{b_2}, \dots)$ . More precisely, we remove in the sequence the trivial parts  $(p^r m)^{b_r}$  whenever  $b_r = 0$ . For an arbitrary sequence  $(m_1^{a_1}, m_2^{a_2}, \dots)$ , set  $\psi((m_1^{a_1}, m_2^{a_2}, \dots)) = (\psi(m_1^{a_1}), \psi(m_2^{a_2}), \dots)$ . By unicity of the  $p$ -adic expansions,  $\psi$  establishes a bijections between (ii) and all the  $p$ -regular sequences of degree  $n$ . Therefore the lemma is proved. Q.E.D.

As in section 12, we will identify any Young diagram  $Y$  of height  $\leq m$  with a dominant polynomial weight of  $G$ . Define the  $S_n$ -modules:  $Sp(Y) = H^0(U, M^{\otimes n})_Y$  and  $S(Y) = T_Y(M^{\otimes n})$ . Indeed  $Sp(Y)$  is the usual Specht module. The following statement is a tilting module version of the classical Schur correspondence, formulated by Green in [G].

**Proposition 14.3:**

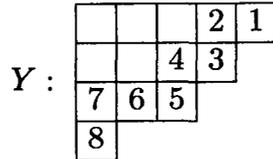
- (i) *The  $S_n$ -module  $Sp(Y)$  is independent of the dimension  $m$  of  $M$  (provided that  $m \geq \text{ht } Y$ ; otherwise  $Sp(Y) = 0$ ).*
- (ii) *If  $Y$  is a  $p$ -regular and  $m \geq \text{ht } Y$ ,  $S(Y)$  is a non-zero simple  $S_n$  module which is independent of  $m$ ; otherwise  $S(Y) = 0$ .*
- (iii) *If  $\dim M \geq n$ , then  $Y \mapsto S(Y)$  is a bijection from the  $p$ -regular Young diagrams  $Y$  of degree  $n$  to the simple  $KS_n$ -modules.*

*Proof:* Under the proviso  $m \geq \text{ht } Y$ , the weight space  $(M^{\otimes n})_Y$  is independent of  $m$ , and therefore  $Sp(Y)$  is also independent of the dimension of  $M$ , what proves the first assertion. In order to prove the last two assertions, we can assume that  $m \geq n$ . By Lemma 11.1(i) and Theorem 14.1, the  $S_n$ -module  $S(Y)$  is simple whenever it is not zero.

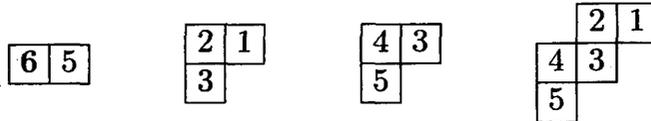
We claim that  $S(Y) = 0$  whenever  $Y$  is not  $p$ -regular. Set  $V = K^n$ ,  $\mathbf{M} = \bigwedge(V \otimes M)$  and  $\epsilon = \sum_{1 \leq k \leq n} \epsilon_i$ . We can identify  $M^{\otimes n}$  with the  $\epsilon$ -weight space of the  $GL(V)$ -module  $\mathbf{M}$ . By Lemma 12.3,  $S(Y)$  is the  $\epsilon$ -weight space of the simple  $GL(V)$ -module  $L_V(Y^\perp)$ . The weight  $Y^\perp$  is not restricted, therefore by Steinberg tensor product Theorem 2.4,  $\epsilon$  is not a weight of  $L_V(Y^\perp)$ . Thus  $S(Y) = 0$  and the claim is proved.

Thanks to the additional assumption  $m \geq n$ ,  $M^{\otimes n}$  contains the regular representation of  $S_n$ . Thus any simple  $S_n$ -module occurs as a subquotient of  $M^{\otimes n}$ . Therefore any  $S_n$ -module is isomorphic to  $S(Y)$ , for some  $p$ -regular Young diagram  $Y$  of degree  $n$ . By Brauer's theory the number of simple  $S_n$ -module equals the number of  $p$ -regular conjugacy classes in  $S_n$ . By Lemma 14.2, this number equals the number of  $p$ -regular Young diagrams  $Y$  of degree  $n$ . Therefore  $Y \mapsto S(Y)$  is a bijection from the  $p$ -regular Young diagrams  $Y$  of degree  $n$  to the simple  $KS_n$ -modules. In particular  $S(Y) \neq 0$  if  $Y$  is  $p$ -regular. Q.E.D.

