8.4 The Fundamental gp of the circle.

Lemma (Path lifting) $p: E \to B$ covering map. $p(e_0) = b_0$. Any path $f: [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.
Proof: Reading exercise

Lemma (lifting of path homotopies) $p: E \to B$ covering. Let $p(e_0) = b_0$. Let $F: I \times I \to B$ be continuous with $F(0,0) = b_0$. There is a ! lifting of F to a continuous map $\tilde{F}: I \times I \to E$ s/t $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy. ~~It is also unique~~ Note: unique

Theorem: $p: E \to B$ covering. $p(e_0) = b_0$. Let f and g be two paths from b_0 to b_1. Let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

Theorem: $\pi_1(S^1, 1) = \mathbb{Z}$.

Proof: $\mathbb{R} \xrightarrow{p} S^1$; $x \mapsto \exp(2\pi i x)$ covering. $p^{-1}(1) = \mathbb{Z}$.
By the preceding theorem, $\exists$ map $\pi_1(S^1, 1) \to \mathbb{Z}$. This map is onto since $\mathbb{R}$ is path connected. It is 1-1 because $\mathbb{R}$ is contractible.
Easy to prove that this map is a homomorphism of gps. $\boxed{\Xi}$

Theorem: $p: (E, e_0) \to (B, b_0)$ be a covering map. If E is path connected, then we have ~~(by path lifting &)~~ (by liftings of paths & path homotopies) a surjective map $\pi_1(B, b_0) \to p^{-1}(b_0)$. [given a loop, lift it to ~~one at~~ a path starting at e_0]. If E is simply connected, this map is an injection.

Def: Universal covering space. = Simply connected covering space
includes path connectedness