

Exercises (1) Find all possible topologies on a set with two or three elements.

transitive
reflexive
& anti-sym.

(2) X (partially) ordered set. The sets $[x, \infty)$ form a base for a topology called the right topology. Arbitrary intersection of open sets is open in the right topology. What is the closure of $\{x\}$?

(b) Show that the right topology is Kolmogoroff, i.e., given two points x, x' either $\exists$ nbd x not containing x' or the other way around.

(c) Let X be a Kolmogoroff space in which arbitrary intersections of opens is open. Show that $x \in \overline{\{x'\}}$ is an order relation (x, x' in X). If this relation is written $\leq$, then the topology is identical with the right topology.

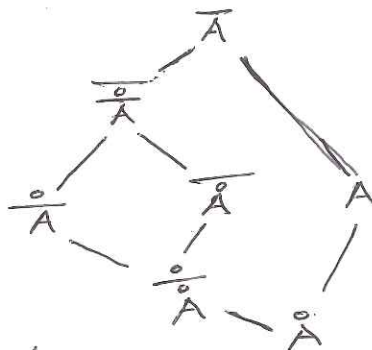
(d) From (c) deduce that a finite Kolmogoroff space has at least one isolated point. Thus in a Kolmogoroff space without isolated points ~~is finite~~ every non-empty open set is infinite.

(3) Set $\alpha(A) := \overset{\circ}{A}$, $\beta(A) := \overline{A}$. Observe $A \subseteq B \Rightarrow \alpha(A) \subseteq \alpha(B)$ & $\beta(A) \subseteq \beta(B)$.

(a) If A is open then $A \subseteq \alpha(A)$; if A is closed then $\beta(A) \subseteq A$.

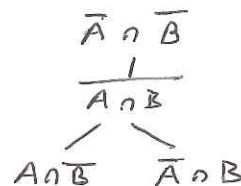
(b) Deduce that $\alpha(\alpha(A)) = A$ and $\beta(\beta(A)) = \beta(A)$ for any subset A .

(c) Give an example of the subset of real line s/t the sets in the following picture are all distinct and no inclusion holds other than the ones indicated (which always hold)



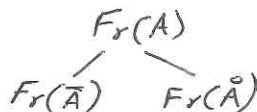
(d) Suppose $U \cap V = \emptyset$ and V is open. Show that $\alpha(U) \cap \overline{V} = \emptyset$ (so $\alpha(U) \cap \alpha(V) = \emptyset$).

(4) Give an example of open sets A & B on the real line s/t the sets in the following picture are all distinct and no inclusion holds other than those indicated (The indicated ones always hold: see part (b))



(b) If A is open, then $A \cap \overline{B} \subseteq \overline{A \cap B}$. On the real line, give an example of two intervals A, B s/t $A \cap \overline{B} \not\subseteq \overline{A \cap B}$.

(5) $F_r(A) \subseteq F_r(A)$; $F_r(\overset{\circ}{A}) \subseteq F_r(A)$. Give an example of A in the real line s/t the three sets in the diagram below are distinct and no inclusion holds other than the ones indicated (which always hold):



(b) $F_r(A \cup B) = (F_r(A) \cap \overline{B}) \cup (F_r(B) \cap \overline{A}) \subseteq F_r(A) \cup F_r(B)$. If $\overline{A} \cap \overline{B} = \emptyset$ then $F_r(A \cup B) = F_r(A) \cup F_r(B)$.

(c) If A & B are open then $(F_r(A) \cap B) \cup (A \cap F_r(B)) \subseteq F_r(A \cap B)$. In general, $F_r(A \cap B) \subseteq (A \cap F_r(B)) \cup (F_r(A) \cap B) \cup (F_r(A) \cap F_r(B))$.

(d) Give an example of A, B open in $\mathbb{R}$ s/t $(F_r(A) \cap B) \cup (A \cap F_r(B))$, $F_r(A \cap B)$, & $(F_r(A) \cap B) \cup (A \cap F_r(B)) \cup (F_r(A) \cap F_r(B))$ all three are distinct.