X top space. $C(X)$ = ring of continuous (important) ★ real-valued functions on X. G.M 1

ZERO-SETS: For $f \in C(X)$, let $Z(f) := f^{-1}(0)$. This is called a zero-set in X.

$Z(X)$ = collection of all zero-sets in X.

Definition: A z-filter $\mathcal{F}$ on X is a collection of subsets of $Z(X)$ s/t

(z-F1) $\mathcal{F}$ is closed under supersets (z-F2) $\mathcal{F}$ has FIP (in particular $X \in \mathcal{F}$ & $\mathcal{F} \neq \emptyset$) (z-F3) $\emptyset \notin \mathcal{F}$.

Subbase of a z-filter: If $\mathcal{G}$ is any collection of subsets of $Z(X)$ having FIP, then $\mathcal{G}$ "generates" a z-filter: just take supersets of finite intersections of elements of $\mathcal{G}$. We say $\mathcal{G}$ is a subbase for the z-filter.

Remark: The notion of z-filters generalizes the notion of filters: If X has the discrete topology, then $Z(X)$ = power set of X, so that z-filters are just filters.

Remark: Given a filter $\mathcal{F}$ in X, $\mathcal{F} \cap Z(X)$ is a z-filter. Every z-filter arises this way: ~~just take~~ since the z-filter has FIP, we may just take the filter in X that it generates.

Theorem: [~~Correspondence between z-filters in X & proper ideals of C(X)~~]

Given a proper ideal I of $C(X)$, the collection $Z[I] := \{Z(f) \mid f \in I\}$ is a z-filter.

Conversely, given a z-filter $\mathcal{F}$, then $\mathcal{J}(\mathcal{F}) := \{f \in C(X) \mid Z(f) \in \mathcal{F}\}$ is a proper ideal of X.

Remark: $\mathcal{J}(Z[I]) \supseteq I$ (inclusion may be strict), $Z[\mathcal{J}(\mathcal{F})] = \mathcal{F}$ (Easy to see)

Proof of Theorem: First Suppose I is a proper ideal. Want to show $Z[I]$ is a z-Filter.

We need to check the three axioms of a z-Filter.

(z-F3) $\emptyset \notin Z[I]$. Suppose $\emptyset \in Z[I]$. Then $\exists f \in I$ s/t $Z(f) = \emptyset \Rightarrow f$ is a unit $\Rightarrow I$ not proper ✗

(z-F2) $Z(f) \cap Z(g) = Z(f^2+g^2)$

(z-F3) If $Z(g) \supseteq Z(f)$ then $Z(fg) = Z(g)$.

Conversely, given a z-filter $\mathcal{F}$, want to show $\mathcal{J}(\mathcal{F}) := \{f \in C(X) \mid Z(f) \in I\}$ is a proper ideal.

① $Z(f) \in \mathcal{F}$ and $Z(g) \in \mathcal{F} \Rightarrow Z(f) \cap Z(g) \in \mathcal{F}$. But $Z(f) \cap Z(g) \subseteq Z(f-g)$

so $Z(f-g) \in \mathcal{F}$ and so $f-g \in \mathcal{J}$.

② $0 \in \mathcal{J}(\mathcal{F})$ since $Z(0) = X \in \mathcal{F}$. $1 \notin \mathcal{J}(\mathcal{F})$ for $Z(f) = \emptyset \notin \mathcal{F}$.

③ $Z(f) \in \mathcal{F}$ and $g \in C[X] \Rightarrow Z(fg) \supseteq Z(f)$ and so $Z(fg) \in \mathcal{F}$ & so $fg \in \mathcal{J}(\mathcal{F})$ QED

Defn: z-ultrafilter is one that is not contained in any strictly larger z-filter.

Remark: z-ultrafilter is just a family of subsets of $Z(X)$ maximal w.r.t. having FIP.

By Zorn's lemma, any family with FIP is contained in a z-ultrafilter.

Theorem: (Correspondence between maximal ideals in $C(X)$ and z-ultrafilters on X)

The maps $I \mapsto Z[I]$ and $\mathcal{F} \mapsto \mathcal{J}(\mathcal{F})$ give a bijective correspondence between maximal ideals of $C(X)$ on the one hand and z-ultrafilters on X on the other hand.

Pf: This follows from the Theorem above and the Remark.

REFERENCE: GILLMAN & JERISON: Rings of Continuous Functions
Van Nostrand, 1966. Chapter 2.