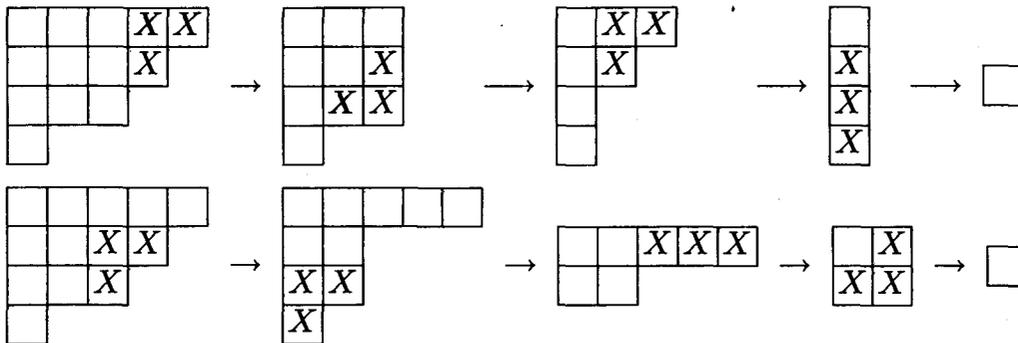
Let  $Y$  be a Young diagram. Its *rim* is the set of boxes of  $Y$  of position  $(i, j)$  such that there are no boxes in position  $(i + 1, j + 1)$ . For example the rim of the diagram  $Y$  below is the set indexed boxes:



A  $p$ -rim of  $Y$  is a connected piece  $Z$  of the rim of  $Y$  of size  $p$  such that  $Y \setminus Z$  is again a Young diagram. A  $p$ -core is a Young diagram  $Y$  which does not contain any  $p$ -rim. In the previous example the  $p$ -rim of  $Y$  for  $p = 2, 3, 5$  are:



There are no 7-rims, therefore  $Y$  is a 7-core. Starting with a Young diagram  $Y$ , we can remove successively  $p$ -rim, until we get a  $p$ -core  $\bar{Y}$ . Although there are usually more than one way to remove  $p$ -rim from  $Y$ , the  $p$ -core  $\bar{Y}$  depends only on  $Y$ . Therefore  $\bar{Y}$  is called the  $p$ -core of  $Y$ . In our previous example, the 3-core of  $Y$  is the one box Young diagram. We show below two different ways to obtain the 3-core  $\bar{Y}$  of  $Y$  by successively removing 3-rims (at each step, the removed 3-rim is indicated by the crossed boxes).



Let  $Y$  be a Young diagram  $Y = (m_1, m_2, \dots)$  of height  $\leq m$ . Denote by  $C_m(Y)$  the set of all Young diagram  $Y'$  of height  $\leq m$  with the same degree and  $p$ -core than  $Y$ . Set  $\epsilon_m(Y) = (-1)^{l_m(Y)}$ , where  $l_m(Y) = \sum_{1 \leq i < j \leq m} [(m_i - m_j + j - i)/p]$ , and where  $[x]$  denotes the integral part of any  $x \in \mathbb{Q}$ . Assume now  $m < p$ . The Young diagram  $Y$  is called  $m$ -small if and only if  $m_1 - m_m \leq p - m$ . It should be noted that the Specht modules are the reductions modulo  $p$  of the simple  $CS_n$ -modules. Thus their Brauer characters are well-understood. Therefore the next statement describes the Brauer character of the simple modular representations  $S(Y)$  for any  $m$ -small Young diagram  $Y$ .

**Theorem 14.4:** *Let  $m < p$  and let  $Y$  be a  $m$ -small Young diagram  $Y$ . Then in  $K_0(S_n)$ , we have:*

$$[S(Y)] = \sum_{Y' \in \mathcal{C}_m(Y)} \epsilon_m(Y') [Sp(Y')].$$

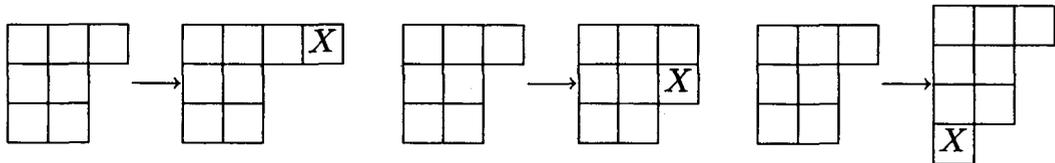
*Proof:* The following combinatorial observations are easy:

(i) As  $Y$  is  $m$ -small, the weight  $\lambda = \lambda(Y)$  is in the fundamental alcove  $C^0$  of  $G$ .

