COVERING SPACES (from Chevalley. Chap II § VI)

Def: X top space is locally connected if the connected nbhds of any point form a FSN.

Remark: X locally connected $\Rightarrow$ every connected component of an open set is open

Proof: $U \overset{\text{open}}{\subseteq} X$, P component of U. Let $p \in P$. By hypothesis $\exists$ conn. nbhd V of p contained in U. Since V connected & P component, we have $V \subseteq P$. So P is a nbhd of each of its points. So P open ▨

CoR: X locally connected $\Rightarrow$ The connected open nbhds form a FSN for any point.

Def: $E \xrightarrow{p} B$ continuous map of top spaces. A subset A of B is evenly covered by p if $p^{-1}(A)$ is not empty & every component of $p^{-1}(A)$ is mapped ~~homeomorphically~~ homeomorphically onto A by p.

Remark: clearly A evenly covered $\Rightarrow$ A connected.

Def: B top. space. A covering space of B is a space $\widetilde{B}$ and a map $p \overset{\text{cont}}{:} \widetilde{B} \to B$ s/t ① $\widetilde{B}$ is connected and locally connected. & ② every point of B has ~~an op~~ a nbhd that is evenly covered by p.

Rmk: If a space admits a covering space then it is connected & locally connected. Conversely, every connected and locally connected space admits at least one covering: namely the trivial one.

Propn: $p : \widetilde{B} \to B$ covering. Then p is open.

Proof: $\widetilde{U} \overset{\text{open}}{\subseteq} \widetilde{B}$. Let $p\tilde{u} = u \in U := p\widetilde{U}$, $\tilde{u} \in \widetilde{U}$. Let V be a nbhd of u that is evenly covered by p. Let $\widetilde{V}$ be the slice of $p^{-1}V$ ~~in which~~ containing $\tilde{u}$. Let $\widetilde{V} \cap \widetilde{U}$ is open in $\widetilde{V}$. Thus the image $p(\widetilde{V} \cap \widetilde{U})$ is open in V, and so a nbhd of u. Since $p(\widetilde{V} \cap \widetilde{U}) \subseteq p\widetilde{U} = U$, this nbhd of u is contained in U and U is thus open. ▨

Lemma 1 $p : \widetilde{B} \to B$ be a continuous map, $\widetilde{B}$ locally connected. Let $\tilde{b} \in \widetilde{B}$, and $b = p\tilde{b}$. Let V be a nbhd of b. Then the component of $p^{-1}V$ that contains $\tilde{b}$ is a nbhd of $\tilde{b}$.

Proof: Let $U \overset{\text{open}}{\text{in}} B$ s/t $b \in U \subseteq V$. Now $\tilde{b} \in p^{-1}U \subseteq p^{-1}V$; $p^{-1}U$ is open since p is continuous. Its component containing $\tilde{b}$ is open ($\because \widetilde{B}$ locally connected). And that component is contained in the component of $p^{-1}V$ containing $\tilde{b}$. QED