FUNDAMENTAL GROUP

HOMOTOPY OF MAPS: $f, g : X \to Y$, $F : X \times I \to Y$ s/t $F_0 = f$ & $F_1 = g$.
This is an equivalence relation.
Question: Why does pasting together $X \times [0,1]$ & $X \times [1,2]$ along $X \times \{1\}$ give a homeomorphism to $X \times [0,2]$?

PATH HOMOTOPY OF PATHS BETWEEN TWO POINTS $F : [0,1] \times [0,1] \to Y$
s/t $F(s,0) = f$, $F(s,1) = g$, $F(0,t) = a$ & $F(1,t) = b$ for all t.
$\simeq_p$ (path homotopy) is an equivalence relation.

Example: Any $f, g : X \to \mathbb{R}^n$ are homotopic. Straight-line homotopy
(How do you show continuity?)

COMPOSITION OF PATHS f: path from x to y & g path from y to z
Then $f * g$ path from x to z.

PROPN Composition is well-defined on path-homotopy classes. It has the following properties: associativity; right & left identities; inverse

Fundamental group definition: $\pi_1(X, x_0) =$ loops at x_0 / path-homotopy

Example: $\pi_1(\mathbb{R}^n, x_0)$ is trivial. (straight-line homotopy)
More generally, convex subsets of $\mathbb{R}^n$ have trivial fundamental gp.

Propn X path conn, $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ but the ~~homomorphic~~ isomorphism need not be unique. So we cannot talk about the fundamental group of a space in general without reference to the base point.

Def: X simply connected if it is path-connected & $\pi_1(X, x_0) = 0$
(independent of the choice of the base point).

~~Lemma~~ Propn: X simply connected. If f and g are both paths from a to b then $f \simeq_p g$.

Proof: $f * \bar{g} \simeq_p e_a$. ~~An~~ Right compose by g. QED

Propn: ~~X ... X, x_0~~ The fundamental gp is a functor from pointed topological spaces to groups.