Exercises: ① For $f, g \in C(X)$, we have $(f,g) \neq C(X)$ iff $Z(f)$ and $Z(g)$ do not meet, hence iff f^2+g^2 — or $|f|+|g|$ — is not a unit in $C(X)$.

② Let I be the ideal in $C(\mathbb{R})$ generated by the single element f where $f(x) = x$. (Thus I is a principal ideal.) Then $\mathcal{I}(Z[I])$ is the maximal ideal $\{g \in C(\mathbb{R}) \mid g(0) = 0\}$. ~~But~~ While $I \subseteq \mathcal{I}(Z[I])$ clearly, the inclusion is strict. E.g. $g(x) = x^{1/3}$ belongs to $\mathcal{I}(Z[I])$ but not to I (since $x \mapsto x^{-2/3}$ is not a continuous function on $\mathbb{R}$).

③ Show the following: ⓐ an ideal M in $C(X)$ is maximal iff $\forall f \in I$ s/t $Z(f)$ meets every member of $Z[M]$, we have $f \in M$.

ⓑ a z-filter $\mathcal{U}$ on X is ultra iff $\forall$ zero set Z that meets every member of $\mathcal{U}$ ~~always~~ belongs to $\mathcal{U}$.

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z-ideals & prime ideals in C(X)

An ideal in $C(X)$ is z-ideal if it is $\mathcal{I}(\mathcal{F})$ for a z-filter $\mathcal{F}$.

Remark: There is a 1-1 bijective correspondence between z-ideals of $C(X)$ and z-filters on X by means of $I \mapsto Z[I]$ and $\mathcal{F} \mapsto \mathcal{I}(\mathcal{F})$.

Remark: ① The intersection of an arbitrary family of z-ideals is a z-ideal.
② z-ideals are reduced and so every z-ideal is an intersection of prime ideals. ($\because Z(f^n) = Z(f)$)

Theorem: The following are equivalent for a z-ideal I in $C(X)$
① I is prime ② I contains a prime ideal ③ $\forall g,h \in C(X)$, if $gh=0$ then $g \in I$ or $h \in I$
④ $\forall f \in C[X]$ there exists a zero set in $Z[I]$ ~~s.t.f~~ on which f does not change sign.

Proof: ① ⇒ ② trivial. ② ⇒ ③ trivial. ③ ⇒ ④ $(f \vee 0)(0 \vee f) = 0$.
④ ⇒ ① Suppose $gh \in I$. Consider $|g| - |h|$. Choose a zero-set Z in $Z[I]$ s/t $|g|-|h|$ does not change sign on it. Suppose $|g| \geq |h|$ on Z. ~~Then every zero set of g on Z is a zero~~
Then every zero of g on Z is a zero of h. Hence $Z(h) \supseteq Z \cap Z(h) = Z \cap Z(gh)$
Thus $Z(h) \in Z[I]$ ($\because Z \cap Z(gh) \in Z[I]$ and $Z[I]$ is a z-filter) ~~Thus gh∈I~~
Thus $h \in I$. QED.

Exercise: Show that every prime ideal in $C(X)$ is contained in a unique maximal ideal.
(Proof: Let M_1 & M_2 be distinct maximal ideals. Then $M_1 \cap M_2$ is not prime and hence does not contain any prime ideal. Note that maximal ideals are z-ideals & intersections of z-ideals is a z-ideal, so the Theorem can be applied.)

Def: A z-filter $\mathcal{F}$ on X is prime if A & B are zero-sets s/t $A \cup B \in \mathcal{F}$ then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

Theorem: P prime in $C(X) \Rightarrow Z[P]$ is prime; $\mathcal{F}$ prime z-filter $\Rightarrow \mathcal{I}(\mathcal{F})$ is prime.

Pf: The second assertion is straightforward. For the first, use the fact that $Z[P] = Z[\mathcal{I}(Z[P])]$ and $\mathcal{I}(Z[P])$ is prime if P is prime (by Theorem) (since $\mathcal{I}(Z[P]) \supseteq P$). QED