I §6 Filters Exercises

① Find all possible filters on a finite set. Identify the ultrafilters

② Let $\mathcal{F}$ be a filter on an infinite set X. Suppose that the intersection of all elts of $\mathcal{F}$ is empty. Then $\mathcal{F}$ is finer than the finite complement filter.

③ $\mathcal{F}_1^{\text{filter}} \cap \mathcal{F}_2^{\text{filter}} = \{ F_1 \cup F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2 \}$

④ X infinite set. Finite complement filter = intersection of elementary filters associated with infinite sequences with distinct elements.

⑤ Let $\mathcal{F}$ & $\mathcal{G}$ be filters so that they have a common upper bound. Then they do have a least upper bound and this consists of sets of the form $F \cap G$, $F \in \mathcal{F}, G \in \mathcal{G}$.

⑥ From filters on X we get topologies on $X \cup \{\omega\} = \tilde{X}$ as already seen.

ⓐ If a topology (on $\tilde{X}$) is finer than the topology $\mathcal{T}(\mathcal{F})$ associated with a filter $\mathcal{F}$ on X then it is either discrete or $\mathcal{T}(\mathcal{F}')$ for $\mathcal{F}'$ finer filter than $\mathcal{F}$ on X. Converse?

ⓑ GLBs are preserved under this association (of topologies to filters)

ⓒ If $\mathcal{G}$ is a subbase of $\mathcal{F}$, then $\mathcal{H} = \tilde{\mathcal{G}} \cup \mathcal{P}(X)$ is a subbase of $\mathcal{T}(\mathcal{F})$.
If $\mathcal{G}$ is a collection of subsets of X (not necessarily a sub-base for a filter) what topology does $\mathcal{H}$ generate?
$\tilde{\mathcal{G}} := \{ F \cup \{\omega\} \mid F \in \mathcal{G} \}$.

ⓓ $\mathcal{F}$ ultra $\iff$ $\mathcal{T}(\mathcal{F})$ is X-maximal (see I.§3.11)

⑦ for $A^{\text{non-empty}} \subseteq X$. $\mathcal{N}_A = \bigcap_{a \in A} \mathcal{N}_a$

⑧ ⓐ Let Φ be a collection of topologies on X. Let $\mathcal{T}_0$ be the intersection of the elements of Φ. For $x \in X$, the nbhd filter $\mathcal{N}_{x,\mathcal{T}_0}$ w.r.t. $\mathcal{T}_0$ is coarser than $\bigcap_{\mathcal{T} \in \Phi} \mathcal{N}_{x,\mathcal{T}}$.

ⓑ On $\mathbb{Q}^2$ let $\mathcal{T}_1$ be the top generated on $\mathbb{Q}^2$ by open intervals w.r.t. lexicographic total order; let $\mathcal{T}_2$ be the similarly defined top with the lexicographic order being the other way around. Show that for $\Phi = \{\mathcal{T}_1, \mathcal{T}_2\}$ and any point x in $\mathbb{Q}^2$, the comparison statement in ⓐ is strict.