(ii) The Young diagrams  $Y \in \mathcal{C}_m(Y)$  correspond exactly to all dominant and polynomial weights of the form  $w(\lambda + \rho) - \rho$  for some  $w \in W_{aff}$ . Moreover  $l_m(Y)$  is the length of  $w$  and therefore  $\epsilon_m(Y) = \epsilon(w)$ .

Hence the theorem follows from Proposition 11.3 (ii). Q.E.D.

It is also possible to use Proposition 11.3 (iii) to write  $[S(Y)]$  as a combination of induced modules. However the index set is an affine Weyl group (instead of a Weyl group), therefore it is not possible to express it in terms of a determinant as in characteristic zero. However, one can derive a combinatorial formula for  $\dim S(Y)$ . Let  $\mathcal{Y}$  be the oriented graph whose vertices are the Young diagrams and whose arrows are  $Y \rightarrow Y'$  if  $Y'$  is obtained by adding one box to  $Y$ . For example, there are three arrows originating in the Young diagram  $(3, 2^2)$ , as shown below (the cross indicates the added box):



For  $m < p$ , let  $\mathcal{Y}_m$  be the set of all  $m$ -small Young diagrams.

**Theorem 14.5:** ([M3]) *Let  $Y \in \mathcal{Y}_m$ . Then the dimension of  $S(Y)$  is the number of oriented paths from  $\emptyset$  to  $Y$  entirely contained in  $\mathcal{Y}_m$ .*

For the proof, see [M3].

Let  $\epsilon$  be the signature representation of  $S_n$ . Since the simple representations of  $S_n$  are indexed by the  $p$ -regular Young diagram of degree  $n$ , the tensor product by  $\epsilon$  induces an involution  $Y \mapsto Y^\epsilon$  on the set of  $p$ -regular Young diagrams, namely we have  $S(Y^\epsilon) = S(Y) \otimes \epsilon$ . In characteristic zero, this involution is simply the usual transposition  $Y \mapsto Y^\perp$ . However, in characteristic  $p$ , the involution  $Y \mapsto Y^\epsilon$  is given by a more complicated rule, which has been conjectured by Mullineux and proved by Kleshchev [K11]. In an unpublished work, Rouquier used the Mullineux algorithm, to prove that the set of small Young diagrams is stable by this involution. However, this can be proved directly.

**Proposition 14.6:** *Let  $m < p$ . For any  $Y \in \mathcal{Y}_m$ ,  $Y^\epsilon$  belongs to  $\mathcal{Y}_{p-m}$ . Moreover the map  $Y \mapsto Y^\epsilon$  induces a bijection from  $\mathcal{Y}_m$  to  $\mathcal{Y}_{p-m}$ .*

*Proof:* For any  $S_n$  module  $X$ , denote by  $\chi_X : S_n \rightarrow K$  be its ordinary character, namely  $\chi_X(g) = \text{Tr } g|_X$  for any  $g \in S_n$ . We have  $\chi_X = \sum_Y [X : S(Y)] \chi_{S(Y)}$ , where  $Y$  runs over the set of  $p$ -regular Young diagrams of degree  $n$  and where  $[X : S(Y)]$  denotes the multiplicity of  $S(Y)$  in a composition series of  $X$ . As the characters  $\chi_{S(Y)}$  are linearly independent, the residues modulo  $p$  of the multiplicities  $[X : S(Y)]$  are completely determined by  $\chi_X$ .

Let  $M$  a vector space of dimension  $m$ , let  $N$  be a vector space of dimension  $m - p$  and let  $Y$  be a  $p$ -regular Young diagram of degree  $n$ . We have  $\chi_{M^{\otimes n}} = \sum_Y \dim T_M(Y) \chi_{S(Y)}$ . It follows from the lemmas 10.4 and 12.4 that:

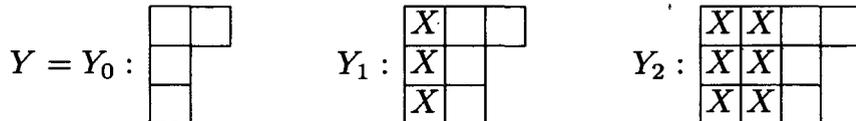
(i)  $[M^{\otimes n} : S(Y)] \neq 0$  modulo  $p$  if and only if  $Y$  is  $m$ -small.

