Some algebraic results

Exercise: Show that in a commutative ring with identity every prime ideal is contained in a minimal prime ideal

Proof: Given a prime ideal $\mathfrak{p}$, consider all prime ideals contained in $\mathfrak{p}$. They form a partially ordered set w.r.t. reverse inclusion, which is inductive. Hence, by Zorn's lemma, there exists minimal prime ideals in $\mathfrak{p}$. QED

Exercise: A prime ideal $\mathfrak{p}$ (in a commutative ring with identity) is minimal iff $\forall\, x \in \mathfrak{p}\ \exists\, y \notin \mathfrak{p}$ and $n \geq 1$ s/t $x^n y = 0$.

Proof: $\Rightarrow$ Suppose not. Then $\{x^n y \mid y \notin \mathfrak{p}, n \geq 1\}$ is a multiplicatively closed set not containing 0 —call it S—. We may choose an ~~maximal~~ ideal maximal w.r.t. not meeting ~~the multiplicative~~ S. This is contained in $\mathfrak{p}$ by construction and prime as is well-known.

$\Leftarrow$ Suppose $\mathfrak{p} \supsetneq \mathfrak{q}$ with $\mathfrak{q}$ prime. Let $x \in \mathfrak{p} \setminus \mathfrak{q}$. Then $x^n y \notin \mathfrak{q}$ for any choice of n and any $y \notin \mathfrak{p}$ (even for $y \notin \mathfrak{q}$) so $x^n y \neq 0$. QED

Exercise: Suppose $x \in X$ s/t $\{x\}$ is a zero-set & x is not isolated. Then the maximal ideal $M_x = \{f \in C(X) \mid f(x) = 0\}$ is not a minimal prime ideal.

Proof: Choose $f \in C(X)$ s/t $f^{-1}(0) = \{x\}$. Such an f exists because $\{x\}$ is a zero set. Suppose $g \in C(X)$ s/t $f^n g \equiv 0$. Then $fg \equiv 0$. Since f is non-vanishing outside of $\{x\}$, g is identically zero outside of $\{x\}$, and so $g \equiv 0$ (being zero on a dense set). ☒ Thus $\cancel{g \notin M_x}$ $g \in M_x$. Use criterion of the previous exercise.

Remark: The proof shows that f as in the proof is a NZD.