Chapter I. Topological Structures §1. Open sets; Nbhds; Closed sets

I.1.E E1

Exercises ① Find all possible topologies on a set with two or three elements.

transitive, reflexive & anti-symm.

② ⓐ X (partially) ordered set. The sets $[x,\infty)$ form a base for a topology called the right topology. Arbitrary intersection of open sets is open in the right topology. What is the closure of $\{x\}$?

ⓑ Show that the right topology is Kolmogoroff, i.e., given two points x, x', $x \neq x'$ either $\exists$ nbhd of x not containing x' or the other way around.

ⓒ Let X be a Kolmogoroff space in which arbitrary intersections of opens is open. Show that $x \in \overline{\{x'\}}$ is an order relation (x, x' in X). If this relation is written $\leq$, then the topology is identical with the right topology.

ⓓ From ⓒ deduce that a finite Kolmogoroff space has at least one isolated point. Thus in a Kolmogoroff space without isolated points ~~is infinite~~ every non-empty open set is infinite.

③ Set $\alpha(A) := \mathring{\overline{A}}$, $\beta(A) := \overline{\mathring{A}}$. Observe $A \subseteq B \Rightarrow \alpha(A) \subseteq \alpha(B)$ & $\beta(A) \subseteq \beta(B)$.

ⓐ If A is open then $A \subseteq \alpha(A)$; if A is closed then $\beta(A) \subseteq A$.

ⓑ Deduce that $\alpha(\alpha(A)) = A$ and $\beta(\beta(A)) = \beta(A)$ for any subset A.

ⓒ Give an example of the subset A of real line s/t the sets in the following picture are all distinct and no inclusion holds other than the ones indicated (which always hold)

$$\begin{tikzcd}[column sep=small]
 & & \overline{A} \arrow[dl, dash] \arrow[drr, dash] & \\
 & \overline{\mathring{\overline{A}}} \arrow[dl, dash] \arrow[dr, dash] & & \\
\mathring{\overline{A}} \arrow[dr, dash] & & \overline{\mathring{A}} \arrow[dl, dash] & A \arrow[ddl, dash] \\
 & \mathring{\overline{\mathring{A}}} \arrow[dr, dash] & & \\
 & & \mathring{A} &
\end{tikzcd}$$

ⓓ Suppose $U \cap V = \phi$ and V is open. Show that $\alpha(U) \cap \overline{V} = \phi$ (so $\alpha(U) \cap \alpha(V) = \phi$).

④ ⓐ Give an example of open sets A & B on the real line s/t the sets in the following picture are all distinct and no inclusion holds other than those indicated (The indicated ones always hold: see part ⓑ)

$$\begin{tikzcd}[column sep=small]
 & \overline{A} \cap \overline{B} \arrow[d, dash] & \\
 & \overline{A \cap B} \arrow[dl, dash] \arrow[dr, dash] & \\
A \cap \overline{B} & & \overline{A} \cap B
\end{tikzcd}$$

ⓑ If A is open, then $A \cap \overline{B} \subseteq \overline{A \cap B}$. On the real line, give an example of two intervals A, B s/t $A \cap \overline{B} \nsubseteq \overline{A \cap B}$.

⑤ ⓐ $Fr(\overline{A}) \subseteq Fr(A)$; $Fr(\mathring{A}) \subseteq Fr(A)$. Give an example of A in the real line s/t the three sets in the diagram below are distinct and no inclusion holds other than the ones indicated (which always hold):

$$\begin{tikzcd}[column sep=small]
 & Fr(A) \arrow[dl, dash] \arrow[dr, dash] & \\
Fr(\overline{A}) & & Fr(\mathring{A})
\end{tikzcd}$$

ⓑ $Fr(A \cup B) = (Fr(A) \cap \overline{\complement B}) \cup (Fr(B) \cap \overline{\complement A}) \subseteq Fr(A) \cup Fr(B)$. If $\overline{A} \cap \overline{B} = \phi$ then $Fr(A \cup B) = Fr(A) \cup Fr(B)$.

ⓒ If A & B are open then $(Fr(A) \cap B) \cup (A \cap Fr(B)) \subseteq Fr(A \cap B)$. ~~and~~ In general, $Fr(A \cap B) \subseteq (A \cap Fr(B)) \cup (Fr(A) \cap B) \cup (Fr(A) \cap Fr(B))$.

ⓓ Give an example of A, B open in $\mathbb{R}$ s/t $(Fr(A) \cap B) \cup (A \cap Fr(B))$, $Fr(A \cap B)$, & $(Fr(A) \cap B) \cup (A \cap Fr(B)) \cup (Fr(A) \cap Fr(B))$ all three are distinct.