For  $g \in S_n$  denotes by  $L(g)$  be the number of cycles of  $g$ . We have  $\chi_{M^{\otimes n}}(g) = m^{L(g)}$ ,  $\chi_{N^{\otimes n}}(g) = (-m)^{L(g)}$  and  $\chi_\epsilon(g) = (-1)^{n+L(g)}$ . We deduce that  $\chi_{M^{\otimes n}} = (-1)^n \chi_\epsilon \chi_{N^{\otimes n}}$ . It follows that:

(ii)  $[M^{\otimes n} : S(Y)] = (-1)^n [N^{\otimes n} : S(Y^\epsilon)]$  modulo  $p$ .

Thus the proposition follows from the assertions (i) and (ii). Q.E.D.

In view of the next statement, fix a Young diagram  $Y$  of degree  $n$  and of height  $\leq m$ . For any  $k \geq 0$ , denote by  $Y_k$  the Young diagram obtained by adding a rectangle of height  $m$  and length  $k$  on the left side of  $Y$ . Here is an example with  $m = 3$  (the added rectangle corresponds with the crossed boxes):



Note that  $Y_k$  is a Young diagram (even when  $\text{ht } Y < m$ ) and  $\text{deg } Y_k = n + km$ . Consider  $S_n$  as a subgroup of  $S_{n+km}$  as usual; henceforth  $S(Y_k)$  can be viewed as  $S_n$ -module by restriction. Therefore we can define the formal series  $\chi_g^Y(z) = \sum_{k \geq 0} \dim S(Y_k)^g z^k$ , for any  $p$ -regular element  $g \in S_n$ , where  $S(Y_k)^g$  is the space of  $g$ -invariant vectors of  $S(Y_k)$ . For  $g = 1$ , the series is simply  $\sum_{k \geq 0} \dim S(Y_k) z^k$ .

Assume now that  $m < p$  and let  $Z(m)$  be the set of complex numbers  $x$  of the form  $x = (\sum_{1 \leq i \leq m} \zeta_i)^m$ , where  $\zeta_1, \dots, \zeta_m$  are  $m$  distinct  $p$ -roots of 1 such that  $\prod_{1 \leq i \leq m} \zeta_i = 1$ .

**Theorem 14.7:** *Assume  $Y$  is a  $m$ -small Young diagram. Then  $\chi_g^Y(z)$  is a rational function with simple poles. More precisely, we have:*

$$\chi_g^Y(z) = \sum_{x \in Z(m)} \frac{a_x^g}{1-xz},$$

for some  $a_x^g \in \mathbb{C}$ .

*Proof:* By restriction, each  $S_{n+mk}$ -module  $S(Y_k)$  can be viewed as

an  $S_n$ -module. Thus denote by  $[S(Y_k)]$  its image in  $K_0(KS_n)$  and set  $\chi^Y(z) = \sum_{k \geq 0} [S(Y_k)] z^k$ . As  $g$  is  $p$ -regular, its action on any  $S_n$ -module  $X$  is semisimple. Therefore the map  $X \mapsto \dim X^g$  induces a linear map  $L_g : \mathbb{C} \otimes K_0(KS_n) \rightarrow \mathbb{C}$  and we have  $\chi_g^Y(z) = L_g \circ \chi^Y(z)$ . Therefore it is enough to prove a similar statement for the  $K_0(KS_n)$ -valued series  $\chi^Y(z)$ , namely  $\chi^Y(z) = \sum_{x \in Z(m)} \frac{a_x}{1-xz}$ , for some  $a_x \in \mathbb{C} \otimes K_0(KS_n)$ .

As before, let  $M$  be an auxiliary vector space of dimension  $m$ . Set  $G' = SL(M)$ ,  $M' = M^{\otimes n}$ ,  $N' = M^{\otimes m}$  and let  $\lambda'$  be the restriction of  $\lambda(Y)$  to the group  $H'$  of diagonal matrices of  $G'$ . The restriction to  $H'$  of the weights  $\lambda(Y_k)$  are all equal to  $\lambda'$ . Note also that  $\lambda'$  is in the fundamental alcove  $C^0$  of  $G'$ . Moreover the commutant of the  $G'$ -module  $M'$  is again  $\phi(KS_n)$  and the  $G'$ -modules  $M'$  and  $N'$  are tilting. Therefore, we can apply Theorem 11.4 to the group  $G'$  and to its tilting modules  $M'$  and  $N'$ .

