Compact Open topology ~~& joint continuity~~

Let now X top space, Y top. space and $\mathcal{Z} \subseteq Y^X$. We have a natural evaluation pairing $\mathcal{Z} \times X \to Y$. Question: for which topologies on $\mathcal{Z}$ is this evaluation map continuous (w.r.t. product top on $\mathcal{Z} \times X$)?

Notation: For $K \subseteq X$ & $U \subseteq Y$, set $\mathcal{Z}(K,U) := \{ f \in \mathcal{Z} \mid f(K) \subseteq U \}$.

Defn of compact open top: Take $\mathcal{Z}(K,U)$, K compact in X, U open in Y to be a subbase. Thus a basis for this top is: $\bigcap_{i=1}^{n} \mathcal{Z}(K_i, U_i)$, $K_i \overset{\text{compact}}{\subseteq} X$, $U_i \overset{\text{open}}{\subseteq} X$.

Theorem: The compact open top $\mathcal{C}$ contains the top $\mathcal{P}$ of pointwise convergence. $\mathcal{C}$ is Hdf if Y is so. $\mathcal{C}$ is regular if Y is regular & elts of $\mathcal{Z}$ are cont.

Proof: The first is clear since singletons are compact. If Y is Hdf, then $\mathcal{P}$ is Hdf, so $\mathcal{C}$ is Hdf. Proof of regularity of $\mathcal{C}$ given that of Y: Let $f \in \mathcal{Z}$ and $Z \overset{\text{open}}{\subseteq} \mathcal{Z}$ s/t $f \in Z$. Want to show $\exists$ cl closed nbhd of f contained in Z. WLOG we may take Z to be a subbasis elt, so put $Z = \mathcal{Z}(K,U)$. Now $f(K)$ is compact $\subseteq U$. By regularity of Y, $\exists$ V closed nbhd of $f(K)$ s/t $f(K) \subseteq V \subseteq U$. Now $\mathcal{Z}(K,V) \subseteq \mathcal{Z}(K,U)$. Since V is a nbhd of $f(K)$, we have $\mathcal{Z}(K,V)$ is a nbhd of f in $\mathcal{Z}$. Moreover it is closed since $\mathcal{Z}(K,V) = \bigcap_{x \in K} \mathcal{Z}(x,V)$ and each $\mathcal{Z}(x,V)$ is $\mathcal{P}$-closed hence $\mathcal{C}$ closed. QED.

Remark: If X is discrete then $\mathcal{C} = \mathcal{P}$. Note that a product of normal/C_I/C_{II} spaces does not necessarily have that property. So there is no hope of proving that $\mathcal{C}$ is normal/C_I/C_{II} ~~from the~~ assuming the corresponding properties for Y.