We claim that  $Z(N') = Z(m)$ . It is clear that  $Z^{1/p}$  consists of conjugacy classes of  $A \in G'$  with  $m$  distinct eigenvalues  $\xi_1, \dots, \xi_m$  such that  $\prod_{1 \leq i \leq m} \xi_i = 1$  and  $\xi_i^p = \xi_j^p$  for any  $i, j$ . Therefore we can write  $\xi_i = \zeta \cdot \zeta_i$ , where  $\zeta^m = 1$  and where the  $\zeta_1, \dots, \zeta_m$  are  $m$  distinct  $p$ -roots of 1 with  $\prod_{1 \leq i \leq m} \zeta_i = 1$ . For such a matrix  $A$ , we have  $\text{ch } N'(A) = (\sum_{1 \leq i \leq m} \xi_i)^m = (\sum_{1 \leq i \leq m} \zeta_i)^m$ , what proves  $Z(N') = Z(m)$ .

It follows from Theorem 11.4 that  $\chi^Y(z) = \sum_{x \in Z(m)} \frac{a_x}{1-xz}$ , for some  $a_x \in \mathbb{C} \otimes K_0(KS_n)$ . Q.E.D.

*Remarks:* We can consider similar series  $\chi_g^Y(z)$  by using representations over a field of characteristic zero. However, these series are usually not rational. Let  $Y$  be a  $m$ -small Young diagram of degree  $n$ , and for any  $k$  denote by  $S_{\mathbb{C}}(Y_k)$  be the simple  $\mathbb{C}S_{n+mk}$  associated with the Young diagram  $Y_k$ . When  $k \rightarrow \infty$ , the space  $S(Y_k)$  is very small compared to its characteristic zero counterpart  $S_{\mathbb{C}}(Y_k)$ . Indeed we have the following asymptotic estimates for  $k \rightarrow \infty$ :

$$\dim S_{\mathbb{C}}(Y_k) \sim C k^{-\alpha} m^{km}, \text{ and } \dim S(Y_k) \sim C' \left| \frac{\sin m\pi/p}{\sin \pi/p} \right|^{km},$$

for some positive constants  $C, C', \alpha$ . The first estimate is an easy corollary of the hook formula. The second estimate is based on the fact that  $(\frac{\sin \pi/p}{\sin m\pi/p})^m$  is the pole of biggest modulus of the rational series  $\chi^Y(z)$ , what follows from Theorem 14.7. Similar generating functions have been considered by Erdmann for  $m = 2$  see [E]. It turns out that for  $m = 2$ , the series  $\chi_g^Y(z)$  are rational for any  $Y^{12}$ .

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<sup>12</sup>E.g. this follows from the fact that any tilting module for  $SL(2)$  is outside a cofinite tilting ideal. It seems unlikely that the series  $\chi_g^Y(z)$  is rational for

Until the end of the section, we will use the following new hypotheses: we fix a  $p$ -regular Young diagram  $Y$  of degree  $n$  and  $M$  is a vector space of arbitrary dimension  $m$ . A *partition* of  $n$  is a sequence  $\mathbf{a} = (a_1, \dots, a_k)$  of positive integers with  $\sum_{1 \leq j \leq k} a_j = n$ . For such a partition  $\mathbf{a}$ , the group  $S_{\mathbf{a}} = \prod_{1 \leq j \leq k} S_{a_j}$  is viewed as a subgroup of  $S_n$  as usual. Following Kleshchev's terminology [K12],  $S(Y)$  is called *completely splittable* if it is semisimple as a  $KS_{\mathbf{a}}$ -module for any partition  $\mathbf{a}$  of  $n$ .

**Lemma 14.8:** *Assume that  $\dim T_M(Y)$  is not divisible by  $p$ , for some vector space  $M$ . Then  $S(Y)$  is completely splittable.*

*Proof:* Note that  $\dim M \geq \text{ht } Y$ , otherwise  $T_M(Y)$  would be zero. Let  $\mathbf{a}$  be any partition of  $n$ . Set  $M_j = M^{\otimes a_j}$  for any  $j$ . By Theorem 14.1,  $S_{a_j}$  generates the commutant of the tilting  $G$ -module  $M_j$ . By definition,  $S(Y) = T_Y^M(\otimes_{1 \leq j \leq k} M_j)$ . Therefore by Proposition 11.2, the  $S_{\mathbf{a}}$ -module  $S(Y)$  is semisimple. Q.E.D.

Whenever  $Y$  is  $m$ -small for some  $m < p$ ,  $\dim T_M(Y)$  is not divisible by  $p$  by Lemma 10.4. Thus Lemma 14.6 provides a simple proof of the following Kleshchev's theorem:

**Theorem 14.9:** (Kleshchev [K11][K12])

*If  $Y$  is  $m$ -small for some  $m < p$ , then  $S(Y)$  is completely splittable.*

*Remark:* The  $m$ -small Young diagrams are considered in [We] in the context of Hecke representations.

**15. Comparison with the quantum case.** Let  $G$  be a reductive group. Denote by  $U_{\eta}$  be the corresponding quantum group at a  $p$ -root of unity  $\eta$ . Tilting modules are defined as well for quantum groups. Denote by  $T_{\eta}(\lambda)$  the tilting  $U_{\eta}$ -module with highest weight  $\lambda$ . Although  $\text{ch } T(\lambda)$  is still unknown, the character of tilting modules  $T_{\eta}(\lambda)$  has been determined by Soergel [So2][So3]. Therefore one should try to compare the tilting module  $T(\lambda)$  with its quantum analog. It follows from Theorem 16.4 that  $\text{Ext}_{\mathbb{C}}^k(T(\lambda), T(\lambda)) = 0$  for  $k = 1, 2$ . By deformation theory, the obstruction of a lifting lies in the  $\text{Ext}^2$ -group and its unicity in the  $\text{Ext}^1$ -group. Therefore  $T(\lambda)$  can be uniquely lifted to a representation of  $U_{\eta}$ , and  $\text{ch } T(\lambda) - \text{ch } T_{\eta}(\lambda)$  is a non-negative linear combination of  $\text{ch } T_{\eta}(\mu)$  for some  $\mu < \lambda$  (see [Je]). The following two conjectures are closely related:

**Conjecture 15.1:** (Andersen [A7]) *If  $(\lambda + \rho)(h_0) < p^2$ , then:*

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all  $Y$  when  $m > 3$

$$\text{ch } T(\lambda) = \text{ch } T_\eta(\lambda).$$

**Conjecture 15.2:** ( $G = GL(V)$ ) Let  $Y$  be a Young diagram with  $\text{deg } Y < p^2$ , and set  $\lambda = \lambda(Y)$ . Then  $\text{ch } T(\lambda) = \text{ch } T_\eta(\lambda)$ .

Let  $\mathcal{H}_n(\eta)$  be the Hecke  $\mathbf{C}$ -algebra of  $S_n$  evaluated at  $\eta$ . The simple modules  $\mathcal{H}_n(\eta)$ -modules are denoted by  $S_\eta(Y)$ , where  $Y$  is a  $p$ -regular Young diagram of degree  $n$ , see [DJ]. The character of  $S_\eta(Y)$  are determined [KL][Ar][Gr]. Therefore it is interesting to know when the characters of  $S(Y)$  and  $S_\eta(Y)$  are equal, or equivalently when their dimensions are the same.

**Conjecture 15.3:** (James) Let  $Y$  be a Young diagram with  $\text{deg } Y < p^2$ . Then  $\dim S(Y) = \dim S_\eta(Y)$ .

Andersen Conjecture implies Lustig Conjecture [A7]. Using the methods of [M3] it is easy to show that Conjecture 15.2 is indeed equivalent to James Conjecture: they are equivalent to the fact that  $M^{\otimes n}$  decomposes in the same way as its quantum analog, whenever  $n < p^2$ , for any vector space  $M$ . James conjecture cannot hold for  $n \geq p^2$ . It should be noted that the condition  $n < p^2$  is exactly the validity domain of Broué's conjecture: for  $n < p^2$ , the  $p$ -Sylow subgroups of  $S_n$  are abelian. These conjectures are unstable, i.e. for a given  $p$  they concern only Young diagrams of bounded size. Based on the clever  $SL(3)$ -computations of [Je], we try the following stable conjecture:

**Conjecture 15.4** Let  $m$  be an integer with  $3 \leq m \leq p$ . Let  $Y = (m_1, \dots, m_m)$  be a Young diagram such that  $m_1 - m_i + (i - 1) < p$  or  $m_i - m_m + (m - i) < p$ , for any  $1 \leq i \leq m$ . Then:

$$\dim S(Y) = \dim S_\eta(Y).$$

For  $m = 3$ , the conjecture holds [JM]:

**Theorem 15.4:** ([JM]) Assume  $p$  odd. Let  $Y = (m_1, m_2, m_3)$  be a Young diagram such that  $m_1 - m_2 \leq p - 2$  or  $m_2 - m_3 \leq p - 2$ . Then:

$$\dim S(Y) = \dim S_\eta(Y).$$

### 16. Appendix: Cohomological criterion for good filtrations.

In section 4, we try to provide the most elementary approach of good filtrations. Especially, we only use the the simplest part of the vanishing theorem of Cline, Parshall, Scott and van der Kallen (Theorem A4) to prove Dönkin Criterion 4.7. In this appendix, we will connect the approach of Section 4 with the usual cohomological description of good filtrations [FP]. For a weight  $\mu \in Q^+$ , we set  $\text{ht}(\mu) = \sum_{i \in I} m_i$  if  $\mu = \sum_{i \in I} m_i \alpha_i$ . This is sometimes called the *height* of  $\mu$ , but this terminology should not be confused with the height of Young diagrams.

**Lemma 16.1:** *Let  $M$  be a  $B$ -module and  $k \geq 0$ . If  $\nu$  is a weight of  $H^k(U, M)$  we have  $\nu \leq \mu$  and  $ht(\mu - \nu) \geq k$  for some weight  $\mu$  of  $M$ .*

*Proof:* Any  $B$ -module  $M$  admits an injective envelope  $I(M)$ : we have  $M \subset I(M)$ ,  $H^0(U, M) = H^0(U, I(M))$  and  $I(M)$  is injective. Set  $Z = I(M)/M$ . For any weight  $\nu$  of  $Z$ , we have  $\nu < \mu$  for some weight  $\mu$  of  $M$ . Choose an injective resolution of  $M$ :  $0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \dots$ , such that  $I_0 = I(M)$ ,  $I_1 = I(Z)$  and  $I_k$  is the injective envelope of the cokernel of  $I_{k-2} \rightarrow I_{k-1}$  for any  $k \geq 2$ . By induction, one proves that for any weight  $\nu$  of  $I_k$  we have  $\nu \leq \mu$  and  $ht(\mu - \nu) \geq k$ , for some weight  $\mu$  of  $M$ . Any weight  $\nu$  of  $H^k(U, M)$  is a weight of  $I_k$  and the lemma follows. Q.E.D.

Recall that  $D : Mod(B) \rightarrow Mod(G)$  is the induction functor from  $B$  to  $G$ . The functor  $D$  is left exact and we denote by  $D^*$  its derived functor. By definition, we have  $DM = D^0M = H^0(G/B, \mathcal{L}(M))$ . The next two lemmas are well-known and they fit in the framework of Zuckerman's functors.

**Lemma 16.2:** *For any  $B$ -module  $M$ , we have:*

$$D^k M = H^k(G/B, \mathcal{L}(M)), \text{ for all } k \geq 0.$$

*Proof:* As the functor  $\mathcal{L}$  is exact, it is enough to prove that  $H^k(G/B, \mathcal{L}(M)) = 0$ , for all  $k > 0$  and any injective  $B$ -module  $M$ . As any indecomposable injective module is a direct summand of  $K[B]$ , we only have to prove the claim for  $M = K[B]$ . Let  $\pi : G \rightarrow G/B$  be the natural projection. We have  $\mathcal{L}(K[B]) = \pi_* \mathcal{O}_G$ . As the variety  $G$  and the morphism  $\pi$  are affine, we have  $R^i \pi_* \mathcal{O}_G = 0$  and  $H^i(G, \mathcal{O}_G) = 0$  for  $i > 0$  by Serre's vanishing theorem. Thus the vanishing of  $H^i(G/B, \mathcal{L}(K[B]))$  follows from Leray's spectral sequence. Q.E.D.

**Lemma 16.3:** *Let  $M$  be a  $G$ -module. We have:*

$$Ext_G^i(\Delta(\lambda), M) = H^i(U, M)_\lambda, \text{ for any } \lambda \in P^+.$$

*Proof:* Let  $N$  be a  $B$ -module. We have  $H^0(G, DN) = H^0(B, N)$ , thus the functor  $H^0(B, -)$  is the composite of the functors  $D$  and  $H^0(G, -)$ . Clearly  $D$  maps injective  $B$ -modules to injective  $G$ -modules. So there is a spectral sequence converging to  $H^*(B, N)$  whose  $E_2^{**}$ -term is  $H^*(G, D^*N)$ .

Assume now that  $N = M \otimes K(-\lambda)$ . Then  $D^k N = M \otimes D^k K(-\lambda) = 0$  for  $k > 0$  by Lemma 16.2 and Kempf's vanishing theorem 3.2. Thus the previous spectral sequence degenerates, and we have  $H^k(G, \nabla(w_0\lambda) \otimes M) = H^k(B, N)$  for all  $k$ . Thus we get:

$$\begin{aligned} H^k(U, M)_\lambda &= H^k(B, M \otimes K(-\lambda)) \\ &= H^k(G, \nabla(w_0\lambda) \otimes M) \end{aligned}$$

$$= \text{Ext}_G^k(\Delta(\lambda), M). \quad \text{Q.E.D.}$$

**Theorem 16.4:** (Cline-Parshall-Scott-van der Kallen vanishing Theorem [CPSV]) *For any  $\lambda, \mu \in P^+$ , we have:*

$$\text{Ext}_G^k(\Delta(\lambda), \nabla(\mu)) = 0, \text{ for all } k > 0.$$

*Proof:* Let  $\lambda, \mu \in P^+$ . We claim that  $H^k(G, \nabla(\lambda) \otimes \nabla(\mu)) = 0$  for any  $k > 0$ . By symmetry of the roles of  $\lambda$  and  $\mu$ , we can assume  $-w_0\mu \not\prec \lambda$ . By the Lemma 16.3,  $H^k(G, \nabla(\lambda) \otimes \nabla(\mu)) = H^k(U, \nabla(\lambda))_{w_0\mu}$  and this last group is 0 by Lemma 16.1. Therefore  $\text{Ext}_G^k(\Delta(\lambda), \nabla(\mu)) = H^k(G, \nabla(-w_0\lambda) \otimes \nabla(\mu)) = 0$ . Q.E.D.

For any  $G$ -module  $M$ ,  $\lambda \in P^+$  and  $k \geq 0$ , set  $h^k(M, \lambda) = \dim H^k(U, M)_\lambda$ . For  $k = 0, 1$ , these numbers have been defined in Section 4, and by Lemma 16.3 the two definitions agree. For a given  $G$ -module  $M$ , almost all numbers  $h^k(M, \lambda)$  are zero (see Lemma A.1) and all of them are  $< \infty$ .

**Theorem 16.5:** *Let  $M$  be a  $G$ -module, and let  $n \geq 0$ .*

$$(i) \text{ If } n \text{ is even, we have } \text{ch } M \leq \sum_{\lambda \in P^+} \sum_{k \leq n} (-1)^k h^k(M, \lambda) \text{ch } \nabla(\lambda),$$

$$(ii) \text{ if } n \text{ is odd, we have } \text{ch } M \geq \sum_{\lambda \in P^+} \sum_{k \leq n} (-1)^k h^k(M, \lambda) \text{ch } \nabla(\lambda).$$

*Proof:* By induction on  $n$ . It follows from the proof of Proposition 4.5 that there exists a short exact sequence  $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$ , where  $X$  has a good filtration. From the vanishing theorem 16.4, we get:

$$h^0(M, \lambda) - h^1(M, \lambda) = h^0(X, \lambda) - h^0(N, \lambda),$$

$$h^k(M, \lambda) = h^{k-1}(N, \lambda), \text{ for } k \geq 2.$$

By Lemma 4.2, we have  $\text{ch } X = \sum_{\lambda \in P^+} h^0(X, \lambda) \text{ch } \nabla(\lambda)$  and, by induction hypothesis,  $\sum_{\lambda \in P^+} \sum_{k \leq n} (-1)^k h^k(N, \lambda) \text{ch } \nabla(\lambda)$  can be compared with  $\text{ch } N$ . The inequality involving  $\text{ch } M$  follows. Q.E.D.

Following Friedlander and Parshall [FP], we say that a  $G$ -module  $M$  has *good dimension*  $\leq m$  if there exists a resolution  $0 \rightarrow M \rightarrow X_0 \rightarrow \dots \rightarrow X_m \rightarrow 0$ , where all  $X_i$  are (finite dimensional)  $G$ -modules with a good filtration.

**Corollary 16.6:** *Let  $m \geq 0$  and let  $M$  be a  $G$ -module. The following assertions are equivalent:*

$$(i) \text{ch } M = \sum_{\lambda \in P^+} \sum_{k \leq m} (-1)^k h^k(M, \lambda) \text{ch } \nabla(\lambda),$$

$$(ii) \text{ } M \text{ has good dimension } \leq m,$$

$$(iii) H^{m+1}(U, M)_\lambda = 0, \text{ for any } \lambda \in P^+.$$

*Proof:* The equivalence (i)  $\iff$  (iii) follows from Theorem 16.5 (apply it for  $n = m$  and for  $n = m + 1$ ). Using a short exact sequence  $0 \rightarrow M \rightarrow$

$X \rightarrow N \rightarrow 0$ , where  $X$  has a good filtration, the equivalence (i)  $\iff$  (ii) follows also by induction over  $m$ . Q.E.D.

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*Université Louis Pasteur, IRMA,  
7, rue René Descartes, 67000  
Strasbourg, FRANCE.  
email: mathieu@math.u-strasbg.fr*

and

*Université Paris 7, 2, place Jussieu, 75005 PARIS.*